Chapter 3

Jacobi manifolds, by I.VAISMAN

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A lecture on Jacobi manifolds[1]

Abstract The lecture exposes the basics of Jacobi manifolds: differentiable manifolds M endowed with a Lie algebra structure of the local type on the ring of functions $C^\infty(M)$. It shows the expression of a Jacobi bracket, and its associated homogeneous Poisson structure. It presents the characteristic foliation and the Lie algebroid of a Jacobi manifold. The latter is used in a geometric quantization theory. Finally, we take a glance at the results about the integration of a Jacobi structure by a contact groupoid, and by a diffeological Lie group. The exposed results are taken from the papers mentioned in the references and they are neither new nor original.

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3.1 Introduction

This is an expository paper, and it does not contain original results of the present author. The framework of this paper is the C^∞ category.

It is well known that Poisson brackets play an essential role in theoretical mechanics and physics. On a Poisson manifold M the ring of differentiable functions $C^\infty(M)$ is endowed with a skew-symmetric Poisson bracket $\{f, g\}$ $(f, g \in C^\infty(M))$ which satisfies the Jacobi identity, and is a derivation with respect to each factor. Information on Poisson manifolds may be found in [21], for instance.

The Jacobi manifolds are defined as the natural generalization which is obtained if the last condition is replaced by the weaker condition that the bracket is of the local type i.e., the bracket is defined by a bilinear, bidifferential operator. In this case we speak of *Jacobi brackets, Jacobi structures*, etc.

Jacobi structures first appeared in the works of K. Shiga and A. Kirillov [11]. Then they were systematically studied by A. Lichnerowicz and his collaborators [17], [8], [5], etc., from the point of view of the Lie algebra cohomology, and of the local geometry.

Recently, M. de León, J. C. Marrero and E. Padrón wrote a series of articles which discuss cohomology and geometric quantization of Jacobi brackets (see [13], [14] and the references therein). In particular, these authors use the Lie algebroid structure of the jet bundle $J^1(M, \mathbf{R})$ of a Jacobi manifold M introduced by Y. Kerbrat and Z. Souici-Benhammadi [10]. On the other hand, [10] also states the integrability of the mentioned Lie algebroid to a local contact Lie groupoid. This theorem was then extended by P. Dazord [4], who also proves a "Lie's third theorem" of the Jacobi-Lie algebra $C^\infty(M)$ to a diffeological Lie group.

3.2 Basic formulas and examples

In spite of the apparently more general definition, the Jacobi brackets are defined by first order differential operators since one has [11], [8]

Proposition 3 *A Jacobi bracket necessarily is of the form*

$$\{f, g\} = \Lambda(df, dg) + f(Eg) - g(Ef), \tag{1.1}$$

where Λ is a bivector field, and E is a vector field on M. Conversely, a bracket defined by (1.1) is a Jacobi bracket iff

$$[\Lambda, \Lambda] = 2E \wedge \Lambda, \quad [\Lambda, E] = L_E\Lambda = 0, \tag{1.2}$$

where the bracket is that of Schouten-Nijenhuis.

Proof. If one assumes that at $x_0 \in M$ a Jacobi bracket has the local coordinate expression

$$\{f, g\} = A^{i_1 \dots i_r j_1 \dots j_s}\Big(\frac{\partial^r f}{\partial x^{i_1} \dots \partial x^{i_r}} \frac{\partial^s g}{\partial x^{j_1} \dots \partial x^{j_s}} - \tag{1.3}$$

$$- \frac{\partial^r g}{\partial x^{i_1} \dots \partial x^{i_r}} \frac{\partial^s f}{\partial x^{j_1} \dots \partial x^{j_s}}\Big) + \text{lower order terms},$$

where $r \geq s,\ r \geq 2$, a technical computation shows that the annulation of the higher order symbol of a sum $\sum_{cycl}\{\{f, g\}, h\}$ at x_0, where

$$jet^{2r-2}_{x_0}(f) = 0, \ \ jet^s_{x_0}(g) = 0, \ \ jet^{s-1}_{x_0}(h) = 0,$$

on well chosen basic covectors at x_0, implies $A^{i_1 \dots i_r j_1 \dots j_s} = 0$, which is a contradiction. See [8] for details.

Therefore, we must have $r = s = 1$, and we deduce the expression (1.1).

Now, the second relation (1.2) follows by noticing that (1.1) implies $Ef = \{1, f\}$, whence the Jacobi identity for $(1, f, g)$ yields

$$E\{f, g\} = \{Ef, g\} + \{f, Eg\}, \tag{1.4}$$

which is equivalent to $L_E\Lambda = 0$.

The first relation (1.2) follows by expressing the Jacobi identity for three functions (f, g, h) via (1.1) and, then, using a known fundamental identity for the Schouten-Nijenhuis bracket e.g., [7], [12]

$$[\Lambda, \Lambda](df, dg, dh) = 2\sum_{cycl} \Lambda(d(\Lambda(df, dg)), dh). \tag{1.5}$$

The fact that (1.2) implies the Jacobi identity follows by using (1.5) in the converse direction. Q. e. d.

