Chapter 4

Quaternionic geometry, by P.PICCINNI

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Solved and unsolved problems in quaternionic geometry[1]

4.1 The birth of H

"...Tomorrow will be the fifteenth birthday of the Quaternions. They started into life, or light, full grown, on the 16th of October, 1843, as I was walking withLady Hamilton to Dublin, and came up to Brougham Bridge. This is to say, I then and there felt the galvanic circuit of thought closed, and the sparks which fell from it were the fundamental equations between i, j, k *exactly such* as I have used them ever since. I pulled out, on the spot, a pocketbook, which still exists, and made an entry, on which, *at the very moment*, I felt that it might be worth my while to expend the labour of at least ten (or it might be fifteen) years to come..."

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William Rowan Hamilton (1805-1865) described with these words his discovery of quaternions (*North British Rev.* **14**, 1858), after long unsuccessful attempts to construct "three dimensional complex numbers". Although the name of Hamilton is related to so important principles and equations in mechanics and symplectic geometry, he was so proud of his quaternions to regard their discovery as his highest contribution to science, and he expected that quaternions should constitute a key method for the development of geometry and physics. Accordingly, quaternions inspired a good part of Hamilton's activity after 1843 (cf. the recent electronic publication of his mathematical papers in [8]), and of course quaternions are the subjects of the treatises [9], [10], where many chapters of geometry and astronomy are presented in the light of the new discovery. It is also reported (cf. [13], p. 911) that Hamilton arrived, in his enthusiasm, to compare the creation of quaternions with that of the infinitesimal calculus.

Of course, this estimation of the power of quaternions did not meet the favor of the history. However, during the last two decades of the twentieth century, the fortune of quaternions encountered a new stage. Although it would be definitely too much to assert that Hamilton's hopes in quaternions were vindicated in recent years, it is certain that a strong increase of interest in quaternions arose exactly in the fields foreseen by Hamilton, namely geometry and physics (cf. the proceedings of the recent meetings [25], [26]).

Thus the usual notation

$$\mathbf{H} = \{q = q_0 + q_1 i + q_2 j + q_3 k,\ q_\alpha \in \mathbf{R}\},$$

of the algebra of quaternions honors Hamilton, and the fundamental equations he discovered in 1843 are:

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

The non commutativity law of $\mathbf{H}$ follows, related to the conjugation $\overline{q} = q_0 - q_1 i - q_2 j - q_3 k$ of quaternions. Namely, for any $q, h \in \mathbf{H}$, one has: $\overline{qh} = \overline{h}\,\overline{q}$.

One of the geometrical applications of $\mathbf{H}$ in the nineteenth century was the description of rotations in $\mathbf{R}^3$ and in $\mathbf{R}^4$, obtained by W. Hamilton (1853) and by A. Hurwitz (1896), respectively. These can be summarized by the two-fold covering spaces

$$Sp(1) \to SO(3), \qquad Sp(1) \times Sp(1) \to SO(4),$$

defined by interpreting rotations of vectors $\vec{x} \in \mathbf{R}^3 \cong \mathrm{Im}\ \mathbf{H}$ or $\mathbf{R}^4 \cong \mathbf{H}$ respectively by multiplications

$$\overline{q}\ \vec{x}\ q, \qquad\qquad q'\ \vec{x}\ q,$$

where $q, q' \in Sp(1) \cong S^3$, the Lie group of unit quaternions.

It is interesting to note that these two elementary applications of $\mathbf{H}$ involve for $n = 1$ the two groups

$$Sp(n), \qquad\qquad Sp(n) \cdot Sp(1),$$

(the latter isomorphic to $SO(4)$ when $n = 1$), much later recovered as defining the two significative *hyperkähler* and *quaternion Kähler* holonomies, related to $\mathbf{H}$.

