Chapter 7

Graph theory, by H.URAKAWA

Selected topics in Geometry
and Mathematical Physics,
Vol. 1, 2002, pp. 189-237.

Graph theory versus Riemannian geometry[1]

Abstract In recent decades, graph theory, in particular infinite graph theory, has been extensively developed. This research area is related to many fields in mathematics and computer sciences. In this lecture note, I want to clarify how Riemannian geometry and Graph Theory have a close relationship with each other. The lecture was given in Potenza, Italy, in September 1999.

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[1] **Hajime URAKAWA**, Mathematics Laboratories, Graduate School of Information Sciences, Tohoku University, Katahira, 2-1-1, Sendai, 980-8577, Japan.

7.1 Discrete Laplacians

In this section, we give a brief description of the discrete Laplacian following [11], [28].

7.1.1 Discrete Laplacians

A graph $G = (V, E)$ is a collection of the set V of vertices, and the set E of edges which connect two vertices. A graph is *finite* (resp. *infinite*) if $\#V < \infty$ (resp. $\#V = \infty$). For two vertices x and y, we write $x \sim y$ if they are connected by an edge. We say y is a neighbor of x and

$$N(x) = \{y \in V : y \sim x\},$$

the set of all neighbors of x. We always assume that any graph is *simple*, i.e., the number of edges connecting two vertices are at most one. Furthermore, we assume that G is *locally finite*, i.e. for each $x \in V$, the number of vertices connecting to x, called the *degree* at x and denoted by $m(x)$, is finite.

A graph $G = (V, E)$ is called *connected* if for all two vertices x and y in V, there exists a finite number of vertices $\{v_i\}_{i=1}^n$ in V satisfying that $v_1 = x$, $v_n = y$ and v_i is connected to v_{i+1} for all $i = 1, 2, \cdots, n-1$.

Recall that the *graph distance* $d(x, y)$ between two vertices x and y in V is defined as follows:

$$d(x, y) = \inf\{\ell(c);\ c \textit{ is a path connecting } x \textit{ and } y\}.$$

Here c is a path connecting x and y if

$$c = [\, c_0, c_1, \cdots, c_{n-1}, c_n \,],$$

where each c_i is a vertex in V, and $c_i \sim c_{i+1}$ for each $i = 0, 1, \cdots, n-1$. In this case, the length of c is n, i.e., $\ell(c) = n$. A graph $G = (V, E)$ is *connected* if each two vertices x and y in V, there exists a path connecting x and y.

Two discrete Laplacian Δ_P (called the *transition Laplacian*) and Δ_A (called the *adjacency Laplacian*) are defined by

$$\Delta_P f(x) = f(x) - \frac{1}{m(x)} \sum_{y \in N(x)} f(y), \tag{7.1}$$

$$\Delta_A f(x) = m(x)f(x) - \sum_{y \in N(x)} f(y),$$

for all $x \in V,\ f \in C(V)$, where $C(V)$ is the space of all real valued functions on V. It holds that

$$m(x)\Delta_P f(x) = \Delta_A f(x),$$

for all $x \in V$. So the spectral properties of the both Laplacians Δ_A and Δ_P are sometimes quite different each other. Usually, these Laplacians are denoted as follows:

$$\Delta_A = D - A, \quad \Delta_P = I - P,$$

where the operator D is the *degree operator* which is defined by

$$Df(x) = m(x)f(x), \quad x \in V,$$

and A is the *adjacency operator* which is defined by

$$Af(x) = \sum_{y \in V} a_{xy} f(y),$$

where a_{xy} $(x,\, y \in V)$ is the number of edges connecting vertices x and y, i.e., in our case,

$$a_{xy} = \begin{cases} 1 & \text{if } x \sim y, \\ 0 & \text{otherwise} \end{cases}$$

The operator I is the identity operator and P is the *tansition operator* which is defined by

$$Pf(x) = \sum_{y \in V} p(x,y) f(y),$$

where $p(x,y)$ is the one-step transition probability for the random walk standing at the vertex x to move to the other vertex y, i.e., in our case,

$$p(x,y) = \begin{cases} \dfrac{1}{m(x)}, & y \sim x, \\ 0, & \text{otherwise.} \end{cases}$$

The following expressions are useful: for all $x \in V$,

$$\Delta_P f(x) = \frac{1}{m(x)} \sum_{y \in N(x)} (f(x) - f(y)), \quad \Delta_A f(x) = \tag{7.2}$$

$$= \sum_{y \in N(x)} (f(x) - f(y)).$$

7.1.2 Differential Geometric Approach.

We first give and fix an orientation on each edge of G. We denote by $e = [x, y]$, the oriented edge e whose origin is x and the terminal is y. The reverse edge to $e = [x, y]$ is denoted by $\overline{e} = [y, x]$. The totality of all oriented edges of $G = (V, E)$ is denoted by $\mathbf{E}$. Note that $\mathbf{E}$ coincides with $\{e,\ \overline{e};\ e \in E\}$ and $\#\mathbf{E} = 2\#E$, where $\#Q$ means the cardinal number of the set Q. Let $C(\mathbf{E})$ is the space of all real valued functions on $\mathbf{E}$.

Let us recall vector analysis or differential geometry, in which the Laplacian is treated systematically.

$-$*Gradient of a function.* For $f \in C(V)$, the *gradient* ∇f is the element in $C(\mathbf{E})$ which is defined by

$$\nabla f(e) = f(y) - f(x), \quad e = [x, y] \in \mathbf{E}.$$

Note that ∇f satisfies that

$$\nabla f(e) = -\nabla f(\overline{e}), \quad e \in \mathbf{E}.$$

$-$*Divergence of a vector field.* An element X in $C(\mathbf{E})$ satisfying that

$$X(e) = -X(\overline{e}), \quad e \in \mathbf{E}$$

is called a *vector field* of a (oriented or directed) graph $G = (V, E)$. We denote by $\mathcal{X}(G)$, the space of all vector fields of $G = (V, E)$. The *divergence* of a vector field $X \in \mathcal{X}(G)$, denoted by $div_P(X)$, or $div_A(X)$ is a function on V defined by

$$div_P(X)(x) = \frac{1}{m(x)} \sum_{\substack{e \in \mathbf{E} \\ o(e) = x}} X(e),$$

$$div_A(X) = \sum_{\substack{e \in \mathbf{E} \\ o(e) = x}} X(e),$$

for all $x \in V$, where we denote by $o(e)$, the origin, and $t(e)$, the terminal of $e \in \mathbf{E}$, respectively.

If we write down the divergence in terms of E, and regarding that

$$\{e \in \mathbf{E};\, o(e) = x\} =$$

$$= \{e \in E;\, o(e) = x\} \cup \{\overline{e} \in E;\, t(e) = x\},$$

we have by definition of the divergence,

$$div_A(X) = \sum_{e \in E,\, o(e)=x} X(e) - \sum_{e \in E,\, t(e)=x} X(e),$$

which interprets the totality of the values of outward flows of X from x, and $div_P(X)$ has also the similar meaning.

$-$*The inner products.* Let us consider the spaces $C_c(V)$ (resp. $C_c(\mathbf{E})$, $\mathcal{X}_c(G)$) of functions $f \in C(V)$ (resp. $X \in C(\mathbf{E})$, $\mathcal{X}(G)$) having finite support. Here the support of f (resp. X) is $\{v \in V;\, f(v) \neq 0\}$ (resp. $\{e \in \mathbf{E};\, X(e) \neq 0\}$). We define the inner products on $C_c(V)$ and $C_c(\mathbf{E})$ as follows:

$$(f_1, f_2)_P = \sum_{x \in V} f_1(x) f_2(x),$$

$$(f_1, f_2)_A = \sum_{x \in V} m(x) f_1(x) f_2(x),$$

and

$$(X_1, X_2) = \frac{1}{2} \sum_{e \in \mathbf{E}} X_1(e) X_2(e),$$

where we count doubly each edge in E, i.e., an oriented edge $e \in \mathbf{E}$ and the reverse $\overline{e} \in \mathbf{E}$, so we multiply by $\frac{1}{2}$.

Let $L^2_P(V)$, $L^2_A(V)$ and $L^2(\mathbf{E})$ be the completion of $C_c(V)$ by the norm $(\ ,\ )_P$, $(\ ,\ )_A$, and $C_c(\mathbf{E})$ by the norm $(\ ,\ )$, respectively. That is,

$$L^2_P(V) = \{f \in C(V);$$

$$(f, f)_P < \infty\},\ L^2_A(V) = \{f \in C(V);\, (f, f)_A < \infty\},$$

and

$$L^2(\mathbf{E}) = \{X \in C(\mathbf{E});\, (\mathbf{X}, \mathbf{X}) < \infty\}$$

respectively.

$-$*Expressions of the Laplacians.* The two Laplacians can be expressed as follows:

$$\Delta_P f(x) = -div_P(\nabla f)(x), \tag{7.3}$$

$$\Delta_A f(x) = -div_A(\nabla f)(x),\ x \in V,\ f \in C(V).$$

Indeed, we have

$$\begin{aligned}\Delta_P f(x) &= -\frac{1}{m(x)}\sum_{y\sim x}(f(y)-f(x))\\ &= -\frac{1}{m(x)}\sum_{y\sim x}\nabla f([x,y])\\ &= -div_P(\nabla f)(x),\end{aligned}$$

and similarly for Δ_A.

$-$ *Green's formula.* The graph version of Green's formula in vector analysis or differential geometry is also formulated in the following way.

Theorem 21 (*Green's formula*)
Let $f \in C(V)$ and $X \in \mathcal{X}(G)$.

(1) *If $f \in C_c(V)$ or $X \in \mathcal{X}_c(G)$, we have*

$$(f, \mathrm{div}_P(X))_P = (f, \mathrm{div}_A(X))_A = -(\nabla f, X),$$

(2) *For $f_1,\ f_2 \in C_c(V)$, we have*

$$(\Delta_P f_1, f_2)_P = (\nabla f_1, \nabla f_2) = (f_1, \Delta_P f_2)_P,$$

$$(\Delta_A f_1, f_2)_A = (\nabla f_1, \nabla f_2) = (f_1, \Delta_A f_2)_A.$$

(3) *In particular, assume that $f \in C_c(V)$. Then*

$$(\Delta_P f, f)_P = (\Delta_A f, f)_A = (\nabla f, \nabla f) \geq 0,$$

$$\Delta_P f = 0 \Leftrightarrow \Delta_A f = 0 \Leftrightarrow \nabla f = 0 \Leftrightarrow f \quad \textit{is constant}.$$

Proof. The assertions (2) and (3) follow immediately from (1). The first equality of (1) is obvious. The proof of the second equality of (1) goes as follows. By the assumption, we may assume that $\#V < \infty$. We have

$$(f, div_A X)_A = \sum_{x\in V} f(x)\, div_A X(x)$$

$$= \sum_{x \in V} f(x) \sum_{\substack{e \in \mathbf{E} \\ o(e) = x}} X(e)$$

$$= \sum_{x \in V} \sum_{\substack{e \in \mathbf{E} \\ o(e) = x}} X(e).$$

On the other hand, we have

$$(\nabla f, X) = \frac{1}{2} \sum_{e \in \mathbf{E}} \nabla f(e)\, X(e)$$

$$= \frac{1}{2} \sum_{e \in \mathbf{E}} (f(t(e)) - f(o(e)))\, X(e)$$

$$= \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} (f(y) - f(x))\, X(e)$$

$$= \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} f(y)\, X([x, y]) - \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} f(x)\, X([x, y]).$$

Here, the second term of the above equality coincides with

$$-\frac{1}{2} \sum_{x \in V} \sum_{e \in \mathbf{E},\, o(e) = x} f(x)\, X(e),$$

and the first term conicides with

$$\frac{1}{2} \sum_{x \in V} \sum_{y \in V} f(y)\, a_{xy}\, X([x, y]) = -\frac{1}{2} \sum_{x \in V} \sum_{y \in V} f(y)\, a_{xy}\, X([y, x])$$

$$= -\frac{1}{2} \sum_{x \in V} \sum_{y \in V} f(x)\, a_{yx}\, X([x, y])$$

(by changing of order of the summations)

$$= -\frac{1}{2} \sum_{x \in V} \sum_{y \sim x} f(x)\, X([x, y])$$

$$= -\frac{1}{2}\sum_{x\in V} \sum_{\substack{e \in \mathbf{E} \\ o(e) = x}} f(x)\, X(e),$$

and hence, we obtain the desired equality. Q.e.d.