Remark 10 *From (1.2) it follows that a Poisson bracket is a Jacobi bracket with $E = 0$. In the case of a general Jacobi bracket the Leibniz property is replaced by*

$$\{f, gh\} = \{f, g\}h + g\{f, h\} + gh(Ef). \tag{1.6}$$

The following theorem (e.g., [5]) provides a very important relationship between Jacobi and Poisson structures

Theorem 10 *Formula (1.1) defines a Jacobi bracket on the manifold M iff formula*

$$P = \frac{1}{t}\Lambda + \frac{\partial}{\partial t} \wedge E \tag{1.7}$$

defines a Poisson bracket on $M \times \mathbf{R}_+$, where $\mathbf{R}_+$ is the real positive t-axis.

Proof. Whether Poisson or not, P defines a bracket

$$\{\phi, \psi\}_P := P(d\phi, d\psi) \quad (\phi, \psi \in C^\infty(M \times \mathbf{R}_+))$$

for which an easy computation yields

$$\{tf, tg\}_P = t\{f, g\} \quad (f, g \in C^\infty(M)), \tag{1.8}$$

where $\{f, g\}$ is defined by (1.1). Therefore, if P is Poisson, (1.1) satisfies the Jacobi identity.

Conversely, if (1.1) satisfies the Jacobi identity, P also does for functions of the form tf. Using (1.5) for P,

$$\sum_{cycl} \{\{\phi, \psi\}_P, \chi\}_P = \frac{1}{2}[P, P](d\phi, d\psi, d\chi),$$

we get $[P, P](d(tf), d(tg), d(th)) = 0$. Since the differentials $\{d(tf)\}$ span $T^*(M \times \mathbf{R}_+)$, P is a Poisson bivector. Q. e. d.

Notice that the proof above also is a second proof of the characterization (1.2) of Jacobi manifolds.

Using the vector field $Z := t(\partial/\partial t)$, it follows that a Poisson bivector P on $M \times \mathbf{R}_+$ is of the form (1.7) iff

$$L_Z P = -P. \tag{1.9}$$

On any manifold, a Poisson bivector which satisfies (1.9) for some vector field Z is called a *homogeneous Poisson structure*, because of the homogeneity of degree -1 of the first term of (1.7) with respect to t. The homogeneous Poisson structures are essential in the study of Poisson manifolds [5]. Computations are simplified by the change of variables $t = e^\tau$ which leads to

$$P = e^{-\tau}(\Lambda + \frac{\partial}{\partial \tau} \wedge E), \quad Z = \frac{\partial}{\partial \tau}. \tag{1.10}$$

Hereafter, we will use (1.10), and identify M with $M \times \{0\}$ in $M \times \mathbf{R}$, where $\mathbf{R}$ is the τ-axis.

The fundamental examples of Jacobi manifolds are the *locally conformal symplectic manifolds* in even dimensions and the *contact manifolds* in odd dimensions [17], [8], [22].

The 2-form Ω defines a locally conformal symplectic structure on M^{2n} if M has an open covering $\{U_\alpha\}$ such that $\Omega/_{U_\alpha} = e^{\sigma_\alpha}\Omega_\alpha$ where Ω_α are symplectic forms on U_α and $\sigma_\alpha \in C^\infty(U_\alpha)$. Thus, the latter define local Poisson brackets, and it turns out that

$$\{f, g\} = e^{-\sigma_\alpha}\{e^{\sigma_\alpha} f, e^{\sigma_\alpha} g\}_\alpha \tag{1.11}$$

is a global Jacobi bracket on M. If the bivector Λ is defined by $\sharp_\Lambda \circ \flat_\Omega = Id.$, and $E := \sharp_\Lambda(d\sigma_\alpha)$, one gets the expression (1.1) of the bracket (1.11). (Our conventions are $\mu(\sharp_\Lambda\theta) = \Lambda(\theta, \mu)$, $\flat_\Omega(X)(Y) = \Omega(X, Y)$, $\forall\theta, \mu \in T^*M$, $\forall X, Y \in TM$.)

The 1-form θ defines a contact structure on M^{2n+1} if $\theta \wedge (d\theta)^n$ is nowhere zero. Then M has the *Reeb vector field* E which is determined by

$$i(E)\theta = 1, \quad i(E)d\theta = 0, \tag{1.12}$$

and $\forall f \in C^\infty(M)$ there exists a vector field X^θ_f defined by

$$i(X^\theta_f)\theta = f, \quad i(X^\theta_f)d\theta = -df + (Ef)\theta. \tag{1.13}$$

Furthermore, if we also put

$$\Lambda(df, dg) := d\theta(X^\theta_f, X^\theta_g), \tag{1.14}$$

we obtain a bracket (1.1) which satisfies the Jacobi identity. Indeed, (1.2) follows by computations which use (1.5), (1.12) and (1.13).