The title of this paragraph indicates also a short introduction, and this is just to advice the reader that the choice of topics contained in this report was guided by the way I viewed the developments of quaternionic geometry, an interest well present in Rome for a long time. My exposition is also based on personal knowledges and tastes, and of course it cannot compete with any of the excellent overviews of several aspects of quaternionic geometry that recently appeared in the volume collecting *Essays on Eintein Manifolds* [7]. I wish to thank Sorin Dragomir for inviting me to talk on this subject at Università della Basilicata and to write the present short survey.

4.2 Quaternionic Analysis

Neither of the two fundamental definitions of regular function of a complex variable seems to be of interest when adapted to functions $f : q \in \mathbf{H} \to h \in \mathbf{H}$ of a quaternionic variable. It is in fact easy to see that the $h = f(q)$ which are holomorphic in quaternionic sense are necessarily linear ($h = aq + b$, if holomorphicity on the left is required). On the other hand, functions $h = f(q)$ which can be represented by quaternionic power series are just those that are sums of power series in four real variables. In other words, the definition of holomorphic quaternionic function is too restrictive and that of analytic quaternionic function is not restrictive enough to hope to build on them a significative theory.

A different approach to quaternionic analysis was developed by R. Fueter starting in 1936 (although some of his ideas are also in a work by Gr. Moisil of 1931). Fueter proposed to define *regular function of a quaternionic variable* any $f : q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbf{H} \to h \in \mathbf{H}$ satisfying the following "Cauchy-Riemann condition":

$$\frac{\partial f}{\partial q_0} + \frac{\partial f}{\partial q_1} i + \frac{\partial f}{\partial q_2} j + \frac{\partial f}{\partial q_3} k = 0.$$

If Dq is the 3-form in $\mathbf{H}$ transformed of dq by the Hodge star, this condition is equivalent to the closeness of the 3-form fDq, so that Stokes' theorem can be set in motion. Thus one can prove a Cauchy theorem, a Cauchy integral formula and the Laurent series for regular $h = f(q)$, as well as a Hartogs' theorem and a Bochner-Martinelli formula for functions of several quaternionic variables.

There are many regular functions of a quaternionic variable, since it is possible to construct them by starting from a harmonic $f_0 : q \in \mathbf{H} \to \mathbf{R}$ or from a complex analytic $g : z \in \mathbf{C} \to \mathbf{C}$). However, the powers $h = q^n$ are not regular (even for $n = 1$, so that the identity function does not enter in quaternionic analysis).

Nowadays, the Fueter's approach to quaternionic analysis seems to be the unique one that was developed (see [30] for a nice survey). Its adaptability to integral representation formulas is the historic reason of the interest of E. Martinelli for $\mathbf{H}$, extended later to the rôle of $\mathbf{H}$ in differential geometry (see the next paragraph). This interest was then transmitted to several students of him, and later shared by other Italian geometers. Among the names one can mention here are those of F. Battaglia, M. Bruni, A. Fino, G. Gentili, S. Marchiafava, E. Musso, K. O' Grady, D. Pertici, F. Podestà, M. Pontecorvo, G.B. Rizza, G. Romani, D. Struppa, F. Tricerri, L. Verdiani,...

New possible developments of quaternionic analysis can be expected by the recent proposal by D. Joyce (see for example his contribution in [26]) to consider regular functions of quaternionic variables on hypercomplex manifolds, and to use them as an approach to a *quaternionic algebraic geometry.*

4.3 Classes of "quaternionic" manifolds

Manifolds M^{4n} equipped with local quaternionic coordinates such that the jacobians of the changes of coordinates belong to the linear group $GL(n, \mathbf{H})$

are of course the first candidates to introduce a quaternionic geometry. However, the discussion in the previous paragraph shows that such manifolds are *locally affine*, i. e. with linear changes of local coordinates. A systematic study of such manifolds was carried out in a work by A. Sommese, 1975, and for $n = 1$ the only compact M^4 carrying such an integrable structure have been shown to be either tori or some particular Hopf surfaces (Ma. Kato, 1980).

On the other hand, by dropping the assumption of integrability, the requirement of having a $GL(n, \mathbf{H})$-structure on the tangent bundle is quite natural, and can be expressed as follows.