7.2 Topology of graphs

Definition 11 *For a finite graph $G = (V, E)$, we define the space of all divergence free vector fields on G, $\mathbf{H}^1(G)$, by*

$$\mathbf{H}^1(G) = \{X \in \mathcal{X}_c(G);$$

$$div_P X(x) = 0\} = \{X \in \mathcal{X}_c(G);\, div_A X(x) = 0\}.$$

We also denote by $\mathbf{H}^0(G)$ the space of constant functions on V. Note that $\dim \mathbf{H}^0(G) = 1$.

The meaning of the condition

$$div_P X(x) = 0$$

or

$$div_A X(x) = 0$$

is that *Kirchhoff's law* holds, i.e., the totality of outward flows of X from x coincides with the one of inward flows of X to x.

Theorem 22
For a connected finite graph $G = (V, E)$, let $b_1(G) = \dim \mathbf{H}^1(G)$.

(1) *Then, we have*

$$1 - b_1(G) = \#V - \#E,$$

say $\chi(G)$, called the Euler number of the graph $G = (V, E)$.

(2) *Let $K(x) = 2 - m(x)$, $x \in V$. Then, we have*

$$\frac{1}{2}\sum_{x\in V} K(x) = \chi(G).$$

Remark 11
If a graph $G = (V, E)$ is regarded as an 1-dimensional simplicial complex $K(G)$, let $C_q(G)$ be its abelian group of q-chains, and let us define $\partial_q : C_q(G) \to C_{q-1}(G)$, the boundary operator in the standard way such as

$$\partial_q \sigma = 0, \quad \sigma \in C_0(G) \quad (q = 0),$$

$$\partial_q \sigma = \sum_{i=1}^{m} c_i\, t(e_i) - \sum_{i=1}^{m} c_i\, o(e_i),$$

$$\sigma = \sum_{i=1}^{m} c_i\, e_i \in C_1(G) \quad (q = 1).$$

Then the q-th homology group $H_q(G)$ is defined by

$$H_q(G) = Z_q(G)/B_q(G),$$

where

$$Z_q(G) = \mathrm{Kernel}(\partial_q), \;\; B_q(G) = \mathrm{Image}(\partial_q).$$

Then it is clear that $\mathbf{H}^1(G) \cong H_1(G) \otimes \mathbf{R}$. So, $b_1(G)$ coincides with the first Betti number of $K(G)$.

Proof. (of Theorem 22) We give here an intuitive proof. Taking an edge $e = (x, y) \in E$, we give the following operation:

(1) split one of the terminal vertices, say x, into two vertices x^+ and x^-,

(2) split the edge e into $e^+ = (x^+, y)$ and $e^- = (x^-, y)$, and

(3) cut each edge connecting x other than e and past to either x^+ or x^-.

Then we get a new graph, say $G' = (V', E')$, where $V' = (V - \{x\}) \cup \{x^+, x^-\}$ and $E' = (E - \{e\}) \cup \{e^+, e^-\}$. Notice that $\#V' = \#V$ and $\#E' = \#E$.

We have a disjoint union:

$$\{z \in V - \{y\};\, z \sim x\} =$$

$$= \{z \in V' - \{y\};\, z \sim x^+\} \cup \{z \in V' - \{y\};\, z \sim x^-\},$$

say, $\{z \in V' - \{y\};\, z \sim x^+\} = \{y_1, \cdots, y_p\}$ and $\{z \in V' - \{y\};\, z \sim x^-\} = \{y_{p+1}, \cdots, y_m\}$, where $m + 1 = m(x)$. We want to give an isomorphism between $\mathbf{H}^1(G)$ and $\mathbf{H^1(G)}$ as follows: For $X \in \mathbf{H}^1(G)$, let define X' by

$$X([x^+, y]) = -\sum_{i=1}^{p} X([x^+, y_i]),$$

$$X([x^-, y]) = -\sum_{i=p+1}^{m} X([x^-, y_i]),$$

$$X'(e) = X(e) \;\; if \;\; e \in \mathbf{E},\, o(e) \neq x^+,\, x^-.$$

Then it is easy to see that $X' \in \mathbf{H}^1(G)$ and the mapping $X \mapsto X'$ gives an isomorphism of $\mathbf{H}^1(G)$ onto $\mathbf{H^1(G')}$. In particular, we have $b^1(G) = b^1(G')$.

Now taking the above operations successively to a given graph $G = (V, E)$, we finially obtain the bouquet graph B_k with k leaves, where $k = 1 - b_1(B_k) = 1 - \chi(G)$.

But, it is easy to see that $\dim \mathbf{H}^1(G) = \dim \mathbf{H}^1(B_k) = k$. We have the desired equality.

For (2), we have

$$\frac{1}{2}\sum_{x \in V} K(x) = \frac{1}{2}\sum_{x \in V} 2 - \frac{1}{2}\sum_{x \in V} m(x) = \#V - \#E = \chi(G).$$

Q.e.d.

7.3 Eigenvalue problems

4.1. Maximum principle.

Definition 12 *A function $f \in C(V)$ is harmonic if $\Delta_P f = 0$ or $\Delta_A f = 0$. $f \in C(V)$ is superharmonic at $x \in V$ if $\Delta_P f \geq 0$ or $\Delta_A f \geq 0$, i.e.,*

$$\frac{1}{m(x)} \sum_{y \in N(x)} f(y) \leq f(x),$$

that is, the mean value of f at the neighborhood $N(x)$ of x is not bigger than the value $f(x)$ of f at x, itself.

The following theorem is obvious, but very useful.

Theorem 23 (*the minimum or maximum principle*)
Assume that $f \in C(V)$ is superharmonic at $x \in V$. If

$$f(x) \le \min\{f(y);\ y \in V,\ y \sim x\}, \tag{7.4}$$

then $f(y) = f(x)$ for all neighbors $y \in N(x)$ of x.

Proof. The condition of superharmonic at x, together with (7.4), implies that

$$\frac{1}{m(x)} \sum_{y \in N(x)} f(y) = f(x),$$

but by means of (4.3), this holds when and only when $f(y) = f(x)$ for all $y \in N(x)$. We have the desired. Q.e.d.

4.2. Free boundary eigenvalue problems.

For a finite graph $G = (V, E)$, one can consider the following free boundary eigenvalue problem for both Laplacians Δ_P or Δ_A:

$$\Delta_P f = \lambda f, \quad \text{or} \quad \Delta_A f = \lambda f. \tag{7.5}$$

For a constant λ is called the *eigenvalue* of Δ_P or Δ_A if there exists $f \in C(V)$, called the *eigenfunction*, satisfies (4.4) and f does not vanish identically on V. Due to Theorem 2.4, every finite graph $G = (V, E)$ with $\#V = n$, admits non-negative n eigenvalues of Δ_P (or Δ_A) counted with multiplicities, say

$$\lambda_1^P \le \lambda_2^P \le \cdots \le \lambda_n^P \quad (\text{or} \quad \lambda_1^A \le \lambda_2^A \le \cdots \le \lambda_n^A).$$

Furthermore, we have

Proposition 9 *Assume that $G = (V, E)$ is finite and connected. Then we have*

(1) $0 = \lambda_1^P < \lambda_2^P \le \cdots \le \lambda_n^P \le 2$, *and*
(2) $0 = \lambda_1^A < \lambda_2^A \le \cdots \le \lambda_n^A \le 2\, m(G)$,

where $m(G) = \max\{m(x);\ x \in V\}$ is the maximum of the degrees of G. The eigenfunctions with the eigenvalue 0 are the constant functions.

Proof. Let $f \in C(V)$ satisfy $f \not\equiv 0$ and $\Delta_P f = \lambda f$. By Theorem 2.4,

$$\lambda(f,f)_P = (\Delta_P f, f)_P = (\nabla f, \nabla f)_P \geq 0. \tag{7.6}$$

Since $(f,f)_P > 0$, we have $\lambda \geq 0$. Assume that $\lambda = 0$. By (7.6), we have $\nabla f = 0$, which implies that f is constant on V since G is connected. Therefore, we have $0 = \lambda_1^P < \lambda_2^P$. That $0 = \lambda_1^A < \lambda_2^A$ follows by the same way.

For the upper bounds of (1) and (2), we have

$$(\nabla f, \nabla f) = \frac{1}{2} \sum_{x\in V} \sum_{y\sim x} (f(y) - f(x))^2 \tag{7.7}$$

$$\leq \sum_{x\in V} \sum_{y\sim x} (f(y)^2 + f(x)^2) \quad (since\ (a-b)^2 \leq 2(a^2+b^2))$$

$$= \sum_{x\in V} \sum_{y\sim x} f(y)^2 + \sum_{x\in V} \sum_{y\sim x} f(x)^2$$

$$= 2 \sum_{x\in V} m(x) f(x)^2,$$

where the last equality follows from that

$$\sum_{x\in V} \sum_{y\sim x} f(x)^2 = \sum_{x\in V} m(x) f(x)^2$$

and

$$\sum_{x\in V} \sum_{y\sim x} f(y)^2 = \sum_{x\in V} \sum_{y\in V} a_{xy}\, f(y)^2 =$$

$$= \sum_{x\in V} \sum_{y\in V} a_{yx}\, f(x)^2 = \sum_{x\in V} \sum_{y\sim x} f(x)^2.$$

Since

$$\sum_{x\in V} m(x) f(x)^2 = (f,f)_P$$

and

$$\sum_{x\in V} m(x) f(x)^2 \leq m(G) \sum_{x\in V} f(x)^2 = m(G)(f,f)_A,$$

we have $\lambda_n^P \leq 2$ and $\lambda_n^A \leq 2\,m(G)$, which are the desired. Q.e.d.

7.4 Graphs with boundaries

Definition 13
A graph $G=(V,E)$ *is said to have a boundary* $\partial G=(\partial V,\partial E)$ *if* V *and* E *have disjoint unions* $V=\dot{V}\cup\partial V$ *and* $E=\dot{E}\cup\partial E$ *satisfying the following two conditions:*

$$e\in\dot{E}\quad\Leftrightarrow\quad x\in\dot{V}\text{ and }y\in\dot{V},\tag{7.8}$$

$$\begin{aligned}e\in\partial E\quad\Leftrightarrow\quad\text{either}\quad & x\in\dot{V}\ \textit{and}\ y\in\partial V,\\ \textit{or}\quad & x\in\partial V\ \textit{and}\ y\in\dot{V},\end{aligned}\tag{7.9}$$

where $\dot{V}$ *(resp.* ∂V*) is called the set of interior vertices (resp. boundary vertices), and* $\dot{E}$ *is called the set of interior edges) (resp. boundary edges) of* $G=(V,E)$*.*

Notice that $\dot{G}=(\dot{V},\dot{E})$ is a subgraph of $G=(V,E)$, called the *interior graph* of $G=(V,E)$.

Now let us consider the *Dirichlet boundary eigenvalue problem* for a finite graph $G=(V,E)$ with boundary $\partial G=(\partial V,\partial E)$. This is the following eigenvalue problem, where Δ stands for the Laplacians Δ_P or Δ_A of $G=(V,E)$.

$$\begin{cases}\Delta f=\mu f & \text{on }\dot{V},\\ f=0 & \text{on }\partial V,\end{cases}$$

The *Neumann boundary eigenvalue problem* is that:

$$\begin{cases}\Delta f=\mu f & \text{on }\dot{V},\\ \Delta f=0 & \text{on }\partial V,\end{cases}$$

The number μ is called the *eigenvalue*, and f is called the *eigenfunction*, if there exists a solution $f\not\equiv 0$ of the both problem, respectively. We have

Proposition 10 *Let* $G=(V,E)=(\dot{V}\cup\partial V,\dot{E}\cup\partial E)$ *be a finite graph with boundary* $\partial G=(\partial V,\partial E)$*. For the Dirichlet or Neumann eigenvalue problem, all the eigenvalues are non-negative. Two eigenfunctions with different eigenvalues are mutually orthogonal.*

Proof. The assertion (2) follows from (1). For (1), we only need the following lemma which is immediately from Theorem 21. The detail is omitted.