It follows easily that, if a Jacobi bracket is given on M, for any fixed nowhere zero function $a \in C^\infty(M)$, the formula

$$\{f, g\}^a := \frac{1}{a}\{af, ag\} \tag{1.15}$$

defines again a Jacobi bracket [8], [5] with

$$\Lambda^a = a\Lambda, \quad E^a = aE + i(da)\Lambda. \tag{1.16}$$

The new bracket is said to be *conformal* with the original bracket.

Conformal transformations are important in the geometry of Jacobi manifolds [5]. We may also see them as follows. Let M_u be Jacobi manifolds with the brackets $\{\,,\,\}_u$ $(u = 1, 2)$. A mapping $\varphi : M_1 \to M_2$ is a *conformal Jacobi morphism* if

$$(1.15') \qquad \{f, g\}_2 \circ \varphi := \frac{1}{a}\{a(f \circ \varphi), a(g \circ \varphi)\}_1,$$

where $f, g \in C^\infty(M_2)$ and a is as in (1.15). If $a = 1$, φ is a *Jacobi morphism*. Furthermore, (1.15′) is equivalent to the fact that the following pairs of tensors of the respective Jacobi structures are φ-related: $(a\Lambda_1, \Lambda_2)$, $(\sharp_{\Lambda_1} da + aE_1, E_2)$. This characterization leads to a corresponding notion of a *conformal Jacobi infinitesimal transformation (vector field)* X on a Jacobi manifold (M, Λ, E) [8]. This is a vector field X which satisfies the conditions

$$(1.17) \qquad L_X\Lambda = b\Lambda, \;\; L_X E = bE + \sharp_\Lambda db \quad (b \in C^\infty(M)).$$

In particular, if $b = 0$ X is a *Jacobi infinitesimal automorphism*.

We finish this section by mentioning the fact that the Jacobi manifolds may be used in the presentation of time dependent Hamiltonian mechanics. Usually, if (q^i, p_i) denote position and momentum coordinates on a phase space T^*N, time dependent mechanics has the phase space $T^*N \times \mathbf{R}$ where $\mathbf{R}$ is the axis of the time t. On this space,

$$(1.18) \qquad \Lambda := \sum_i \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} + \frac{\partial}{\partial t} \wedge (\sum_i p_i \frac{\partial}{\partial p_i}), \;\; E := \frac{\partial}{\partial t}$$

defines a Jacobi structure [9].

Conversely, and following [18], let (M, Λ, E) be a Jacobi manifold with a nowhere vanishing vector E. Then a *time function* is defined as a function $t \in C^\infty(M)$ with no critical points, and such that $Et = 1$. A Jacobi manifold endowed with a time function may be seen as a generalized, time-dependent phase space. Indeed, on such a manifold, if we put

$$(1.19) \qquad \Lambda_0 := \Lambda - (\sharp_\Lambda dt) \wedge E,$$

it follows easily that Λ_0 is a Poisson structure ($[\Lambda_0, \Lambda_0] = 0$), and the vector field

$$(1.20) \qquad X^0_f := \sharp_{\Lambda_0}(df) + E$$

will define the trajectories of the dynamical system with Hamiltonian function f. In the classical case (1.18), we regain the classical trajectories, and the last term of (1.20) gives t the role of time.

3.3 The characteristic foliation

For any function $f \in C^\infty(M)$, where M is a Jacobi manifold with the bracket (1.1), one has a *Hamiltonian vector field* defined by

$$X_f := (pr_{TM}(X^P_{e^\tau f}))_{\tau=0}, \tag{2.1}$$

where $X^P_{e^\tau f}$ is the Poisson-Hamiltonian vector field on $M \times \mathbf{R}$ i.e., with (1.10),

$$X^P_{e^\tau f} = \sharp_\Lambda df + fE - (Ef)\frac{\partial}{\partial \tau}. \tag{2.2}$$

Therefore, we get

$$X_f = \sharp_\Lambda df + fE, \quad \{f, g\} = X_f g - g(Ef). \tag{2.3}$$

Furthermore, one has

Proposition 4 *The Hamiltonian vector fields satisfy*

$$X_{\{f,g\}} = [X_f, X_g], \tag{2.4}$$

$$[X_f, E] = -(Ef)E - \sharp_\Lambda d(Ef), \tag{2.5}$$

$$L_{X_f}\Lambda = -(Ef)\Lambda. \tag{2.6}$$

Proof. Since in view of (1.8) we have $X^P_{e^\tau\{f,g\}} = [X^P_{e^\tau f}, X^P_{e^\tau g}]$, (2.2) implies (2.4), as well as

$$E\{f, g\} = X_f(Eg) - X_g(Ef), \tag{2.7}$$

which is equivalent to the known property $L_E\Lambda = 0$. The relations (2.5), (2.6) follow similarly from $L_{X^P_{e^\tau f}} P = 0$, which is known from Poisson geometry. Q. e. d.