Definition 2
An *almost hypercomplex structure* on a smooth $4n$-dimensional manifold M^{4n} is an ordered triple $H = (I_1, I_2, I_3)$ of almost complex structures satisfying the quaternionic identities $I_\alpha \circ I_\beta = I_\gamma$ for $(\alpha, \beta, \gamma) = (1, 2, 3)$ and cyclic permutations. If the structures I_1, I_2, I_3 are integrable, H is said to be a *hypercomplex structure.*

Note that here the integrability condition corresponds to the existence of a torsionless connection preserving H, that is much weaker than requiring the integrability of the $GL(n, \mathbf{H})$-structure. If $H = (I_1, I_2, I_3)$ is an almost hypercomplex structure, any triple (J_1, J_2, J_3) obtained from (I_1, I_2, I_3) by multiplying by a matrix of $SO(3)$ is again an almost hypercomplex structure. Moreover, there is a set of *compatible almost complex structures* $J = a_1 I_1 + a_2 I_2 + a_3 I_3$, where a_1, a_2, a_3 are functions satisfying $a_1^2 + a_2^2 + a_3^2 = 1$. The following classes of manifolds are also of interest.

Definition 3
An *almost quaternionic structure* on M^{4n} is a rank 3 subbundle Q of the endomorphism bundle $\mathrm{End}(TM)$, locally spanned by almost hypercomplex structures $H = (I_1, I_2, I_3)$ which are related by $SO(3)$-matrices on the intersections of trivializing open sets. A *quaternionic structure* on the manifold M^{4n} is an almost quaternionic structure such that there exists a torsionless connection ∇ of TM which, when extended to $\mathrm{End}(TM)$, preserves Q, i.e. $\nabla Q \subseteq Q$.

Note again that the existence of the Q-preserving torsionless connection ∇ is not equivalent with the integrability of Q as a G-structure. If an almost quaternionic structure Q is fixed on M^{4n}, the local bases (I_1, I_2, I_3) of section of Q are also called *local compatible almost hypercomplex structures*, and any local $J = a_1 I_1 + a_2 I_2 + a_3 I_3$ with $a_1^2 + a_2^2 + a_3^2 = 1$ is a *local compatible almost complex structure*. Next, a Riemannian metric g on the hypercomplex manifold (M, H) is said to be *hyperhermitian*, respectively *hyperkähler*, if it is Hermitian, respectively, Kählerian, with respect to I_1, I_2, I_3. Similarly, on an quaternionic manifold (M, Q), g is *quaternion Hermitian* if it is Hermitian with respect to the local bases (I_α) of Q, and *quaternion Kähler* if moreover Q is parallel with respect to the Levi-Civita connection. If M is only almost hypecomplex or almost quaternionic, then almost hyperhermitian and almost quaternion Hermitian are similarly defined.

The *twistor space* Z, defined for any almost quaternionic (M, Q), is the manifold of the compatible almost complex structures on the tangent spaces of M. Thus, Z is the S^2-bundle associated with the vector bundle Q, equipped with the metric which makes orthonormal the local bases $H = (I_1, I_2, I_3)$ [4], [27].

The origin of both hyperkähler and quaternion Kähler manifolds can be traced back to the following celebrated:

Theorem 14 (M. Berger, 1955) *[4] The possible holonomy representations of simply connected irreducible and non-symmetric Riemannian manifolds are given by the following list:*

$$SO(n), \qquad U(n), SU(n), \qquad Sp(n), Sp(n) \cdot Sp(1), \qquad G_2, Spin(7).$$

More explicit evidence to the group:

$$Sp(n) \cdot Sp(1) = \frac{Sp(n) \times Sp(1)}{\pm(I, 1)}$$

was given a few years later by E. Martinelli through his study of the Fubini-Study metric of $\mathbf{H}P^n$, and the consequent proposal to consider $Sp(n) \cdot Sp(1)$ as a structure group of a worthly to study "generalized" quaternionic geometry [20], [21]. In these works, the rôle of a "fundamental" 4-form $\Omega = \omega_I^2 + \omega_J^2 + \omega_K^2$ is recognized, globally defined in terms of the Kähler 2-forms of the local almost complex structures I, J, K. The structures we just defined are summarized in the following tables.