Lemma 4 *Under the same situation Proposition 10, for all functions* f, f_1 *and* f_2 *in* $C(V) = C(\dot{V} \cup \partial V)$, *we have*

$$\sum_{x \in \dot{V}} m(x) f(x) \Delta_P f(x) + \sum_{x \in \partial V} m(x) f(x) \Delta_P f(x) = (\nabla f, \nabla f), \tag{7.10}$$

$$\left(\sum_{x \in \dot{V}} m(x) f(x) \Delta_P f(x) + \right.$$

$$\left. + \sum_{x \in \partial V} m(x) f(x) \Delta_P f(x) = (\nabla f, \nabla f) \right),$$

$$\sum_{x \in \dot{V}} m(x) f_1(x) (\Delta^P f_2)(x) = \sum_{x \in \dot{V}} m(x) (\Delta^P f_1)(x) f_2(x)$$

$$- \sum_{x \in \partial V} m(x) \left\{ f_1(x)(\Delta^P f_2)(x) - (\Delta^P f_1)(x) f_2(x) \right\},$$

$$\Big(\sum_{x \in \dot{V}} f_1(x)(\Delta^A f_2)(x) = \sum_{x \in \dot{V}} (\Delta^A f_1)(x) f_2(x) \tag{7.11}$$

$$- \sum_{x \in \partial V} \left\{ f_1(x)(\Delta^A f_2)(x) - (\Delta^A f_1)(x) f_2(x) \right\} \Big).$$

In the following, we only deal the Dirichlet eigenvalue problem. We denote the set of eigenvalues with their multiplicities by

$$\mu_1^P(G) \le \mu_2^P(G) \le \cdots \le \mu_k^P(G)$$

$$(\textit{resp. } \mu_1^A(G) \le \mu_2^A(G) \le \cdots \le \mu_k^A(G)),$$

where $k = \#\dot{V}$. We also denote by ψ_i^P (resp. ψ_i^A) the corresponding eigenfunctions with the eigenvalues μ_i^P (resp. μ_i^A) which are orthonormal with respect to the inner product $(\ ,\ )_P$ (resp. $(\ ,\ )_A$) $(i = 1, 2, \cdots, k)$.

Definition 14 *We define by* $C_0(V)$, *the set of all functions* $f \in C(V)$ *satisfying that* $f(x) = 0$ *for all* $x \in \partial V$.

Proposition 11 *$\mu_1^P > 0$ and $\mu_1^A > 0$.*

Proof. We show $\mu_1^A > 0$. That $\mu_1^P > 0$ can be proved by the similar way.

By Lemma 4,

$$\mu_1^A = (\Delta_A \psi_1^A, \psi_1^A)_A = (\nabla \psi_1^A, \nabla \psi_1^A) \geq 0.$$

Assume that $\mu_1^A = 0$. Since $\nabla \psi_1^A = 0$, ψ_1^A is constant in $V = \dot{V} \cup \partial V$. Since ψ_1^A vanishes on ∂V, ψ_1^A vanishes everywhere on V, which contradicts that ψ_1^A is the eigenfunction. Q.e.d.

Theorem 24 (*the minimax principle*)

(1) *The eigenvalues μ_1^P and μ_1^A are characterized as follows:*

$$\mu_1^P = \inf\left\{ \frac{(\nabla f, \nabla f)}{(f,f)_P};\ 0 \not\equiv f \in C_0(V) \right\},$$

$$\mu_1^A = \inf\left\{ \frac{(\nabla f, \nabla f)}{(f,f)_A};\ 0 \not\equiv f \in C_0(V) \right\}.$$

(2) *Furthermore, if $0 \not\equiv f \in C_0(V)$ satisfies*

$$\mu_1^P = \frac{(\nabla f, \nabla f)}{(f,f)_P} \quad (resp. \quad \mu_1^A = \frac{(\nabla f, \nabla f)}{(f,f)_A}),$$

then f is the eigenfunction of the Dirichlet boundary eigenvalue problem of Δ_P (resp. Δ_A) with the eigenvalue μ_1^P (resp. μ_1^A).

Proof. The first statement follows by the definition. We prove the second for Δ_A. The proof is the similar for Δ_P. Assume that $0 \not\equiv f \in C_0(V)$ satisfies

$$\mu_1^A = \frac{(\nabla f, \nabla f)}{(f,f)_A}. \tag{7.12}$$

The minimality of λ_1^A and (7.12) impliy that for all $\varphi \in C_0(V)$, we have

$$0 = \frac{d}{dt}\Big|_{t=0} \frac{(\nabla f + t\nabla\varphi, \nabla f + t\nabla\varphi)}{(f + t\varphi, f + t\varphi)_A}$$

$$= 2\frac{(\nabla f, \nabla\varphi)(f,f)_A - (\nabla f, \nabla f)(f,\varphi)_A}{{(f,f)_A}^2}$$

$$= 2\frac{(\nabla f, \nabla \varphi) - \mu_1^A\,(f,\varphi)_A}{(f,f)_A},$$

which implies that

$$0 = (\nabla f, \nabla\varphi) - \mu_1^A\,(f,\varphi)_A =$$
$$= (\Delta^A f, \varphi)_A - \mu_1^A\,(f,\varphi)_A = (\Delta^A f - \mu_1^A f, \varphi)_A.$$

Here since $\varphi \in C_0(V)$ is arbitrary, we obtain

$$\Delta^A f - \mu_1^A\, f = 0 \quad on \quad \dot{V},$$

which is the desired. Q.e.d.

Theorem 25

Let $G = (V,E)$ be a finite graph with boundary $\partial G = (\partial V, \partial E)$. Assume that the interior $\dot{G} = (\dot{V}, \dot{E})$ is connected. Then, for both the Laplacians $\Delta = \Delta^P$ and Δ^A,

(1) *the first eigenfunction ψ_1 of the Dirichlet eigenvalue problem of G for Δ is positive everywhere on $\dot{V}$ or negative everywhere on $\dot{V}$.*

(2) *The multiplicity of the first eigenvalue the Dirichlet eigenvalue problem of G for Δ is one.*

Proof. We give a proof for Δ^A. The proof goes by the similar way in the case of Δ^P. By definition of $(\ ,\ )_A$, for each $f \in C(V)$, we have

$$(f,f)_A = (|f|,|f|)_A\ , \tag{7.13}$$

$$(\nabla f, \nabla f) = \frac{1}{2}\sum_{e=[x,y]\in\mathbf{E}} (f(y) - f(x))^2 \tag{7.14}$$

$$\geq \frac{1}{2}\sum_{e=[x,y]\in\mathbf{E}} (|f(y)| - |f(x)|)^2 \quad = (\nabla|f|, \nabla|f|)\,.$$

In particular, taking $f \in C_0(V) \subset C(V)$ as the first eigenfunction of the Dirichlet eigenvalue problem for Δ^A of G with the eigenvalue μ_1^A, we have

$$\mu_1^A = \frac{(\nabla f, \nabla)}{(f,f)_A} \geq \frac{(\nabla|f|, \nabla|f|)}{(|f|,|f|)_A}. \tag{7.15}$$

By the minimality of μ_1^A, the equality holds for (5.13). Thus, by (2) of Theorem 5.8, $|f|$ is also the eigenfunction corresponding to the eigenvalue μ_1^A. Therefore, we have

$$\Delta^A|f| = \mu_1^A\,|f| \geq 0 \quad (on \quad \dot{V}).$$

That is, $|f|$ is super harmonic at $\dot{V}$. By Theorem 23 (the maximum principle), we have

$$|f| > 0 \quad (on \quad \dot{V}).$$

Because, if $|f|$ vanish at some vertex $x \in \dot{V}$, $|f|$ vanish everywhere on $\dot{V}$ because of the connectivity of $\dot{V}$ and Theorem 23. Thus, $|f|$ vanish identically on V, which is a contradiction.

Furthermore, assume that f takes both positive and negative values on $\dot{V}$. By means of connectivity of $\dot{V}$, there exists $e \in \dot{E}$ satisfying that $f(x)f(y) < 0$, where x and y are the end vertices of e. Then (7.14) is strictly inequality, i.e.,

$$(\nabla f, \nabla f) > (\nabla |f|, \nabla |f|).$$

Therefore, we have

$$\mu_1^A = \frac{(\nabla f, \nabla f)}{(f,f)_A} > \frac{(\nabla |f|, \nabla |f|)}{(|f|,|f|)_A},$$

which contradicts that $|f|$ is the eigenfunction with the eigenvalue μ_1^A . Thus, the sign of f does not change on $\dot{V}$.

For (2), assume that the multiplicity of μ_1^A is bigger than one, and let f_1 and f_2 be linearly independent eigenfunctions with the eigenvalue μ_1^A. Take any vertex x in $\dot{V}$. Define a linear map A of $\mathbf{R^2}$ into $\mathbf{R}$ by

$$A(a,b) = af_1(x) + bf_2(x).$$

Since A is never injective, Ker$(A) \neq \{(0,0)\}$. So, choose $(a_0,b_0) \in Ker(A) - \{(0,0)\}$ and define a function φ on V by $\varphi = a_0 f_1 + b_0 f_2$. Then, $\varphi \not\equiv 0$. Because, if so, this contradicts that $\{f_1, f_2\}$ is linearly independent since $(a_0,b_0) \neq (0,0)$. Notice that $\varphi(x) = 0$ because of $(a_0,b_0) \in Ker(A)$, this contradicts (1) of Theorem 25 since φ is the eigenfunction with the eigenvalue μ_1^A. Q.e.d.

7.5 Discrete Barta's theorem

In this section, we want to show

Theorem 26
Let $G = (V,E)$ be a finite graph with boundary $\partial G = (\partial V, \partial E)$. Let μ_1^P (resp. μ_1^A) be the first eigenvalue of the Dirichlet eigenvalue problem for Δ^P (resp.

Δ^A). *Let* f *be a function on* V *such that* f *is positive* $\dot{V}$ *and equal to zero on* ∂V*. Then*

$$\inf \frac{\Delta^P f}{f} \leq \mu_1^P \leq \sup \frac{\Delta^P f}{f}$$

$$\left(resp. \quad \inf \frac{\Delta^A f}{f} \leq \mu_1^A \leq \sup \frac{\Delta^A f}{f} \right).$$

Proof. The proof goes by the similar way as in Barta's theorem for a Riemannian manifold. Let ψ_1 be the first eigenfunction of the Dirichlet eigenvalue problem for Δ^A. For Δ^P, the proof goes by the similar way. By (1) of Theorem 25, we may assume $\psi_1 > 0$ everywhere on $\dot{V}$. We can write $\psi_1 = f + h$, and then $h = 0$ on ∂V. Then, we have

$$\mu_1^A = \frac{\Delta^A \psi_1}{\psi_1} = \frac{\Delta^A (f+h)}{f+h} \tag{7.16}$$

$$= \frac{\Delta^A f}{f} + \frac{\Delta^A (f+h)}{f+h} - \frac{\Delta^A f}{f}$$

$$= \frac{\Delta^A f}{f} + \frac{f \, \Delta^A h - h \, \Delta^A f}{f(f+h)}.$$

Note here that if a function ψ on V vanishing on ∂V satisfies $(\mathbf{1}, \psi)_{\mathbf{A}} = \sum_{\mathbf{x} \in \mathbf{V}} \psi(\mathbf{x})$
$= 0$, where $\mathbf{1}$ is a constant function on V such that $\mathbf{1}(\mathbf{x}) = \mathbf{1} \, (\mathbf{x} \in \mathbf{V})$, then either $\psi \equiv 0$ or ψ changes its sign on $\dot{V}$. However, we have $f(f+h) > 0$ on $\dot{V}$, and

$$(\mathbf{1}, f \, \Delta^A h - h \, \Delta^A f)_A = (f, \Delta^A h)_A - (h, \Delta^A f)_A$$
$$= (\nabla f, \nabla h) - (\nabla h, \nabla f) = 0,$$

whence either $f \, \Delta^A h - h \, \Delta^A f \equiv 0$ or $\frac{f \, \Delta^A h - h \, \Delta^A f}{f(f+h)}$ changes its sign on $\dot{V}$. The former immediately implies that $\mu_1 = \frac{\Delta^A f}{f}$ by (7.16). In the latter case, at $x \in \dot{V}$ where the sign is negative, we have by (7.16) again

$$\mu_1 \leq \frac{\Delta^A f}{f}(x) \leq \sup \frac{\Delta^A f}{f}.$$

At $y \in \dot{V}$ where the sign is positive, we have also

$$\inf \frac{\Delta^A f}{f} \leq \frac{\Delta^A f}{f}(y) \leq \mu_1.$$

We have the desired. Q.e.d.