Corollary 2 *Any Hamiltonian vector field is a conformal Jacobi infinitesimal automorphism.*

Now, let $\mathcal{C}(M)$ be the distribution with singularities spanned by the Hamiltonian vector fields with respect to the Jacobi structure of M. (2.4) shows that $\mathcal{C}(M)$ is involutive, and (2.5), (2.6) show that it is invariant by Hamiltonian flows. Hence, by the Sussmann-Frobenius theorem (e.g., [21]) $\mathcal{C}(M)$ is a *foliation with singularities*, called the *characteristic foliation* of the Jacobi structure. If the connected components of M are *characteristic leaves*, the Jacobi structure is called *transitive*. If $\mathcal{C}(M)$ has no singular points, the structure is *regular*.

Formulas (2.3) show that $\mathcal{C}(M)$ is also spanned by $im\,\sharp_\Lambda$ and E, and that the computation of the Jacobi bracket is an operation along the characteristic leaves. In other words, the Jacobi bracket on M induces transitive Jacobi structures on the characteristic leaves. From (1.11)-(1.14) it follows that the locally conformal symplectic, and the contact manifolds have transitive Jacobi structures. One shows that these are the only ones.

Namely, since Λ is skew symmetric, a transitive Jacobi manifold (M^{2n}, Λ, E) is even dimensional iff $E \in im\,\sharp_\Lambda$, and it is odd dimensional iff the previous condition does not hold.

In the even dimensional case, since Λ is non degenerate, we may define the forms $\Omega := \sharp_\Lambda^{-1}\Lambda$, $\epsilon := -2\sharp_\Lambda^{-1}E$. The local coordinate expression of the Schouten-Nijenhuis bracket (e.g., [21]), and (1.2), yield

$$d\Omega = -\sharp_\Lambda[\Lambda, \Lambda] = \epsilon \wedge \Omega. \tag{2.8}$$

Furthermore, since $L_E\Lambda = 0$,

$$L_E\Omega = di(E)\Omega + i(E)d\Omega = -\frac{1}{2}d\epsilon = 0. \tag{2.8'}$$

These results show that M is a locally conformal symplectic manifold with the local symplectic forms $e^{-\sigma_\alpha}\Omega$ where $d\sigma_\alpha = \epsilon$.

For an odd-dimensional transitive Jacobi manifold (M^{2n+1}, Λ, E) we must have $rank\,\Lambda = 2n$, and the equations

$$i(E)\theta = 1, \;\; i(\sharp_\Lambda\xi)\theta = 0 \;\; (\forall \xi \in T^*M) \tag{2.9}$$

provide a well defined 1-form θ on M. Furthermore, we may see that θ satisfies

$$\eta := i(E)d\theta = 0, \;\; \zeta := i(\sharp_\Lambda\xi)d\theta = (i(E)\xi)\theta - \xi. \tag{2.10}$$

Indeed, one can check that

$$i(E)\eta = 0,\ i(E)\zeta = 0,\ i(\sharp_\Lambda\xi)\eta = 0, \quad i(\sharp_\Lambda\sigma)\zeta = i(\sharp_\Lambda\sigma)(\xi(E)\theta - \xi). \tag{2.11}$$

The first three relations are immediate. The last follows, for instance, by using (1.2) and (2.9) in an important equality of Gelfand and Dorfman [7]:

$$\theta(\sharp_\Lambda[\xi,\sigma]) = \theta([\sharp_\Lambda\xi, \sharp_\Lambda\sigma]) + \frac{1}{2}[\Lambda,\Lambda](\theta,\xi,\sigma). \tag{2.12}$$

In (2.11), (2.12), ξ, σ are arbitrary 1-forms and

$$[\xi,\sigma] := L_{\sharp_\Lambda\xi}\sigma - L_{\sharp_\Lambda\sigma}\xi - d(\Lambda(\xi,\sigma)). \tag{2.13}$$

Now, it follows from (2.9), (2.10) that θ is a contact form on M, and the Jacobi structure is that of a contact manifold.

It is interesting to note that the previous results imply an equivalent definition of the Jacobi structures as indicated by

Theorem 11 *A Jacobi structure on M is equivalent with a generalized foliation of M by locally conformal symplectic and/or contact leaves such that $\forall f, g \in C^\infty(M)$ the leafwise brackets $\{f, g\}$ are functions in $C^\infty(M)$.*

Following is an example which shows how the various leaves can appear. Take $M = \mathbf{R}^{3n+1}$, and

$$\Lambda = \sum_{i=1}^{n} u_i \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + (t\frac{\partial}{\partial t}) \wedge (\sum_{j=1}^{n} p_j \frac{\partial}{\partial p_j}), \quad E = t\frac{\partial}{\partial t}, \tag{2.14}$$

where (u_i, q^i, p_i, t) are the coordinates in $\mathbf{R}^{3n+1}$. It is easy to check that (1.2) holds, and one sees that the Jacobi structure (2.14) has both even and odd-dimensional characteristic leaves.