TABLE 1. Structures related to **H**

M^{4n}	$GL(n, \mathbf{H})$ *almosthypercomplex*	$GL(n, \mathbf{H}) \cdot Sp(1)$ *almostquaternionic*
M^{4n}	$Gl(n, \mathbf{H})$ *with* $T^{\nabla} = 0$ *hypercomplex*	$Gl(n, \mathbf{H}) \cdot Sp(1)$ *with* $T^{\nabla} = 0$ *quaternionic*

TABLE 2. Riemannian structures related to **H**

(M^{4n}, g)	$Sp(n)$ *with* $T^{\nabla} = 0$ *hyperhermitian*	$Sp(n) \cdot Sp(1)$ *with* $T^{\nabla} = 0$ *quat.Hermitian*
(M^{4n}, g)	*with* $Hol^g \subset Sp(n)$ *hyperkaehler*	*with* $Hol^g \subset Sp(n) \cdot Sp(1)$ *quat.Kaehler*

Since $Sp(n) \subset SU(2n)$, any hyperkähler metric on a M^{4n} is Ricci-flat. More generally, quaternion Kähler metrics can be shown to be Einstein, and as such they give rise to three quite different situation according to the sign of the scalar curvature $s > 0$, $s = 0$, or $s < 0$.

4.4 Hyperkähler Manifolds

The theory of hyperkähler manifolds M^{4n} has become a beautiful and interesting field of research, with ideas of algebraic and differential geometry and motivations arising in gauge theory, like Nahm's equations and monopole moduli spaces. Such a theory used to appear much more restrictive until the late seventies, when it was unclear even the existence of significative examples, besides the flat ones, in particular for $n \geq 2$.

One has to mention of course that the existence of a hyperkähler metric on any $K3$ surface is an immediate consequence of Calabi-Yau theorem (1976), due to the vanishing of the first Chern class and to the isomorphism $Sp(1) \cong SU(2)$. Although some properties of these hyperkähler metrics on $K3$ surfaces have been described in special cases (like in the work of D. Alekseevsky - M. Graev, 1990), no explicit expressions of such metrics are known, "and it seems likely that no such formulae exist; that is, that these metrics are transcendental objects that admit no exact algebraic description" ([12], p. 160). Two series of compact manifolds $K^{[n]}$ and K_n of real dimension $4n$, $n \geq 2$, and admitting large families of hyperkähler matrics were constructed by A. Beauville in 1982, generalizing an earlier construction of $K^{[2]}$ by A. Fujiki. These examples, having a second Betti number $b_2 = 23$ and $b_2 = 7$ respectively, remain the only known series of compact hyperkähler manifolds. Two new compact examples have been recently constructed by K. O' Grady (1997, 2000), of real dimension $4n = 20$ and $4n = 12$ and with second Betti number $b_2 \geq 24$ and $b_2 = 8$, respectively. The existence of a hyperkähler metric on a compact M^{4n} gives strong restrictions to the Euler characteristic χ, the signature τ and the Betti numbers, like [28]:

$$n\chi(M) \text{ is divisible by } 24,$$

$$\chi(M), \tau(M), b_{2n}(M) \text{ are even unless } 4n \text{ is divisible by } 32.$$

On the non compact front, in 1979 E. Calabi constructed a hyperkähler metric on $T^*(\mathbf{C}P^n)$, the cotangent bundle of complex projective spaces, extending a similar construction given the year before by Eguchi-Hanson in the case $n = 1$. It was in that occasion that Calabi proposed the term "hyperkähler", describing the existence of a S^2-family of complex structures each of which is Kähler with respect to the given metric. Hyperkähler metrics were later shown to exist on cotangent bundles of more general classes of Kähler manifolds, in particular by writing explicit hyperkähler potentials on T^* of Hermitian symmetric spaces (O. Biquard - P. Gauduchon, 1995). The most general result in this direction is probably the recent independent constructions by B. Feix and D. Kaledin of a hyperkähler metric in an open neighborhood of the zero section on the cotangent bundle of any complex