As an application of Theorem 26, we have

Theorem 27
Let $G = (V, E)$ *be a connected graph, and* $T_d = (V(T_d), E(T_d))$, *the homogeneous regular tree of degree* d. *For all* $r > 0$, *let* $B_r(x_0)$ (*resp.* V_r^d) *be the distance ball of radius* r *in* G (*resp.* T_d), *i.e.,* $B_r(x_0) = \{x \in V;\ d(x, x_0) < r\}$ *and* $V_r^d = \{y \in V(T_d);\ d(y, o) < r\}$ *for some fixed vertices* $x_0 \in V$ *and* $o \in V(T_d)$. *Then the first eigenvalues* $\mu_1^P(B_r(x_0))$ *and* $\mu_1^A(B_r(x_0))$ *of the Dirichlet eigenvalue problems of* $B_r(x_0)$ *are estimated as follows:*

(1) *If* $m(x) \leq d$ *for all* $x \in V$, *then*

$$\mu_1^P(B_r(x_0)) \leq \mu_1^P(V_r^d) \quad \textit{and} \quad \mu_1^P(B_r(x_0)) \leq \mu_1^P(V_r^d)$$

for all $r > 0$.

(2) *If* G *is a tree and* $m(x) \geq d$ *for all* $x \in V$, *then*

$$\mu_1^P(B_r(x_0)) \geq \mu_1^P(V_r^d) \quad \textit{and} \quad \mu_1^P(B_r(x_0)) \geq \mu_1^P(V_r^d)$$

for all $r > 0$.

(For a proof, see [31].)

As its Corollaries, we have

Corollary 3
Let $G = (V, E)$ *be an infinite connected graph. We denote by* $\mathrm{Spec}_P(G)$ (*resp.* $\mathrm{Spec}_A(G)$) *the set of the spectrum of the Laplacian* Δ^P (*resp.* Δ^A). *Then the infimum of their spectra can be estimated as follows:*

(1) *If* $m(x) \leq d$ *for all* $x \in V$, *then*

$$\inf \mathrm{Spec}(\Delta^P) \leq 1 - \frac{2\sqrt{d-1}}{d}$$

$$\left(\textit{resp.} \quad \inf \mathrm{Spec}(\Delta^A) \leq d - 2\sqrt{d-1}\right).$$

(2) *If* G *is a tree and* $m(x) \geq d$ *for all* $x \in V$, *then*

$$\inf \mathrm{Spec}(\Delta^P) \geq 1 - \frac{2\sqrt{d-1}}{d}$$

$$\left(resp. \quad \inf \mathrm{Spec}(\Delta^A) \geq d - 2\sqrt{d-1} \right).$$

The essential spectra of the Laplacian for an infinite graph can be estimated as follows:

Corollary 4
Let $G = (V, E)$ be an infinite connected graph.
(1) *Assume that*

$$m(x) \geq d \quad for\ all \quad x \in V.$$

Then the essential spectrum of the Laplacian Δ^P (resp. Δ^A) meets with the interval $[0, 1 - \frac{2\sqrt{d-1}}{d}]$ (resp. $[0, d - 2\sqrt{d-1}]$).
(2) *Let G be an infinite tree and satisfy*

$$\lim_{r \to \infty} \inf_{x \notin B_r(x_0)} m(x) = \infty.$$

Then the essential spectrum of the Laplacian Δ^P coincides with a single set $\{1\}$, and the one of Δ^A is empty.

For a finite graph, we have

Corollary 5
Let $G = (V, E)$ be a finite connected graph. Let $d(G)$ be the diameter, and $m(G) = \sup\{m(x); x \in V\}$. Assume that $d(G) \geq 4$. Then the m-th eigenvalue of Δ^P (resp. Δ^A), denoted by λ_m^P (resp. λ_m^A), $m = 1, 2, \cdots, [d(G)/2]$, can be estimated as follows:

$$\lambda_m^P \leq \mu_1^P(V_r^d) \leq 1 - \frac{2\sqrt{m(G)-1}}{m(G)} \cos\left(\frac{\pi}{\frac{d(G)}{2m} + 1}\right)$$

$$\left(resp. \quad \lambda_m^A \leq \mu_1^A(V_r^d) \leq m(G) - 2\sqrt{m(G)-1} \cos\left(\frac{\pi}{\frac{d(G)}{2m} + 1}\right)\right),$$

where we take $d = m(G)$ and $r = \frac{d(G)}{2m}$ in V_r^d.

(For a proof, see also [31].)

7.6 Cheeger constants

In this section, we define the Cheeger constant and give estimation of the infimum of the spectra of the Laplacians for an infinite graph $G = (V, E)$.

Definition 15 *We first denote by* $\mu_0^P(G)$ *(resp.* $\mu_0^A(G)$*), the infimum of the spectrum of the Laplacian* Δ^P *(resp.* Δ^A*) on* $L^2_P(V)$ *(resp.* $L^2_A(V)$*), that is,*

$$\mu_0^P(G) = \inf Spec(\Delta^P) = \inf \left\{ \frac{(\nabla f, \nabla f)}{(f,f)_P};\ 0 \not\equiv f \in C_c(V) \right\},$$

$$\left(resp.\quad \mu_0^A(G) = \inf Spec(\Delta^A) = \inf \left\{ \frac{(\nabla f, \nabla f)}{(f,f)_A};\ 0 \not\equiv f \in C_c(V) \right\} \right),$$

where $C_c(V)$ *is the space of all functions on* V *with finite supports.*

Definition 16 *We define two kinds of the isoperimetric constants, so called Cheeger constsnts,* $i_P(G)$ *(resp.* $i_A(G)$*) for an infinite graph* $G = (V, E)$ *as follows:*

For any finite subset $S \subset V$*, its boundary* ∂S *is defined by*

$$\partial S = \{e \in E;\ e = (x, y),\ x \in S \text{ and } y \notin S\}.$$

We put

$$A(S) = \sum_{x \in S} m(x), \quad L(\partial S) = \#(\partial S),$$

where $\#(Q)$ *is the cardinality of a set* Q*. Then,*

$$i_P(G) = \inf \left\{ \frac{L(\partial S)}{A(S)};\ \emptyset \neq S \subset V \text{ and } \#(S) < \infty \right\},$$

where we take the infimum over all non-empty finite subsets S *of* V*. And by the same way, we define*

$$i_A(G) = \inf \left\{ \frac{\#(\partial S)}{\#(S)};\ \emptyset \neq S \subset V \text{ and } \#(S) < \infty \right\}.$$

Theorem 28 (*cf. [9],[10],[12]*) *Let $G = (V, E)$ be an infinite graph. Then we have*

(1) $\frac{1}{2} i_P(G)^2 \leq \mu_0^P(G) \leq i_P(G)$.

(2) *Assume that* $m(G) = \sup\{m(x);\, x \in V\} < \infty$. *Then we have*

$$\frac{1}{2m(G)} i_A(G)^2 \leq \mu_0(G) \leq i_A(G)\,.$$

Remark 12 *The above estimations are not sharp. Indeed, Mohar [19], [20], [21], and Tan [27] gave the following estimation:*

$$1 - \sqrt{1 - i_P(G)^2} \leq \mu_0^P(G).$$

On the other hand, it is known (cf. [26]) that

$$m(G) - \sqrt{m(G)^2 - i_A(G)^2} \leq \mu_0^A(G).$$

For other lower and upper bounds of the spectrum Spec(Δ^P), *see [34].*

Remark 13 *For a finite graph $G = (V, E)$, one can define the Cheeger constants as follows:*

$$i_P(G) = \inf\left\{ \frac{L(\partial S)}{A(S)};\, \emptyset \neq S \subset V \text{ and } \#(S) \leq \frac{1}{2}\#(V) \right\},$$

$$i_A(G) = \inf\left\{ \frac{\#(\partial S)}{\#(S};\, \emptyset \neq S \subset V \text{ and } \#(S) \leq \frac{1}{2}\#(V) \right\}.$$

Then, Mohar showed that if $G = (V, E)$ is a finite graph with $n := \#(V) \geq 4$, then the first eigenvalue λ_1^A of Δ^A satisfies

$$i_A(G) \leq \sqrt{\lambda_1^A\,(2\,m(G) - \lambda_1^A)}.$$

Moreover, it holds (cf. [27]) that, under the same assumption, the first eigenvalue λ_1^P of Δ^P satisfies

$$i_P(G) \leq \sqrt{\lambda_1^P\,(2 - \lambda_1^P)},$$

and the equality holds if and only if $G = K_{1,n-1}$ which is the star garph of n vertices (see also [1] and [3]).

Proof. (of Theorem 28) We only show (1), and (2) can be shown by the similar way. For every $f \in C_c(V)$ with $(f,f)_P = 1$, we shall show $i_P(G) \leq \sqrt{2(\nabla f, \nabla f)}$.

We may first assume $f \geq 0$ because we have by the same way as the proof of Theorem 25,

$$(\nabla f, \nabla f) \geq (\nabla |f|, \nabla |f|), \quad \textit{and} \quad (f,f)_P = (|f|,|f|)_P.$$

(*the first step*) We give an orientation at each edges, and let $\mathbf{E}$ be the set of all oriented edges of G. For each $e \in \mathbf{E}$, let $o(e)$ be the origin of e, and $t(e)$ the terminal of e. Define $A = \sum_{e \in \mathbf{E}} |f(o(e))^2 - f(t(e))^2|$. Then we can show that

$$A \leq \sqrt{2(\nabla f, \nabla f)}. \tag{7.17}$$

Because we have

$$A^2 = \left(\sum_{[x,y] \in \mathbf{E}} (f(x)+f(y))\, |f(x)-f(y)| \right)^2$$

$$\leq \sum_{[x,y]\in \mathbf{E}} (f(x)+f(y))^2 \sum_{[x,y]\in \mathbf{E}} |f(x)-f(y)|^2$$

(by Cauchy-Schwarz)

$$= \sum_{[x,y]\in \mathbf{E}} \{f(x)^2 + 2f(x)f(y) + f(y)^2\}\,(\nabla f, \nabla f)$$

$$\leq 2 \sum_{[x,y]\in \mathbf{E}} \{f(x)^2 + f(y)^2\}\,(\nabla f, \nabla f)$$

$$= 2 \sum_{x \in V} m(x) f(x)^2\,(\nabla f, \nabla f)$$

$$= (f,f)_P\,(\nabla f, \nabla f) = (\nabla f, \nabla f).$$

(*the second step*) Since $f \in C_c(V)$, we may write

$$\{f(x);\, x \in V\} = \{\beta_0 = 0 < \beta_1 < \cdots < \beta_N\}.$$

Putting $K_i = \{x \in V;\, f(x) \geq \beta_i\}$, we have

$$\partial K_i = \{e = (x,y) \in E;\, f(x) \geq \beta_i \;\; \textit{and} \;\; f(y) < \beta_i\},$$

and by definition of $i_P(G)$,

$$i_P(G)A(K_i) \leq L(\partial K_i). \tag{7.18}$$

(*the third step*) Here we have

$$A \geq i_P(G). \tag{7.19}$$

In fact, we have

$$\begin{aligned} A &= \sum_{[x,y]\in \mathbf{E}} |f(x)^2 - f(y)^2| \\ &= \sum_{i=1}^{N} \sum_{\substack{f(x)=\beta_i \\ x\in V}} \sum_{\substack{y\sim x \\ f(y)<\beta_i}} \left|f(x)^2 - f(y)^2\right|. \end{aligned}$$