In [5], the authors also make a detailed local analysis of a Jacobi structure. Namely, let x_0 be a point of the Jacobi manifold (M, Λ, E), and S the characteristic leaf through x_0. Take an open neighbourhood U of x_0 in M, and the corresponding homogeneous Poisson structure P of (1.10) on $U \times \mathbf{R}$. Then, one may look at the leaf of P through x_0, and its transversal Poisson structure defined by Weinstein's structure theorem [23]. A thorough analysis shows that if S is even-dimensional, U is conformally Jacobi equivalent with the product of a neighbourhood V of x_0 in S, and a transversal Jacobi structure which vanishes at x_0. If S is odd-dimensional, the original Jacobi structure on U is equivalent to a product of V with a transversal, homogeneous Poisson structure.

3.4 Geometric quantization

(**The Lie algebroid and geometric quantization**)

In Poisson geometry, several authors discovered a bracket of 1-forms which turned out to be of a fundamental importance (e.g., [21]) namely, the bracket (2.13) for a Poisson bivector Λ i.e., in the case where $[\Lambda, \Lambda] = 0$. Further studies showed that the natural framework for many aspects concerning Poisson structures is that of Lie algebroids [19]. The equivalence between Jacobi structures and homogeneous Poisson structures, described in Section 1, leads to a corresponding development for the Jacobi manifolds.

We recall that a Lie algebroid is a vector bundle $p : A \to M$, endowed with a Lie algebra structure on its space ΓA of global cross sections, and with an *anchor morphism* $a : A \to TM$ such that

$$a[s_1, s_2]_A = [as_1, as_2],$$

$$[s_1, fs_2]_A = (as_1)(f)s_2 + f[s_1, s_2]_A \qquad (f \in C^\infty(M)). \tag{3.1}$$

The basic point is that calculus on manifolds (exterior differential calculus, Lie differentiation, Schouten-Nijenhuis bracket, etc.) can be transposed to Lie algebroids by using $[s_1, s_2]_A$ and (as) in the way the usual bracket and vector fields are used in usual calculus (e.g., [12], [19]).

For a Poisson manifold M, the associated Lie algebroid is T^*M with the bracket of 1-forms mentioned at the beginning of this section. For a Jacobi manifold M, an associated Lie algebroid was defined in [10], [13], and the underlying vector bundle is that of the 1-jets of functions on M. We define this structure as follows.

We identify the jet bundle $J^1M := J^1(M, \mathbf{R})$ with the vector bundle $T^*(M \times \mathbf{R})/_{\tau=0}$, where the notation is that of (1.10), by

$$(\xi_0 = f(x^i), \xi_i = \partial_{x_i} f) \leftrightarrow [\xi = e^\tau(\xi_0 d\tau + \sum_i \xi_i dx^i)]_{\tau=0}, \tag{3.2}$$

where $f \in C^\infty(M)$ and (x^i) are local coordinates on M. Accordingly, the sections of J^1M are identified with the 1-forms ξ of the right hand side of (3.2). We call them *homogeneous 1-forms* on $M \times \mathbf{R}$, since they are characterized by

$$L_{\frac{\partial}{\partial\tau}} \xi = \xi. \tag{3.3}$$

Now, we may use the bracket of 1-forms associated with the Poisson structure P of (1.10), and define

$$\{\phi, \psi\} = \{\phi, \psi\}_P, \tag{3.4}$$

where $\phi, \psi \in \Gamma J^1 M$ are identified with the homogeneous 1-forms:

$$\phi = e^\tau(f d\tau + \alpha),\ \psi = e^\tau(g d\tau + \beta), \quad f, g \in C^\infty(M),\ \alpha, \beta \in \Gamma T^*M. \tag{3.5}$$

The result is

$$\{\phi, \psi\} = e^\tau((\{f, g\} - \Lambda(df - \alpha, dg - \beta))d\tau + L_{\sharp_\Lambda \alpha}\beta - L_{\sharp_\Lambda \beta}\alpha - \tag{3.6}$$
$$-d(\Lambda(\alpha, \beta) + f L_E \beta - g L_E \alpha - \alpha(E)\beta + \beta(E)\alpha),$$

and this exactly is the bracket of [10].

Now, the following facts first proven in [10] are immediate

Proposition 5 *i). The mapping* $j^1(f) \mapsto e^\tau(f d\tau + df)$ *is a Lie algebra homomorphism* $(C^\infty(M), \{\ ,\ \}) \to (J^1 M, \{\ ,\ \})$. *ii).* $J^1 M$, *with the bracket (3.6), and with the morphism* $\rho := pr_{TM} \circ \sharp_P/_{\tau=0}$ *as the anchor, is a Lie algebroid.*

Proof. i) follows from (3.6). ii) follows from the fact that $\sharp_P : T^*(M \times \mathbf{R}) \to T(M \times \mathbf{R})$ is the anchor of the Lie algebroid of P, and the use of the formula

$$\sharp_P(e^\tau(f d\tau + \alpha)) = \sharp_\Lambda \alpha + f E - \alpha(E)\frac{\partial}{\partial \tau} \tag{3.7}$$

in the computation of the relevant brackets. Q. e. d.