Kähler manifold (see the contribution by Kaledin in [26]). The quotient construction by reduction of symplectic and Kähler geometry can be pursued in the hyperkähler setting as well, and this was done in [11]. See also the survey of M. Atiyah [3], where the author tells how his view on the interest of hyperkähler manifolds "was radically changed by the discovery (in [11]) of a very simple and beautiful quotient construction which generates vast numbers of hyperkähler manifolds in a natural way". Consider a group of G of authomorphisms of a hyperkähler manifold M^{4n}, and look at the symplectic forms $\omega_{I_1}, \omega_{I_2}, \omega_{I_3}$ on M. Then in good cases (for example, if G is semisimple) the symplectic moment maps $\mu_{I_1}, \mu_{I_2}, \mu_{I_3}$ exist and they can be combined into a single hyperkähler moment map:

$$\mu : M \to \mathsf{g}^* \otimes \mathbf{R}^3,$$

where g is the Lie algebra of G. Then:

Theorem 15 [*11*] *If G acts freely and properly on $\mu^{-1}(0)$, then the quotient $\mu^{-1}(0)/G$ is hyperkähler.*

This theorem produces very interesting examples already starting from the simplest choice $M = \mathbf{H}^n = \mathbf{C}^n \times \mathbf{C}^n$. One can recover the Calabi metric on $T^*(\mathbf{C}P^{n-1})$ as a quotient of the flat space acted on by $G = S^1$, $e^{it}(z, w) = (e^{it}z, e^{-it}w)$. The hyperkähler cotangent bundles of complex Grassmannians can be obtained as well as hyperkähler quotient, by acting by $G = U(n)$ on a $\mathbf{H}^{n(n+m)}$. Again the flat space $M = \mathbf{H}^2 = \mathbf{C}^2 \times \mathbf{C}^2$, but acted on by $G = \mathbf{R}$, $t(z_1, z_2, w_1, w_2) = (e^{it}z_1, e^{-it}z_2, w_1 + t, w_2)$, produces the Taub-NUT hyperkähler metric on $\mathbf{R}^4$, also related to a family of self dual Einstein metrics with negative scalar curvature [6].

4.5 Quaternion Kähler Manifolds

Recall that quaternion Kähler manifolds M^{4n}, being Einstein, belong to three different classes, according to the sign of the scalar curvature:

$$\begin{cases} s > 0, & \text{positive (if also complete)} \\ s = 0, & \text{locally hyperkähler} \\ s < 0, & \text{negative} \end{cases}$$

The case $s > 0$ relates rapidly with complex algebraic geometry, via the twistor space Z, defined for the larger class of almost quaternionic manifolds (cf. paragraph 3 as well as Table 1). Z has real dimension $4n + 2$, it is equipped with a natural almost complex structure, integrable if and only if M is quaternionic, and the fibers of $Z \to M$ are rational curves [27], [4], [2]. The complex manifold Z is of special interest when M is quaternion Kähler, and then a much richer structure can be defined on Z. If the scalar curvature s of the quaternion Kähler M is non-zero, then Z admits a complex contact structure, such that the line bundle $L = TZ/D$, quotient of the tangent bundle by the contact distribution, restricts on the fibers of $Z \to M$ to $\mathcal{O}(2)$. Moreover, for $s > 0$, a Kähler Einstein metric with positive scalar curvature is naturally defined on Z, making $Z \to M$ a Riemannian submersion with totally geodesic fibers, and the Kähler 2-form is proportional to the curvature of a natural connection on L, so that L is an ample line bundle [27]. All of this makes the twistor space Z of a positive quaternion Kähler M a *contact Fano manifold*, a key property to prove the following:

Theorem 16 [*17*] *For any n there are - up to isometries and rescaling - only finitely many positive quaternion Kähler manifolds* M^{4n}.