Furthermore, if $x \sim y$, $f(x) = \beta_i$ and $f(y) = \beta_{i-k} < \beta_i$, then

$$[x,y] \in \partial K_i \cap \partial K_{i-1} \cap \cdots \cap \partial K_{i-k+1} \quad \textit{and}$$

$$f(x)^2 - f(y)^2 =$$
$$= ({\beta_i}^2 - {\beta_{i-1}}^2) + ({\beta_{i-1}}^2 - {\beta_{i-2}}^2) + \cdots + ({\beta_{i-k+1}}^2 - {\beta_{i-k}}^2).$$

Then, we have

$$\begin{aligned} A &= \sum_{i=1}^{N} \sum_{[x,y]\in\partial K_i} ({\beta_i}^2 - {\beta_{i-1}}^2) \\ &= \sum_{i=1}^{N} L(\partial K_i)({\beta_i}^2 - {\beta_{i-1}}^2) \\ &\geq i_P(G) \sum_{i=1}^{N} A(K_i)({\beta_i}^2 - {\beta_{i-1}}^2) \\ &= i_P(G)\left\{\sum_{i=1}^{N} A(K_i){\beta_i}^2 - \sum_{i=1}^{N-1} A(K_{i+1}){\beta_i}^2\right\}, \end{aligned}$$

where note $\beta_0 = 0$ in the last equality. Since

$$K_i - K_{i+1} = \{x \in V;\, \beta_i \leq f(x) < \beta_{i+1}\} =$$

$$= \{x \in V;\, f(x) = \beta_i\},$$

we have

$$A(K_i) - A(K_{i+1}) = A(K_i - K_{i+1}) = \sum_{\substack{x \in V \\ f(x) = \beta_i}} m(x).$$

Therefore, we obtain

$$\begin{aligned} A &\geq i_P(G) \sum_{i=1}^{N} \sum_{\substack{x \in V \\ f(x) = \beta_i}} m(x) {\beta_i}^2 \\ &= i_P(G) \sum_{x \in V} m(x) f(x)^2 \\ &= i_P(G)(f, f)_P = i_P(G), \end{aligned}$$

which is (7.19).

Altogether with (7.17), (7.18) and (7.19), we obtain

$$i_P(G)^2 \leq 2(\nabla f, \nabla f).$$

At last, we prove the inequality

$$\mu_o^P(G) \leq i_P(G).$$

For every nonempty finite subset S of V, we define a function $f \in C_c(V)$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

Then, we have

$$(f, f)_P = A(S) \quad \textit{and} \quad (\nabla f, \nabla f) \leq L(\partial S),$$

hence we have

$$\mu_0^P \leq \frac{(\nabla f, \nabla f)}{(f, f)_P} \leq \frac{L(\partial S)}{A(S)}.$$

Since S is arbitrary, we obtain $\mu_0^P \leq i_P(G)$. Q.e.d.

7.7 Exponential volume growth

(Exponential volume growth and spectra)

Definition 17 *For an infinite graph $G = (V, E)$, we define the exponential volume growths $\tau_P(G)$ (resp. $\tau_A(G)$) by*

$$\tau_P(G) = \limsup_{r\to\infty} \frac{1}{r} \log V(r),$$

$$\left(\tau_A(G) = \limsup_{r\to\infty} \frac{1}{r} \log(\#(B_r(x_0)))\right),$$

where

$$V(r) = A(B_r(x_0)) = \sum_{x\in B_r(x_0)} m(x),$$

and $B_r(x_0) = \{x \in V;\, d(x, x_0) < r\}$ $(r > 0)$ for some fixed vertex $x_0 \in V$. Note here that the definitions of $\tau_P(G)$ and $\tau_A(G)$ are independent on a choice of a fiexd vertex x_0.

Proposition 12 *We have the following inequalities:*

$$\tau_P(G) \geq \log(i_P(G) + 1), \quad \text{and} \quad \tau_A(G) \geq \log(i_A(G) + 1).$$

Proof. We only give a proof of the former, because it is the same for the latter. By definition, we have

$$\begin{aligned} e^{\tau_P(G)} &= \limsup_{r\to\infty} V(r)^{\frac{1}{r}} \\ &= \limsup_{r\to\infty} \frac{V(r+1)}{V(r)} \\ &= \limsup_{r\to\infty} \left(\frac{V(r+1) - V(r)}{V(r)} + 1\right) \\ &= \limsup_{r\to\infty} \left(\frac{\#(\partial B_r(x_0))}{V(r)} + 1\right) \\ &\geq i_P(G) + 1. \end{aligned}$$

Q.e.d.

Theorem 29 *For every infinite graph $G = (V, E)$,*

(1) (*cf. Fujiwara, [12]*) *we have*

$$\mu_0^P(G) \leq 1 - 2\, e^{\tau_P(G)/2}(1 + e^{\tau_P(G)})^{-1}.$$

(2) *If $m(G) = \sup\{m(x);\, x \in V\} < \infty$, we have*

$$\mu_0^A(G) \leq m(G)\left(1 - 2\, e^{\tau_A(G)/2}(1 + e^{\tau_A(G)})^{-1}\right).$$

In particular, if $\tau_P(G) = 0$, then $\mu_0^P(G) = 0$. If $\tau_A(G) = 0$, then $\mu_A(G) = 0$.

Remark 14 *(a) For every real number τ,*

$$1 - 2e^{\tau/2}(1 + e^{\tau})^{-1} = \frac{(1 - e^{\tau/2})^2}{1 + e^{\tau}},$$

which is a monotone increasing function in τ, say $f(\tau)$. And,

$$\lim_{\tau \to 0} f(\tau) = 0$$

and

$$\lim_{\tau \to \infty} f(\tau) = 1.$$

(b) Theorem 29 (1) sharpens the results due to [10] and [24].

Definition 18
An infinite graph $G = (V, E)$ is subexponential if $\tau_P(G) = 0$ or $\tau_A(G) = 0$.

Example 3 *The d dimensional integer lattice*

$$\mathbf{Z}^d = \{(m_1, \cdots, m_d);\, m_i \in \mathbf{Z}\ (i = 1, \ldots, d)\}$$

is a graph if two points $(m_1, \cdots, m_d)$ and $(m_1', \cdots, m_d')$ are connected by an edge if there exists i with $i = 1, \cdots, d$ such that $|m_i - m_i'| = 1$ and $m_j = m_j'$ for all $j = 1, \cdots, d$ with $j \neq i$. Since $V(r)$ and $\#(B_r(x_0))$ have polynomial growths as $r \to \infty$, $\mathbf{Z}^d$ is subexponential, i.e., $\tau_P(\mathbf{Z}^d) = 0$ and $\tau_A(\mathbf{Z}^d) = 0$.

Before going into the proof of of Theorem 29 (2), we need some lemmas.

Lemma 5 *If a real number α satisfies that $\tau_A(G) < 2\alpha$, then*

$$(e^{-\alpha\rho}, e^{-\alpha\rho})_A < \infty,$$

where ρ is the distance function from the fiexed vertex x_0, i.e., $\rho(x) = d(x, x_0)$, $(x \in V)$.

Proof. Indeed, we have

$$(e^{-\alpha\rho}, e^{-\alpha\rho})_A = \sum_{x\in V} e^{-2\alpha\rho(x)}$$

$$= \sum_{r=0}^{\infty} \{\#(B_{r+1}(x_0)) - \#(B_r(x_0))\}\ e^{-2\,\alpha\,r}$$

$$= \left(1 - e^{-2\alpha}\right) \sum_{r=0}^{\infty} \#(B_{r+1}(x_0))e^{-2\,\alpha\,r}.$$

Since $\tau_A(G) < 2\alpha$, we can choose $\tau_A(G) < \beta < 2\alpha$. Then, by definition of $\tau_A(G)$, for all large r,

$$\#(B_r(x_0)) \le e^{\beta r},$$

which implies

$$\sum_{r=0}^{\infty} e^{\beta r} e^{-2\alpha r} < \infty.$$

We have the desired result. Q.e.d.

For each α and $j = 1, 2, \cdots$, define functions h_j and f_j on V by

$$h_j(x) = \begin{cases} \alpha\rho(x), & \rho(x) \le j, \\ 2\alpha j - \alpha\rho(x), & \rho(x) > j, \end{cases}$$

$$f_j(x) = e^{h_j(x)}.$$

Then we have

Lemma 6 *If two vertices x and y are neighbors, then for each $j = 1, 2, \cdots$,*

$$(f_j(x) - f_j(y))^2 \le \frac{(1 - e^{\alpha})^2}{1 + e^{2\alpha}} \left(f_j(x)^2 + f_j(y)^2\right).$$

Proof. If $x \sim y$, $\rho(x) = \rho(y)$, or $|\rho(x) - \rho(y)| = 1$. In the case $\rho(x) = \rho(y)$, we have the desired. In the case $|\rho(x) - \rho(y)| = 1$, we may assume that $\rho(y) = \rho(x) + 1$. We give the proof dividing two cases.

Case 1. $\rho(x) \le j - 1$: In this case, $\rho(y) \le j$, so we have by definition of f_j,

$$f_j(y) = e^{\alpha\rho(y)} = e^{\alpha} f_j(x),$$

and then we have

$$(f_j(x) - f_j(y))^2 = (1 - e^{\alpha})^2 f_j(x)^2$$

$$= \frac{(1-e^{\alpha})^2}{1+e^{\alpha}} \left(f_j(x)^2 + e^{2\alpha} f_j(y)^2\right)$$

$$= \frac{(1-e^{\alpha})^2}{1+e^{\alpha}} \left(f_j(x)^2 + f_j(y)^2\right).$$

Case 2. $\rho(x) \ge j$. In this case, $\rho(y) \ge \rho(x) + 1 \ge j + 1$. Then we have

$$f_j(y) = e^{2\alpha j - \alpha\rho(y)} = e^{-\alpha} e^{2\alpha j - \alpha\rho(x)} = e^{-\alpha} f_j(x).$$

By the similar way as the Case 1, we have the desired inequality. Q.e.d.

Lemma 7
If $(e^{-\alpha\rho}, e^{-\alpha\rho})_A < \infty$, *then, for each* $j = 1, 2, \cdots$, $(f_j, f_j)_A < \infty$.

Proof. We have

$$(f_j, f_j)_A = \sum_{x \in B_{j+1}(x_0)} f_j(x)^2 + \sum_{x \notin B_{j+1}(x_0)} f_j(x)^2$$

$$\le e^{2\alpha j} \#(B_{j+1}(x_0)) + e^{4\alpha j} (e^{-\alpha\rho}, e^{-\alpha\rho})_A < \infty.$$

Q.e.d.

Lemma 8 *Under the same assumption of Lemma 7, we have*

$$(\nabla f_j, \nabla f_j) \le m(G) \frac{(1-e^{\alpha})^2}{1+e^{2\alpha}} (f_j, f_j)_A \,.$$

Proof. Indeed, we have, by Lemma 6,

$$(\nabla f_j, \nabla f_j) = \frac{1}{2}\sum_{x\sim y}(f_j(x) - f_j(y))^2$$

$$\leq \frac{1}{2}\frac{(1-e^{\alpha})^2}{1+e^{2\alpha}}\sum_{x\sim y}\left(f_j(x)^2 + f_j(y)^2\right)$$

$$= \frac{(1-e^{\alpha})^2}{1+e^{2\alpha}}\sum_{x\in V} m(x)f_j(x)^2$$

$$\leq m(G)\,\frac{(1-e^{\alpha})^2}{1+e^{2\alpha}}(f_j, f_j)_A\,.$$

Proof. (of Theorem 29) For each vertex $x \in V$, let χ_x be a function on V defined by

$$\chi_x(y) = \begin{cases} 1, & y = x, \\ 0, & \neq x. \end{cases}$$

Then we have

$$(\nabla\chi_x, \nabla\chi_x) = \frac{1}{2}\sum_{z\sim y}(\chi_x(z) - \chi_x(y))^2 = m(x),$$

and $(\chi_x, \chi_x)_A = 1$. Therefore, we obtain that $\mu_0^A(G) \leq m(G)$. So we may assume $\tau_A(G) < \infty$.