In what follows we will describe some interesting structures connected with homogeneity and the Lie algebroid $J^1 M$.

¿From (3.7) we may see that $\sharp_P/_{\tau=0}$ determines the Jacobi structure (Λ, E). This suggests an interpretation of the Jacobi structures as Hamiltonian structures on a *Gelfand-Dorfman complex* [6]. Namely, let $\chi_0(M)$ be the Lie algebra of the vector fields of the form

$$X + f\frac{\partial}{\partial \tau} \quad (X \in \Gamma TM,\ f \in C^\infty(M)) \tag{3.8}$$

on $M \times \mathbf{R}$. Let us extend the homogeneity condition (3.3) to k-forms, and denote by $\wedge^k_0(M)$ the space of homogeneous k-forms on $M \times \mathbf{R}$,

$$\wedge^k_0(M) = \{\lambda = e^\tau(\alpha + \beta \wedge d\tau)\} \quad (\alpha \in \wedge^k M,\ \beta \in \wedge^{k-1} M). \tag{3.9}$$

Then, $(\oplus_k \wedge_0^k (M), d, i(X + f(\partial/\partial\tau))$, with the usual operators d, i, is a chain complex which satisfies the Gelfand-Dorfman conditions of [6]. (But, not an exterior algebra!). It follows easily that a Jacobi structure on M is equivalent with a Hamiltonian structure on this Gelfand-Dorfman complex. Indeed, such a Hamiltonian structure is equivalent with a homogeneous Poisson structure of the type (1.10).

A look at (1.10) and (3.8) shows that the notion of a homogeneous k-vector field Q should be characterized by the condition

$$L_{\frac{\partial}{\partial\tau}} Q = -(k-1)Q \tag{3.10}$$

or, equivalently,

$$Q = e^{-(k-1)\tau}(Q_1 + \frac{\partial}{\partial\tau} \wedge Q_2), \;\; (Q_1 \in \mathcal{V}^k(M), \; Q_2 \in \mathcal{V}^{k-1}(M)), \tag{3.11}$$

where $\mathcal{V}^k(M)$ is the space of the k-vector fields on M. We will indeed use this definition of homogeneity. It is compatible with the Schouten-Nijenhuis bracket, and if we denote the spaces of homogeneous k-vector fields on $M \times \mathbf{R}$ by $\mathcal{V}_0^k(M)$, $\mathcal{V}_0 = \oplus_k \mathcal{V}_0^k$, endowed with the Lichnerowicz differential $\sigma_P := -[P, \;]$ (e.g., [21]) is a cochain complex. It follows that

$$\sigma_P(Q) = -e^{-k\tau}(-[\Lambda, Q_1] + \Lambda \wedge Q_2 + (k-1)E \wedge Q_1 + \tag{3.12}$$

$$+\frac{\partial}{\partial\tau} \wedge ([\Lambda, Q_2] - (k-2)E \wedge Q_2 + [E, Q_1])).$$

The cohomology spaces of this complex were studied by Lichnerowicz [17] where they were obtained as the 1-*differentiable Chevalley-Eilenberg cohomology*. $H^1(\mathcal{V}_0, \sigma_P)$ is the quotient space of the space of conformal Jacobi infinitesimal transformations by the space of Hamiltonian vector fields.

On the other hand, there are also cohomology spaces of the Lie algebroid J^1M as of any Lie algebroid [12]. The cochains which define these spaces are the cross sections of the exterior powers $\wedge^k(J^1M)^*$ of the dual vector bundle of J^1M, and the coboundary is the exterior differential associated with the anchor and bracket of the Lie algebroid structure

$$(\sigma A)(s_0, ..., s_k) = \sum_{i=0}^{k} (-1)^i (\rho s_i) A(s_0, ...\hat{s}_i, ..., s_k) + \tag{3.13}$$

$$+\sum_{i<j=1}^{k}(-1)^{i+j}A(\{s_i,s_j\},s_0,...,\hat{s}_i,...,\hat{s}_j,...s_k).$$

In (3.13), we may see the arguments s_i as homogeneous 1-forms on $M\times\mathbf{R}$. Accordingly, A may be seen as follows:

$$A=[e^{-k\tau}(A_1+\frac{\partial}{\partial\tau}\wedge A_2)]_{\tau=0},\quad A_1\in\mathcal{V}^k(M),\ A_2\in\mathcal{V}^{k-1}(M), \tag{3.14}$$

and σA, where A is given by (3.14), may be interpreted as the value of the Lichnerowicz differential $\sigma(e^{-k\tau}A)=-[P,e^{-k\tau}A]$ at $\tau=0$. A straightforward computation yields

$$\sigma A=(-[\Lambda,A_1]+kE\wedge A_1+\Lambda\wedge A_2)+ \tag{3.15}$$

$$+\frac{\partial}{\partial\tau}\wedge([\Lambda,A_2]-(k-1)E\wedge A_2+[E,A_1]).$$

Formula (3.15), and the corresponding cohomology spaces were introduced in [14] under the name of *Lichnerowicz-Jacobi (JL) cohomology*. One important point is that there exist mappings $\sharp:\wedge^kT^*M\to\wedge^k(J^1M)^*$ which, using the above notation, are defined by

$$(\sharp\lambda)(s_1,...,s_k)=(-1)^k\lambda(\rho s_1,...,\rho s_k), \tag{3.16}$$

and these mappings induce homomorphisms $\sharp$ of the de Rham cohomology of M into the JL-cohomology.