This theorem is only one of the arguments supporting the following:

Conjecture 1 (LeBrun - Salamon) *Any positive quaternion Kähler* M^{4n} *is symmetric.*

The positive symmetric quaternion Kähler manifolds were classified by J. Wolf (1965). For any compact simple Lie group G with trivial center there is a positive quaternion Kähler "Wolf space" $G/K \cdot Sp(1)$, with twistor space $G/K \cdot U(1)$. Here $K \subset Sp(n)$ and G has Lie algebra $\mathsf{g} = \mathsf{k} + \mathsf{sp}(1) + \mathsf{m}$ with $[\mathsf{k} + \mathsf{sp}(1), \mathsf{m}] \subset \mathsf{m}$. More explicitly, the Wolf spaces are:

$$\mathbf{H}P^n, \quad Gr_2(\mathbf{C}^{n+1}), \quad Gr_4(\mathbf{R}^{n+1}),$$

$$\frac{G_2}{SO(4)}, \frac{F_4}{Sp(3)Sp(1)}, \frac{E_6}{SU(6)Sp(1)}, \frac{E_7}{Spin(12)Sp(1)}, \frac{E_8}{E_7Sp(1)},$$

where Gr denotes the Grassmannian and the oriented Grassmannian, in complex and real vector spaces, respectively. Thus the Wolf spaces are homogeneous spaces of the groups:

$$Sp(n+1),\ SU(n+1),\ SO(n+1),\ G_2,\ F_4,\ E_6,\ E_7,\ E_8.$$

The above conjecture has been proved under any of the following hypotheses:

(a) if M^{4n} is homogeneous (Alekseevsky, 1968); (b) if $n = 1$ (Hitchin, 1987) or $n = 2$ (Poon - Salamon, 1991); (c) if $b_2 > 0$ or if $n \leq 4$ and $b_4 = 1$ (LeBrun - Salamon, 1994). Very recently, two independent preprints appeared containing partial proofs of the LeBrun - Salamon conjecture, although some gaps to obtain a complete proof seem still unsolved.

4.6 Quotients in Quaternion Kähler Geometry

Hyperkähler and quaternion Kähler geometries are related by remarkable fibrations, roughly described in Table 3 (the left column indicates the prototypes). These fibrations over manifolds M^{4n} were introduced, going from the bottom to the top of the table, by L. Berard Bergery and S. Salamon (1980), by M. Konishi (1975), by A. Swann (1990).

TABLE 3. Fibrations among structures related to **H**

$\mathbf{H}^{n+1} - 0$	$C(\mathcal{S})^{4n+4}$	*hyperkaehlercone*
$\downarrow$	$\downarrow \mathbf{R}^+$	$\downarrow$
S^{4n+3}	$\mathcal{S}^{4n+3}$	$3 -$ *Sasakianbundle*
$\downarrow$	$\downarrow \mathbf{S^1}$	$\downarrow$
$\mathbf{C}P^{2n+1}$	Z^{4n+2}	*twistors*
$\downarrow$	$\downarrow \mathbf{S^2}$	$\downarrow$
$\mathbf{H}P^n$	M^{4n}	*positive quat. Kaehler*

For each of the four structures listed in Table 3 there is a reduction theorem allowing to reproduce the geometry on quotients, provided there is an action of a Lie group preserving the structure. These theorems were obtained between the late eighties and the early nineties. One of them, for the hyperkähler case, is of course Theorem 4.1, and the others are due to Boyer - Galicki - Mann for 3-Sasakian bundles, to Hitchin for twistors, and toGalicki - Lawson for quaternion Kähler structures (cf. [7] for their explicit descriptions). For example, the diagonal action of $U(1)$ yields, at 3-Sasakian and quaternion Kähler levels:

$$\begin{array}{ccc} S^{4n+3} & \overset{U(1)}{\Longrightarrow} & \frac{SU(n+1)}{S(U(n-1)\times U(1))} \\ \downarrow & & \downarrow \\ \mathbf{H}P^n & \overset{U(1)}{\Longrightarrow} & Gr_2(\mathbf{C}^{n+1}), \end{array} \tag{4.1}$$