Now we assume $m(G)\frac{(1-e^{\tau_A(G)/2})^2}{1+e^{\tau_A(G)}} < \mu_0^A(G)$. We want to derive a contradiction. The function $f(\tau) = \frac{(1-e^{\tau/2})^2}{1+e^{tau}}$ is monotone increasing, so we can choose a real number α satisfying that $\tau_A(G) < 2\alpha$ and $m(G)f(\alpha) < \mu_0^A(G)$. Then by Lemma 5, we have $(e^{-\alpha\rho}, e^{-\alpha\rho})_A < \infty$. By Lemma 7, we have $(f_j, f_j)_A < \infty$. By Lemma 8, we have

$$\frac{(\nabla f_j, \nabla f_j)}{(f_j, f_j)_A} \leq m(G)\frac{(1-e^{\alpha})^2}{1+e^{2\alpha}} = m(G)f(\alpha) < \mu_0^A(G),$$

which contradicts the definition of $\mu_0^A(G)$. We obtain Theorem 29 (2). Q.e.d.

7.8 Heat kernel and Green kernel

Let $G = (V, E)$ be a locally finite infinite graph. Let $p(x, y)$ be the one step transition probability of the random walk standing at $x \in V$ to move to $y \in V$, i.e.,

$$p(x,y) = p_{xy} = \begin{cases} \dfrac{1}{m(x)}, & y \sim x, \\ 0, & \text{otherwise.} \end{cases}$$

We define the operator P acting on $C(V)$, by

$$Pf(x) = \sum_{y \in V} p(x,y) f(y) = \frac{1}{m(x)} \sum_{y \sim x} f(y), \quad (x \in V),$$

for each $f \in C(V)$. Let us recall the transition Laplacian Δ^P which is defined by $\Delta_P = I - P$, i.e.,

$$\begin{aligned} (\Delta^P f)(x) = f(x) - Pf(x) = \\ = f(x) - \frac{1}{m(x)} \sum_{y \sim x} f(y), \quad (x \in V). \end{aligned}$$

Definition 19 *For each $n = 0, 1, 2, \cdots$, $p^n(x, y)$ $(x, y \in V)$ is defined inductively by*

$$p^n(x,y) = \sum_{x \in V} p(x,z) p^{n-1}(z,y).$$

Here $p^0(x, y)$ is

$$p^0(x,y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases}$$

We call $p^n(x, y)$ the heat kernel.

Remark that

$$p^1(x,y) = \sum_{z \in V} p(x,z) p^0(z,y) = p(x,y).$$

The function $p^n(x, y)$ is the n-th step transition probability of the random walk standing at x to move to y. Note here that

$$p^{n+1}(x,y) - p^n(x,y) + (\Delta^P)_x p^n(x,y) = 0. \tag{7.20}$$

Indeed, we have by definition

$$\begin{aligned}(\Delta^P)_x p^n(x,y) &= p^n(x,y) - \sum_{x\in V} p(x,z)p^n(z,y)\\ &= p^n(x,y) - p^{n+1}(x,y).\end{aligned}$$

Q.e.d.

Definition 20 *The Green kernel (or Green function) is the following series:*

$$G(x,y) = \sum_{n=0}^{\infty} p^n(x,y) \quad (x,\, y \in V).$$

The above series is not always convergent. If there exist x and y in V such that $G(x,y) < \infty$, (then $G(x,y) < \infty$ for all x and y), the graph $G = (V,E)$ is said to be transient, and otherwise, recurrent.

Statistically, $G(x,y)$ is the expectation of visiting time of the random walk starting at x to y.

If $G = (V,E)$ is transient, it holds that for all x and y in V,

$$(\Delta^P)_x G(x,y) = p^0(x,y) = \begin{cases} 1 & x = y, \\ 0 & x \neq y. \end{cases} \tag{7.21}$$

Indeed, we have

$$\begin{aligned}(\Delta^P)_x G(x,y) &= G(x,y) - \sum_{x\in V} p(x,z)G(z,y)\\ &= G(x,y) - \sum_{n=0}^{\infty}\sum_{z\in V} p(x,z)p^n(z,y)\\ &= G(x,y) - \sum_{n=0}^{\infty} p^{n+1}(x,y)\\ &= p^0(x,y).\end{aligned}$$

For estimations of the heat kernel and the Green kernel, see [30], [33], [35].

7.9 Martin kernel and Martin boundary

In this section, we only consider a transient infinite graph $G = (V, E)$. It is known (see [15]) that if $F(x, y)$ is the probability of visiting to y of the random walk starting at x, then the following three hold:

$$\begin{cases} F(x,x) = 1 & x \in V, \\ G(x,y) = F(x,y)G(y,y) & x, y \in V, \\ F(x,y)F(y,z) \le F(x,z) & x,\ y,\ z \in V. \end{cases} \tag{7.22}$$

Definition 21 *We always fix a vertex $o \in V$. The function K_y $(y \in V)$ dfined by*

$$K_y(x) = K(x,y) = \frac{G(x,y)}{G(o,y)} \quad (x \in V)$$

is called the Martin kernel of $G = (V, E)$.

Proposition 13 *The Martin kernel satisfies the following properties:*

(1) *For each $y \in V$, K_y is bounded on V.*
(2) *For each $y \in V$, K_y is harmonic on the set $\{x \in V;\, x \neq y\}$.*
(3) *$K_y(o) = 1$.*

Proof. (3) is clear. For (1), we only see that for all x and y in V, by (10.1),

$$K(x,y) = \frac{G(x,y)}{G(y,y)}\frac{G(y,y)}{G(o,y)} = \frac{F(x,y)}{F(o,y)} \le \frac{1}{F(o,y)} < \infty. \tag{7.23}$$

For (2), it suffices to see by (7.21),

$$(\Delta^P)_x K(x,y) = \begin{cases} 0, & x \neq y, \\ \dfrac{1}{G(0,x)}, & x = y. \end{cases} \tag{7.24}$$

We have Proposition 13. Q.e.d.

Definition 22 *For a transient infinite graph $G = (V, E)$, we define a new distance $\rho(u, v)$ $(u,\ v \in V)$ as follows:*

$$\rho(u,v) = \sum_{x \in V} q(x)F(o,x)|K(x,u) - K(x,v)| \quad (u,\ v \in V),$$

where $q(x)$ $(x \in V)$ *is a weight function on* V *which satisfies that*

$$0 < q(x) < \infty, \quad and \quad \sum_{x \in V} q(x) < \infty.$$

As such a function, one can find in such a way that, numbering all elements of V *as* $\{u_1, u_2, \cdots\}$*, and define* q *by*

$$q(u_i) = 2^{-i} \quad (i = 1, 2, \cdots),$$

since $\sum_{x \in V} q(x) = \sum_{i=1}^{\infty} 2^{-i} < \infty$.

Lemma 9 *The function* $\rho(u, v)$ *is a distance in* V*.*

Proof. By the third equation of (7.22) and (7.23), we have

$$\rho(u, v) \leq \sum_{x \in V} q(x) F(o, x) \{|K(x, u)| + |K(x, v)|\} \leq$$

$$\leq 2 \sum_{x \in V} q(x) < \infty.$$

The triangle inequality and $\rho(u, v) = \rho(v, u)$ are clear. Furthermore, we can see that

$$\rho(u, v) = 0 \Longleftrightarrow K_u = K_v \Longleftrightarrow u = v.$$

By Definition 22, one can see immediately the first equivalence. For the second equivalence, assume that $K_u = K_v$ and $u \neq v$. We want derive a contradiction. Indeed, we have

$$K(u, v) = \sum_{z \in V} p(u, z) K(z, v)$$

(since K_v is harmonic on $\{x \in V;\, x \neq v\}$)

$$= \sum_{z \in V} p(u, z) K(z, u) \quad (since\ K_u = K_v)$$

$$= K(u, u) - (\Delta^P)_u K(u, u) \quad (by\ definition\ of\ \Delta^P)$$

$$= K(u, u) - \frac{1}{G(o, u)} \quad (by\ (7.24))$$

$$< K(u, u),$$

which contradicts the assumption $K_u = K_v$. Q.e.d.

Proposition 14 *For a sequence* $\{y_n\}_{n=1}^{\infty}$ *of vertices in* V*, a necessary and sufficient condition to be a Cauchy sequence in the metric space* (V, ρ) *is that for each* $x \in V$*, the sequence of real numbers,* $\{K_{y_n}(x)\}_{n=1}^{\infty}$ *is also a Cauchy sequence. In particular, the topology of* V *induced from the distance* ρ *does not depend on a choice of* $q(x)$ $(x \in V)$*.*

Remark 15 *A sequence* $\{y_n\}$ *in a metric space* (V, ρ) *is a Cauchy sequence if if* $\rho(y_n, y_m) \to 0$ *as* m*,* $n \to \infty$*.*

Proof. Assume that $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (V, ρ). By definition of ρ,

$$\{q(x)F(o,x)K(x,y_n)\}_{n=1}^{\infty}$$

is also a Cauchy sequence in the real line, for each fixed $x \in V$. Since $q(x)F(o,x) > 0$, $\{K(x,y_n)\}_{n=1}^{\infty}$ is also a Cauchy sequence.

Conversely, assume that for each $x \in V$, $\{K(x,y_n)\}_{n=1}^{\infty}$ is a Cauchy sequence. Let $\epsilon > 0$ be an arbitrarily small number. Take a finite subset S of V in such a way that the following holds:

$$\sum_{x \in V-S} q(x) \le \frac{\epsilon}{4}.$$

there exists a large number $N > 0$ satisfying that for each $x \in S$ and for all n, $m \ge N$,

$$|K(x,y_n) - K(x,y_m)| \le \frac{\epsilon}{2\sum_{z \in S} q(z)},$$

since S is a finite set and the assumption that $\{K(x,y_n)\}_{n=1}^{\infty}$ is a Cauchy sequence. Then, we have

$$\begin{aligned}
\rho(y_n, y_m) &= \sum_{x \in S} q(x)F(o,x)|K(x,y_n) - K(x,y_m)| \\
&+ \sum_{x \in V-S} q(x)F(o,x)|K(x,y_n) - K(x,y_m)| \\
&\le \left\{\sum_{x \in S} q(x)F(o,x)\right\} \frac{\epsilon}{2\sum_{z \in S} q(z)} + 2\sum_{x \in V-S} q(x) \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

which implies that $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (V, ρ). Q.e.d.

Definition 23 *Let V^* be the completion of the metric space (V,ρ). We denote by the same letter ρ, the induced metric on V^* from ρ. The complement V_∞ of V in V^* is called the Martin boundary of $G=(V,E)$. That is, V^* is the set of all Cauchy sequences $Y=\{y_n\}_{n=1}^\infty$ in V with respect to ρ, and the inclusion of V into V^* is given by the mapping which assigns $v\in V$ to the sequence $\{v,\, v,\, \cdots\}\in V^*$.*

By means of Proposition 14 , V_∞ is characterized as follows: V_∞ is identified by the set of all $g\in C(V)$ of which is the limit

$$g=\lim_{n\to\infty}K_{y_n}$$

where $\{y_n\}_{n=1}^\infty$ is a sequence of mutually distinct vertices in V.

As $\{y_n\}_{n=1}^\infty$ is Cauchy if and only if $\{K(x,y_n)\}_{n=1}^\infty$ is so, for all $x\in V$, due to Proposition 14, one can define for each Cauchy sequence $Y=\{y_n\}_{n=1}^\infty\in V^*$,

$$K_Y(x)=K(x,Y)=\lim_{n\to\infty}K(x,y_n)\quad (x\in V). \tag{7.25}$$

Furthermore, the distance ρ on V^* is given by

$$\rho(Y,Y')=\lim_{n\to\infty}\rho(y_n,y_n') \tag{7.26}$$

$$\left(=\sum_{x\in V}q(x)F(o,x)|K(x,Y)-K(x,Y')|\right),$$

for $Y=\{y_n\}_{n=1}^\infty$ and $Y'=\{y_n'\}_{n=1}^\infty$. Because $\{\rho(y_n,y_n)\}_{n=1}^\infty$ is covergent since

$$|\rho(y_n,y_n)-\rho(y_m,y_m)|\le$$

$$\le|\rho(y_n,y_n)-\rho(y_n,y_m)|+|\rho(y_n,y_m)-\rho(y_m,y_m)|$$

$$\le 2\rho(y_n,y_m)\to 0,$$

as $n,\, m\to\infty$.