In [14] the JL-cohomology was used in the study of a generalization of geometric quantization to Jacobi manifolds. Essentially, this is the procedure which sends every $f\in C^\infty(M)$ to the operator

$$\hat{f}(u):=\nabla_{X_f}u+2\pi\sqrt{-1}fu\quad(u\in\Gamma(K)), \tag{3.17}$$

where K is a Hermitian line bundle over M, and ∇ is a Hermitian partial connection on K, defined along the characteristic leaves of the Jacobi structure. Furthermore, one must ask the *Dirac principle*

$$\widehat{\{f,g\}}=\hat{f}\circ\hat{g}-\hat{g}\circ\hat{f} \tag{3.18}$$

to hold, and this happens iff

$$\frac{\sqrt{-1}}{2\pi}Curvature_{\nabla}(X_f,X_g)(u)=\Lambda(df,dg)u. \tag{3.19}$$

It is not quite clear how to handle the partial connections above. In [21], [14] this notion is replaced by that of a *contravariant derivative* which is the analog of a covariant derivative, with the Lie algebroid J^1M used instead of TM. That is, we have *derivatives* $D_s u$ for $s \in J^1M$ and $u \in \Gamma(K)$ which are $\mathbf{R}$-bilinear, and satisfy

$$D_{fs}u = fD_s u, \quad D_s(fu) = fD_s u + ((\rho s)f)u \quad (f \in C^\infty(M)). \tag{3.20}$$

Then one takes $\nabla_{X_f} u := D_{j^1 f} u$.

Now, the left hand side of (3.19) may be seen as a bivector of the form (3.14) with a vanishing second term. As such, it is σ-closed and it defines a JL-cohomology class which is the $\sharp$-image of the first Chern class of the line bundle K. Hence, by a classical theorem of Weil-Kobayashi (e.g., [21]) the bundle K and the required contravariant derivative exist iff the previous JL-class is the image of an integral cohomology class.

But, the problem is a bit more complicated since, in fact, the JL-class of Λ vanishes:

$$\Lambda = \sigma(\frac{\partial}{\partial \tau}).$$

Hence, a solution seems to exist always. Namely, the trivial bundle K with the contravariant derivative

$$D_{[e^\tau(\alpha + f d\tau)]_{\tau=0}}(1) = -2\pi\sqrt{-1}.$$

Obviously, this is not what we want since we get the trivial result $\hat{f} = X_f$.

The conclusion is that in order to obtain a good geometric quantization of a Jacobi bracket we have to find a closed 2-form Ω on M which represents an integral cohomology class, a vector field A on M and a function $f \in C^\infty(M)$ such that

$$\sharp\Omega = \sigma(A + \frac{\partial}{\partial \tau}), \tag{3.21}$$

and such that the corresponding operators (3.17) are not trivial i.e., $\hat{f} \neq X_f$. Concrete conditions of non triviality are given in [14].

3.5 Integration of Jacobi structures

In this section we briefly describe the integration theory of the Lie algebroid of a Jacobi manifold M, and of the Jacobi algebra $C^\infty(M)$ itself, developed by

Y. Kerbrat and Z. Souici-Benhammadi [10], C. Albert [1], and P. Dazord [4]. The first step of the theory was inspired by Weinstein's theory of symplectic groupoids [24] [2], while the latter notion is replaced by that of a *contact groupoid*. Then, the second step, due to Dazord [4], consists of integrating the Lie algebra $C^\infty(M)$ to a diffeological Lie group in the sense of J. M. Souriau [20], if the Lie algebroid J^1M integrates to a global contact Lie groupoid.

Recall that a *groupoid* is the set of morphisms of a small category Γ where each morphism has an inverse. The groupoid operation is composition of morphisms, usually written in the opposite order, and it is defined on a subset $\Gamma^{(2)} \subseteq \Gamma \times \Gamma$. The set of unit morphisms is a subset $\Gamma_0 \subseteq \Gamma$, and one has two projections $\alpha : \Gamma \to \Gamma_0$, $\beta : \Gamma \to \Gamma_0$ which send a morphism to its right and left unit, respectively. If everything is differentiable (Γ is not necessarily Hausdorff but Γ_0 is), if α, β are submersions, and the inverse mapping is a diffeomorphism, Γ with all the previously mentioned structure is a *Lie groupoid*. On the other hand, if we have a similar structure of manifolds and mappings, but composition is defined only on an open neighbourhood of Γ_0, we get a *local Lie groupoid*. Every (local) Lie groupoid has a well defined corresponding Lie algebroid over Γ_0, with the underlying vector bundle isomorphic to the normal bundle of Γ_0 in Γ, and the bracket induced by the bracket of "left invariant vector fields" on Γ [2].