where we denote by $\Longrightarrow$ the reduction procedure. Similarly, the diagonal action of $Sp(1)$ gives:

$$\begin{array}{ccc} S^{4n+3} & \overset{Sp(1)}{\Longrightarrow} & \frac{SO(n+1)}{SO(n-3)\times Sp(1))} \\ \downarrow & & \downarrow \\ \mathbf{H}P^n & \overset{Sp(1)}{\Longrightarrow} & Gr_4(\mathbf{R}^{n+1}). \end{array} \tag{4.2}$$

Thus the two series of *classical* Wolf spaces:

$$Gr_2(\mathbf{C}^{n+1}), \qquad Gr_4(\mathbf{R}^{n+1}),$$

can be obtained as quaternion Kähler quotients of $\mathbf{H}P^n$. The following problem is then quite natural:

Problem 1 *Can the exceptional Wolf spaces:*

$$\frac{G_2}{SO(4)}, \quad \frac{F_4}{Sp(3)Sp(1)},$$

$$\frac{E_6}{SU(6)Sp(1)}, \quad \frac{E_7}{Spin(12)Sp(1)}, \quad \frac{E_8}{E_7Sp(1)}$$

be also obtained as quaternion Kähler quotients of some $\mathbf{H}P^n$*?*

For the first exceptional Wolf space, $G_2/SO(4)$, of real dimension 8, a (partially) positive answer is given by the following result:

Proposition 6 *[14] The real Grassmannian $Gr_4(\mathbf{R}^7)$ is acted on by the diagonal subgroup*

$$U(1) \subset U(3) \subset SO(6) \subset SO(7),$$

and the quaternion Kähler quotient is an orbifold $\mathbf{Z_3} \backslash \mathbf{G_2}/\mathbf{SO(4)}$.

By combining this with the $Sp(1)$-reduction from $\mathbf{H}P^6$ we obtain:

$$\begin{array}{ccccc} S^{27} & \stackrel{Sp(1)}{\Longrightarrow} & \frac{SO(7)}{SO(3)\times Sp(1)} & \stackrel{U(1)}{\Longrightarrow} & \mathbf{Z_3}\backslash \mathbf{G_2}/\mathbf{Sp(1)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{H}P^6 & \stackrel{Sp(1)}{\Longrightarrow} & Gr_4(\mathbf{R}^7) & \stackrel{U(1)}{\Longrightarrow} & \mathbf{Z_3}\backslash \mathbf{G_2}/\mathbf{SO(4)}. \end{array} \tag{4.3}$$

The above $U(1) \times Sp(1)$-quotient can be deformed by introducing weights in the $U(1)$-action, and suitable choices of the weights allow to obtain smooth 11-dimensional quotients at the 3-Sasakian level. The same is true by using a similar action on the Hopf fibration $S^{31} \to \mathbf{H}P^7$, yielding now 15-dimensional manifolds. These "exceptional" 11 and 15-dimensional manifolds, constructed in [5], are the first examples that are 3-Sasakian and neither homogeneous nor toric. Moreover, the 15-dimensional examples are deformations of an action that, read on the quaternion Kähler basis, gives as quotient an orbifold $\mathbf{Z_2}\backslash$CAYLEY.

The symbol CAYLEY was introduced in papers on calibrations to denote the symmetric space

$$\frac{Spin(7)}{(Sp(1) \times Sp(1) \times Sp(1))/\mathbf{Z_2}},$$

isometric to the Wolf space $Gr_4(\mathbf{R}^7)$. As such, CAYLEY can be also obtained by a $Sp(1)$-reduction of $\mathbf{H}P^6$. However, the reduction described in the diagram

$$\begin{array}{ccccc} S^{31} & \stackrel{Sp(1)}{\Longrightarrow} & \frac{SO(7)}{SO(4)\times Sp(1)} & \stackrel{U(1)}{\Longrightarrow} & \mathbf{Z_2}\backslash \frac{\mathbf{Spin(7)}}{\mathbf{Sp(1)}\times\mathbf{Sp(1)}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{H}P^7 & \stackrel{Sp(1)}{\Longrightarrow} & Gr_4(\mathbf{R}^8) & \stackrel{U(1)}{\Longrightarrow} & \mathbf{Z_2}\backslash \text{CAYLEY} \end{array} \tag{4.4}$$

is interesting not only for its implications in 3-Sasakian geometry, but also for the following further discussion.