It is easy to see that

(1) for a sequence $\{Y_n\}_{n=1}^\infty$ in V^*, it is a Cauchy sequence in V^* if and only if $\{K_{Y_n}\}_{n=1}^\infty$ is a Cauchy sequence in the real line for each $x\in V$.

(2) For two $Y = \{y_n\}_{n=1}^{\infty}$ and $Y' = \{y'_n\}_{n=1}^{\infty}$ in V^*, that they are equivalent is that $\lim_{n\to\infty} \rho(y_n, y'_n) = 0$ if and only if $K_Y = K_{Y'}$ if and only if $\rho(Y, Y') = 0$, by definition.

Furthermore, we have

Proposition 15 *The metric space* (V^*, ρ) *is compact.*

Proof. Assume that $\{Y_n\}_{n=1}^{\infty}$ is any sequence in V^*. Then we have by (7.22),

$$K(x, Y_n) \leq \sup_{y \in V} K(x, y) \leq \frac{1}{F(o, x)} < \infty \quad (x \in V).$$

By the diagonal argument, one can choose a subsequence

$$\{Y_{n_k}\}_{k=1}^{\infty}$$

satisfying that

$$\{K(x, Y_k)\}_{k=1}^{\infty}$$

is a Cauchy sequence in the real line for each $x \in V$. Because, we enumerate all elements in V as $V = \{u_1, u_2, \cdots\}$. Then we can choose subsequence $\{Y_n^1\}_{n=1}^{\infty}$ of $\{Y_n\}_{n=1}^{\infty}$ satisfying that $K(u_1, Y_n^1)\}_{k=1}^{\infty}$ is a Cauchy sequence, and then one can choose a subsequence $\{Y_n^2\}_{n=1}^{\infty}$ of $\{Y_n^1\}_{n=1}^{\infty}$ satisfying that $\{K(u_2, Y_n^2)\}_{n=1}^{\infty}$ is a Cauchy sequence, and so on. Continue this process. Then for the subsequence $\{Y_k^k\}_{k=1}^{\infty}$ of $\{Y_n\}_{n=1}^{\infty}$, $\{K(u_i, Y_k^k)\}_{k=1}^{\infty}$ is a Cauchy sequence in the real line for all u_i. Due to Proposition 10.8, the subsequence $\{Y_k^k\}_{k=1}^{\infty}$ of $\{Y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (V^*, ρ), which is convergent, because of the completeness of (V^*, ρ). Q.e.d.

We can state (cf. [15])

Theorem 30 (Poisson-Martin representation theorem)
Assume that an infinite garph $G = (V, E)$ *is transient. For each non-negative harmonic function* h *on* V*, there exists a Borel measure on* V_∞ *such that*

$$h(x) = \int_{V_\infty} K_Y(x) d\mu(Y) \quad (x \in V).$$

Conversely, for each Borel measure μ *on* V_∞ *the above integral* h *gives a non-negative harmonic function on* V*.*

Definition 24 *A non-negative harmonic funtion h on V is called minimal if the only non-negative harmonic function h_1 on V satisfying*

$$h_1(x) \leq h(x) \quad (x \in V)$$

is a constant multiple of h. Then let us define

$$V_{1,\infty} = \{Y \in V_\infty;\ K_Y \mathit{is\ minimal}\},$$

$$V_{0,\infty} = V_\infty - V_{1,\infty} \quad (\mathit{the\ complement\ of} V_{1,\infty} \mathit{\ in\ } V_\infty),$$

and a Borel measure μ on V_∞ is called regular if $\mu(V_{0,\infty}) = 0$.

Then we can state (cf. [15])

Theorem 31 *Assume that $G = (V, E)$ is a transient infinite graph. Then for any non-negative harmonic function h on V there exists a unique regular Borel measure μ on V_∞ satisfying that*

$$h(x) = \int_{V_\infty} K_Y(x) d\mu(Y) \quad (x \in V).$$

7.10 End compactification

For a transient infinite graph $G = (V, E)$, we can construct another compactification, so called the end boundary.

Definition 25 *A sequence $Y = \{y_i\}_{i=0}^{\infty}$ of vertices in V is called a ray if it satisfies that $y_n \neq y_m$ for all $n \neq m$, and $y_i \sim y_{i+1}$ for all $i = 0, 1, 2, \cdots$.*

Two rays $Y = \{y_i\}_{i=0}^{\infty}$ and $Y' = \{y'_i\}_{i=0}^{\infty}$ are equivalent if there exist positive numbers n and k such that $y_i = y'_{i+k}$ for all $i \geq n$.

The equivalence classes, say $[Y]$, are called end, and the totality of all ends is called the end boundary, and denoted by $V(\infty)$.

Theorem 32 (cf.[*8*]) *Assume that a locally finite, infinite graph $G = (V, E)$ is transient. We can give the following topology, called the* cone topology *on the union $V \cup V(\infty)$ of V and the end boundary $V(\infty)$ which becomes compact by this topology: for every ray $Y = \{y_i\}_{i=0}^{\infty}$ and finite subset F of E, let $C_F(Y)$ be the connected component of the complement of F in E of which corresponding set of vertices, denoted by the same letter $c_F(Y)$, contains all but except finite number of y_i. For a finite subset F of E, we denote by $C_F([Y]) = C_F(Y)$ since $C_F(Y) = C_F(Y')$ for two equivalent Y and Y'.*

The cone topology *on $V \cup V(\infty)$ is that*

- *For each $x \in V$, the single set $\{x\}$ is open,*
- *For each $[Y] \in V(\infty)$, the family consisting of subsets $(C_F([Y]) - U) \cup [Y]$ of $V \cup V(\infty)$, is a basis of neighborhood of $[Y]$, where F runs over all finite subsets E and U runs over all finite subsets of V.*

Remark 16

Here two edges e and e' of a subset G of E are to be connected in G if there exits a path

$$c = [v_0, v_1, \cdots, v_{n+1}]$$

satisfying

$$e = (v_0, v_1), \quad e' = (v_n, v_{n+1})$$

and each edge (v_i, v_{i+1}) belongs to G, the maximal subsets of G in which elements are connected mutually in G are said the connected components of G.

Definition 26 *A ray $Y = \{y_i\}_{i=0}^{\infty}$ is said geodesic if for all m and n with $m < n$, the path $c = [y_m. y_{m+1}, \cdots, y_n]$ is the shortest pathconnecting y_m and y_n, i.e., $\ell(c) = n - m$.*

For all $x \in V$ and $\omega \in V(\infty)$, there exists a ray $Y = \{y_i\}_{i=0}^{\infty}$ which is a geodesic and $[Y] = \omega$. This Y is unique up to the equivalence and is called the geodesic ray starting at x and reaching ω.

It is known (cf. [8]) that

Proposition 16 *The end boundary* $V(\infty)$ *coincides with the totality of all geodesic rays starting at a fixed vertex* $o \in V$*.*

Furthermore, it is known (cf. [8]) that

Theorem 33 *Assume that a locally finite, infinite tree* $G = (V, E)$ *is transient.*

(1) *Then the Martin boundary coincides with the end boundary, i.e.,* $V_{1,\infty} = V_\infty = V(\infty)$*.*

(2) *A sequence* $\{y_i\}_{i=0}^\infty$ *of vertices in* V *converges to* $\omega \in V(\infty)$ *if and only if* $(o, [y_i, \omega]_o) \to \infty$ *as* $i \to \infty$*. Here* d *is the graph distance of* $G = (V, E)$ *and for two elements* ω *and* η *in* $V \cup V(\infty)$*,* $p \in V$ *is the confluence of* ω *and* η*, denoted by* $[\omega, \eta]_o$*, if* p *is the last common vertex in the geodesic ray from* o *to* ω*, and the one from* o *to* η*.*

For the end boundary $V(\infty)$, the *end Dirichlet* bonudary problem can be settled: For a given function φ on $V(\infty)$ which is continuous with respect to the cone topology, find a solution $f \in C(V)$ satisfying that

$$\begin{cases} \Delta^P f(x) = 0, & x \in V, \\ \lim_{x \to \omega} f(x) = \varphi(\omega), & \omega \in V(\infty), \end{cases} \tag{7.27}$$

where the limit is taken in the cone topology on $V \cup V(\infty)$.

For this problem, it is known (cf. [8]) that

Theorem 34 *Assume that* $G = (V, E)$ *is a locally finite infinite transient graph and its Green kernel* $G(x, y)$ vanishes at infinity*, i.e., for each* $\epsilon > 0$*, the set* $\{x \in V; G(x, o) < \epsilon\}$ *is finite set for a fixed vertex* $o \in V$*. Then for every continuous function* φ *on* $V(\infty)$*, there exists a unique solution* $f \in C(V \cup V(\infty))$ *of the end Dirichlet problem.*

Finally, we give examples.

Example 4 (1) *For the* d *dimensional integer lattice* $\mathbf{Z^d}$*, it is transient if and only if* $d \geq 3$ *(cf. Polya). In this case,* $V_\infty = V(\infty)$ *is a single set, i.e., every non-negative harmonic function on* $\mathbf{Z^d}$ *is constant.*

(2) *For the homogeneous regular tree of degree d with* ($d \geq 3$), *is always transient, and the Green kernel is given (cf. [4], [30]) by*

$$G(x,y) = \frac{d-1}{d-2}\left(\frac{1}{d-1}\right)^{d(x,y)} \qquad (x,\, y \in V(T_d)),$$

where $d(x,y)$ is the graph distance between x and y. The Martin boundary coincides with the end boundary $V_\infty = V(\infty)$, and this can be identified with the Cantor set in the circle $S^1 = [0,1]/\{0,1\}$.

For the Green kernel, we have (see for example, [30]) that

Theorem 35 *Assume that $G = (V,E)$ is a locally finite infinite garph.*

(1) *If $m(x) \leq d$ for all $x \in V$, then we have*

$$G(x,y) \geq \frac{d-1}{d-2}\left(\frac{1}{d-1}\right)^{d(x,y)} \qquad (x,\, y \in V).$$

(2) *If $G = (V,E)$ is an infinite tree and satisfies $m(x) \geq d \geq 3$ for all $x \in V$, then we have*

$$G(x,y) \leq \frac{d-1}{d-2}\left(\frac{1}{d-1}\right)^{d(x,y)} \qquad (x,\, y \in V).$$

Corollary 6
$G = (V,E)$ is an infinite tree and $m(x) \geq 3$ for all $x \in V$. Then the end Dirichlet problem has a unique solution f for every continuous function φ on $V(\infty)$.

7.11 From discreteness to continuum

In this section, we want to show what is the completion and the Martin boundary of the homogeneous regular tree T_d with $d = q + 1 \geq 3$.

Let $\mathcal{Q}$ be the set of all symbols

$$0.a_1a_2\cdots a_k,$$

where $a_1,\, a_2,\, \cdots,\, a_k \in \{0,\, 1,\, 2,\, \cdots,\, q-1\}$. We may assume $0. \in \mathcal{Q}$, and may regard $u = 0.a_1a_2\cdots a_k$ as a rational number $\pi(u)$ expanded by q in such a way that

$$\pi(u) = a_1\, q^{-1} + a_2\, q^{-2} + \cdots + a_k\, q^{-k} \in [0,1].$$

And let $-\mathcal{Q}$ be the set of all symbols $-\,0.a_1a_2\cdots a_k$, where $0.a_1a_2\cdots a_k$ runs over the set $\mathcal{Q}$. We may assume $-\,0. \in -\mathcal{Q}$ and may also regard $u = -\,0.a_1a_2\cdots a_k$ as

$$\pi(u) = -\,a_1\, q^{-1} - a_2\, q^{-2} - \cdots - a_k\, q^{-k} \in [-1,0].$$

We consider the following infinite graph $G = (V,E)$ of which the set V of vertices is $V = \mathcal{Q} \cup (-\mathcal{Q})$, and the set E of edges is given by the following relations:

(1) $0.a_1a_2\cdots a_k \sim 0.a_1a_2\cdots a_kx$,
where $x \in \{0,\, 1,\, 2,\, \cdots,\, q-1\}$,
(2) $-\,0.a_1a_2\cdots a_k \sim -\,0.a_1a_2\cdots a_kx$,
where $x \in \{0,\, 1,\, 2,\, \cdots,\, q-1\}$,
(3) $-\,0. \sim 0.$,
respectively.

Then the obtained infinite graph $G = (V,E)$ can be identified naturally with the homogeneous regular tree T_d of degree $d = q+1$.

Recall that the Green kernel $G(x,y)$ of T_d is given by

$$G(x,y) = \frac{q}{q-1} q^{-d(x,y)} \quad (x,\, y \in V), \tag{7.28}$$

and the Martin kernel $K(x,y)$ is given by

$$K(x,y) = \frac{G(x,y)}{G(o,y)} = q^{d(o,y)-d(x,y)} \quad (x,\, y \in V), \tag{7.29}$$

and the probability $F(x,y)$ of visiting to y for the random walk starting at x is given by

$$F(x,y) = \frac{G(x,y)}{G(y,y)} = q^{-d(x,y)} \quad (x,\, y \in V). \tag{7.30}$$

We may give the weight function $q(x)$ $(x \in V)$ as

$$q(x) = q^{-2d(o,x)} \quad (x \in V). \tag{7.31}$$

Then the distance $\rho(u, v)$ $(u,\, v \in V)$ given by

$$\rho(u, v) = \sum_{x \in V} q(x) F(o, x) |K(x, u) - K(x, v)| \quad (u,\, v \in V)$$

can be estimated below as follows:

Theorem 36 *We have*

$$\frac{q(q^2 - 1)}{q^3 - 1} |\pi(u) - \pi(v)|^2 \le \rho(u, v) \quad (u,\, v \in \mathcal{Q}),$$

where $|\, x \,|$ *is the absolute value of a real number* x.

We have immediately by Theorem 36

Corollary 7
If a sequence $\{u_n\}_{n=1}^{\infty}$ *of vertices in* $\mathcal{Q}$ *is a Cauchy sequence with respect to the distance* ρ, *then* $\{\pi(u_n)\}_{n=1}^{\infty}$ *is a Cauchy sequence in* $[0, 1]$ *with respect to the distance* $|x - y|$, $(x,\, y \in [0, 1])$. *Therefore, from the completion of* $T_d = \mathcal{Q} \cup (-\mathcal{Q})$, *we can reconstruct the closed interval* $[0, 1]$.

Proof. (of Theorem 36) In the proof, we identify u and $\pi(u)$ for $u \in \mathcal{Q}$. Notice that, by (7.29), for $u,\, v \in \mathcal{Q}$, and $x \in -\mathcal{Q}$,

$$|K(x, u) - K(x, v)| = \left| q^{d(o,u) - d(x,u)} - q^{d(o,v) - d(x,v)} \right| =$$

$$= \left| q^{-d(o,x)} - q^{-d(o,x)} \right| = 0,$$

where we take $o = 0. \in \mathcal{Q}$. By this and together with (7.30), (7.31) and the definition of $\rho(u, v)$, we have for $u,\, v \in \mathcal{Q}$,

$$\rho(u, v) = \sum_{x \in \mathcal{Q}} q^{-3d(o,x)} \left| q^{d(o,u) - d(x,u)} - q^{d(o,v) - d(x,v)} \right| \tag{7.32}$$

$$= \sum_{x\in\mathcal{Q}} q^{-4d(o,x)} \left| q^{2d(o,[x,u]_o)} - q^{2d(o,[x,v]_o)} \right| ,$$

where $[x,u]_o$ is the confluence of x and u, and

$$d(x,u) = d(o,x) + d(o,u) - 2d(o,[x,u]_o)$$

for x, $u \in \mathcal{Q}$.

For u, $v \in \mathcal{Q}$, let

$$u = 0.a_1 \cdots a_p a_{p+1} \cdots a_{p+s},$$

and

$$v = 0.a_1 \cdots a_p a'_{p+1} \cdots a'_{p+t} ,$$

with $a_{p+1} \neq a'_{p+1}$. So we have $[u,v]_o = 0.a_1 \cdots a_p$. For $r \geq p+1 = d(o,[u,v]_o)+1$, let A_r be the set of all vertices x in $\mathcal{Q}$ which follow from $0.a_1 \cdots a_p a_{p+1}$ with $d(o,x) = r$. Then we have

Lemma 10 (1) *For all $x \in A_r$, we have*

$$d(o,[u,v]_o) + 1 = p+1 \leq d(o,[x,u]_o) = d(o,[x,v]_o).$$

(2) *We have* $\#(A_r) = q^{r-(p+1)}$.

Proof. (1) All $x \in A_r$ can be written as

$$x = 0.a_1 \cdots a_p a_{p+1} x_{p+2} \cdots x_r$$

for some $x_{p+2}, \cdots, x_r \in \{0,\, 1,\, q-1\}$. Then, we have

$$[x,u]_o = 0.a_1 \cdots a_p a_{p+1} y_{p+2} \cdots y_r,$$

for some $y_{p+2}, \cdots, y_r \in \{0,\, 1,\, \ldots,\, q-1\}$, and

$$[x,v]_o = 0.a_1 \cdots a_p = [u,v]_o.$$

Then, we have $d(o,[x,u]_o) \geq p+1 = 1 + d(o,[u,v]_o)$ and $d(o,[x,v]_o) = p = d(o,[u,v]_o)$.

The statement (2) is clear. Q.e.d.

By Lemma 10, we have, for $x \in A_r$,

$$\left| q^{2d(o,[x,u]_o)} - q^{2d(o,[x,v]_o)} \right| = \left| q^{2d(o,[x,u]_o)} - q^{d(o,[u,v]_o)} \right|$$

$$\geq \left|q^{2(p+1)} - q^{2p}\right| = q^{2p}(q^2-1).$$

Therefore, we have

$$\begin{aligned}
\rho(u,v) &\geq \sum_{r=p+1}^{\infty} q^{-4r} \sum_{x\in A_r} \left|q^{2d(o,[x,u]_o)} - q^{2d(o,[x,v]_o)}\right| \\
&\geq \sum_{r=p+1}^{\infty} \#(A_r) q^{-4r}(q^2-1)q^{2p} \\
&= \sum_{r=p+1}^{\infty} d^{r-(p+1)} q^{-4r}(q^2-1)q^{2p} \\
&= \frac{q^2-1}{q(q^3-1)} q^{-2p} \\
&= \frac{q^2-1}{q(q^3-1)} q^{-2d(o,[u,v]_o)} \\
&\geq \frac{q(q^2-1)}{q^3-1} |u-v|^2,
\end{aligned}$$

since

$$\begin{aligned}
|u-v| &= \left|0.a_1\cdots a_p a_{p+1}\cdots a_{p+s} - 0.a_1\cdots a_p a'_{p+1}\cdots a'_{p+t}\right| \\
&= \left|0.0\cdots 0(a_{p+1}-a'_{p+1})\cdots\right| \\
&\leq qq^{p+1}.
\end{aligned}$$

because of $a_{p+1} - a'_{p+1} \neq 0$.

Bibliography

[1] N. Alon, *Eigenvalues and expanders*, Combinatorics, 6(1986), 83-96.

[2] G.A. Baker, *Drum shapes and isospectral graphs*, J. Math. Physics, 7(1966), 2238-2242.

[3] F. Bien, *Construction of telephone networks by group representations*, Notices Amer. Math. Soc., 38(1989), 5-22.

[4] P. Cartier, *Fonctions harmoniques sur un arbre*, Sympos. Math., 9(1972), 203-270.

[5] F.R.K. Chung, *Eigenvalues of graphs*, Proc. I.C.M., 1994, Zürich, 1333-1342, Birkhäuser, 1995.

[6] F.R.K. Chung, *Spectral Graph Theory*, CBMS, Regional Conf. Series Math., 92(1997), Amer. Math. Soc.

[7] D.M. Cvetkovic & M. Doob & H. Sachs, *Spectra of Graphs*, Academic Press, 1978.

[8] D.I. Cartwright & P.M. Soardi & W. Woess, *Martin and end compactifications for non locally finite graphs*, Trans. Amer. Math. Soc., 338(1993), 679-693.

[9] J. Dodziuk, *Difference equation, isoperimetric inequalities and transience of certain random walks*, Trans. Amer. Math. Soc., 284(1984), 787-794.

[10] J. Dodziuk & W.S. Kendall, *Combinatorial Laplacians and isoperimetric inequality* in *From Local Times to Global Geometry*, Control and Physics, Ed. K.D. Elworthy, (1986), 68-75.

[11] J. Friedman, *Some geometric aspects of graphs and their eigenfunctions*, Duke Math. J., 69(1993), 487-525.

[12] K. Fujiwara, *Growth and the spectrum of the Laplacian of an infinite graph*, Tohoku Math. J., 48(1996), 293-302.

[13] Y. Higuchi, *A remark on exponential growth and the spectrum of the Laplacian*, to appear.

[14] Y. Higuchi & T. Shirai, *Isoperimetric constants of (d, f)-regular planar graphs*, to appear.

[15] J.G. Kemeny & J.L. Snell & A.W. Knap, *Denumerable Markov Chains*, Van Nostrand, 1966.

[16] A. Katsuda & H. Urakawa, *The first eigenvalue of the discrete Dirichlet problem for a graph*, J. Combin. Math. Combin. Comput., 27(1998), 217-225.

[17] A. Katsuda & H. Urakawa, *The Faber-Krahn type isoperimetric inequalities for a graph*, Tohoku. Math. J., 51(1999), 267-281.

[18] M. Kotani & T. Shirai & T. Sunada, *Asymptotic behavior of the transition probability of a random walk on an infinite graph*, J. Funct. Analysis, 159(1998), 684-689.

[19] B. Mohar, *The spectrum of an infinite graph*, Linear Algebra Appl., 48(1982), 245-256.

[20] B. Mohar, *Isoperimetric inequalities, growth, and the spectrum of graphs*, Linear Algebra Appl., 103(1988), 119-131.

[21] B. Mohar, *Isoperimetric numbers of graphs*, J. Combin. Theory, Ser. B, 47(1989), 274-291.

[22] B. Mohar & W. Woess, *A survey on spectra of infinite graphs*, Bull. London Math. Soc., 21(1989), 209-234.

[23] S. Nayatani & H. Urakawa, *Spectrum of the Schroedinger operator on a complete manifold*, J. Funct. Analysis, 112(1993), 459-479.

[24] Y. Ohno & H. Urakawa, *On the first eigenvalue of the combinatorial Laplacian for a graph*, Interdisciplinary Inform. Sci., 1(1994), 33-46.

[25] M.A. Picardello, *Harmonic Analysis and Discrete Potential Theory*, Prenum Press, 1992.

[26] T. Sunada & P.W. Sy, *Spectral analysis on weighted graphs*, 1997 (in Japanese).

[27] Jingson Tan, *The spectrum of combinatorial Laplacian for a graph*, Doctoral Dissertation, Tohoku University, 2000.

[28] H. Urakawa, *Laplacian and network*, Shokabo, Tokyo (in Japanese), 1996.

[29] H. Urakawa, *Spectra of the discrete and continuous Laplacians on graphs and Riemannian manifolds*, Interdisciplinary Inform. Sci., 3(1997), 95-109.

[30] H. Urakawa, *Heat kernel and Green kernel comparison theorems for infinite graphs*, J. Funct. Analysis, 146(1997, 206-235.

[31] H. Urakawa, *Eigenvalue comparison theorems of the discrete Laplacians for a graph*, Geometriae Dedicata, 74(1999), 95-112.

[32] H. Urakawa, *A discrete analogue of the harmonic morphism*, preprint.

[33] H. Urakawa, *A discrete analogue of the harmonic morphism and Green kernel comparison theorems*, Glasgow Math. J., 42(2000), 319-334.

[34] H. Urakawa, *The spectrum of an infinite graph*, to appear in Canadian J. Math., 2000.

[35] H. Urakawa, *The Cheeger constant, the heat kernel and the Green kernel of an infinite graph*, to appear.

[36] W. Woess, *Random walks on infinite graphs and groups - A survey on selected topics*, Bull. London Math. Soc., 26(1994), 1-60.

[37] W. Woess, *Random walks on infinite graphs and groups*, Cambridge University Press, 2000.