Consider $\mathbf{H}P^{15}$, acted on by $Sp(2)$. The quaternion Kähler quotient is now a finite quotient of a 20-dimensional manifold, for whose a geometric interpretation can suggest the notation BICAYLEY. It is the symmetric space

$$\frac{Spin(9)}{(Sp(2) \times Sp(1) \times Sp(1))/\mathbf{Z_2}},$$

again isometric to a Wolf space, namely to $Gr_4(\mathbf{R}^9)$. Our diagram is now (up to finite quotients):

$$\begin{array}{ccc} S^{63} & \overset{Sp(2)}{\Longrightarrow} & \frac{Spin(9)}{Sp(2)\times Sp(1)} \\ \downarrow & & \downarrow \\ \mathbf{H}P^{15} & \overset{Sp(2)}{\Longrightarrow} & \text{BICAYLEY.} \end{array} \tag{4.5}$$

We are now ready to describe an action leading to the second exceptional Wolf space, namely

$$\frac{F_4}{Sp(3) \cdot Sp(1)},$$

of real dimension 28, that again for reasons related to its geometric realization we denote by TRICAYLEY. The projective space $\mathbf{H}P^{23}$ is acted on by a group $Sp(2) \times U(1)^6$, yielding as quaternion Kähler quotient (and again up to finite quotients) TRICAYLEY. Our last diagram is thus:

$$\begin{array}{ccc} S^{95} & \overset{Sp(2)\times T^6}{\Longrightarrow} & \frac{F_4}{Sp(3)} \\ \downarrow & & \downarrow \\ \mathbf{H}P^{23} & \overset{Sp(2)\times T^6}{\Longrightarrow} & \text{TRICAYLEY.} \end{array} \tag{4.6}$$

The reductions described by the above diagrams were obtained by regarding the Wolf spaces related to $Spin(7)$, $Spin(9)$, F_4 as exceptional Grassmannians in some geometry involving the Cayley numbers, and observing that their associated Stiefel manifolds are contained in the zero set of the associated moment maps at the 3-Sasakian level [23], [24]. Of course, the way to deal with the Wolf spaces of the groups E_6, E_7, E_8 is still long, and these methods might be not the most appropriate.

To conclude this short survey, I spend a few words on my motivation for Problem 6.1, that comes from the special emphasis I viewed on $\mathbf{H}P^n$ in my approach to quaternionic geometry. The Riemannian geometry of $\mathbf{H}P^n$ led E. Martinelli, about forty years ago, to propose $Sp(n) \cdot Sp(1)$ as the right choice for a "generalized" quaternion Hermitian geometry. The best understood geometry related to this group is today the positive quaternion Kähler geometry, via twistors and the fibrations described in Table 3.

Now a positive answer to Problem 6.1 would make the quaternionic projective spaces responsible, through the reduction procedure, of the geometry of all the Wolf spaces, that of course assuming the LeBrun - Salamon conjecture means of all the positive quaternion Kähler manifolds. Even in this latter (apparently) wider situation, several results show peculiar features of $\mathbf{H}P^n$. Just to remain where Italian contributions were given, examples are the character to be "locally projective" of manifolds with an integrable $GL(n,\mathbf{H}) \cdot Sp(1)$-structure [18], and the characterizations of $\mathbf{H}P^n$ as the unique positive quaternion Kähler manifold with zero characteristic class $\varepsilon \in H^2(M;\mathbf{Z_2})$ [19], [27], or admitting non-isometric authomorphisms [1], [16].

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