Furthermore, a Lie groupoid Γ with a symplectic form σ such that the graph of the composition of Γ is a Lagrangian submanifold of $\Gamma \times \Gamma \times (-\Gamma)$ is a *symplectic groupoid*. Then Γ_0 is a Lagrangian submanifold, endowed with a Poisson structure whose Lie algebroid is that of Γ. Weinstein's integration theorem tells that every Poisson manifold can be obtained in this way from a Hausdorff, local symplectic groupoid [24], [2].

There are various versions of the notion of a *contact groupoid* [15, 16, 10][4]. Here, we choose the one given in [10]. Namely, a Lie groupoid Γ is a *contact groupoid* if it is endowed with a contact form θ, and a real function f such that the following relation holds on $\Gamma^{(2)}$

$$m^*\theta = (\beta^* f)(\alpha^*\theta) + \beta^*\theta \tag{4.1}$$

(m is composition in Γ). One also has a similar notion of a *local contact groupoid*. The following result is stated in [10]: the vector and bivector fields defined on the unit manifold Γ_0 of a contact groupoid Γ by

$$E_0 = -\beta_*(\sharp_\Lambda df), \quad \Lambda_0(x_0) = \alpha_*(\Lambda(x_0)), \tag{4.2}$$

where $x_0 \in \Gamma_0$, and Λ is the bivector of the Jacobi structure of the contact structure θ (see Section 1), exist and define a Jacobi structure on Γ_0. Furthermore, the Lie algebroid of $(\Gamma_0, E_0, \Lambda_0)$ is isomorphic to the Lie algebroid of the Lie groupoid Γ. A similar but more general result is proven in [4].

The converse problem is integrability i.e., given a Jacobi manifold $(\Gamma_0, \Lambda_0, E_0)$, find a contact groupoid with the Jacobi manifold of units equivalent to the given manifold. This is not always possible but one can prove [10], [4]

Theorem 12 *Let* $(\Gamma_0, \Lambda_0, E_0)$ *be a Jacobi manifold. Then, there exists a local contact groupoid which has the given manifold as its Jacobi manifold of units.*

The proof of Dazord [4] uses the following strategy. Consider the homogeneous Poisson manifold $(S := \Gamma_0 \times \mathbf{R}, P_0, Z_0)$, where P_0 and Z_0 are defined as in (1.10). Following Weinstein's theorem, let $(\tilde{\Gamma}, \sigma)$ be the local symplectic groupoid with the Poisson manifold of units S. Since S is homogeneous, the flow $exp(tZ_0)$ is a flow of Poisson mappings $(S, P_0) \to (S, e^{-t}P_0)$, and, by results on symplectic groupoids [2], it can be lifted to a flow of symplectic mappings $(\tilde{\Gamma}, \sigma) \to (\tilde{\Gamma}, e^t\sigma)$. Then, if Γ is the quotient of $\tilde{\Gamma}$ by the lifted flow, with the natural projection $\pi : \tilde{\Gamma} \to \Gamma$, one obtains a structure of a local Lie groupoid on Γ with the composition $xy = \pi(\tilde{x}\tilde{y})$ defined whenever there are composable elements $\tilde{x}, \tilde{y} \in \tilde{\Gamma}$ such that $x = \pi\tilde{x}, y = \pi\tilde{y}$. Furthermore, if the lifted flow has the tangent vector field Z, one must have $L_Z\sigma = \sigma$, whence $\sigma = d\theta$ for the 1-form $\theta = i(Z)\sigma$ on $\tilde{\Gamma}$, and θ descends to a structure of local contact groupoid on Γ, with the Jacobi manifold of units Γ_0.

In particular, if $\tilde{\Gamma}$ is a global symplectic groupoid, Γ is a global contact groupoid. In what follows, we assume that we are in this case precisely.

The previous results tell us how to "integrate" the Lie algebroid of a Jacobi manifold. In Dazord's paper [4] this is used in order to integrate the Jacobi-Lie algebra of functions $C^\infty(\Gamma_0)$ to a certain kind of generalized Lie group.

Namely, [20], [3], [4] a *diffeological space* is a set S endowed with a *diffeology* $\mathcal{D} = \cup_{p=1}^{\infty}\mathcal{D}_p(S)$, where $\mathcal{D}_p(S)$ consists of mappings from open sets of $\mathbf{R}^p$ to S, called *p-slices*, such that the following axioms hold:
1) all constant mappings belong to the diffeology;
2) the composition of a p-slice with a C^∞ mapping from an open set of $\mathbf{R}^q$ to an open set of $\mathbf{R}^p$ is a q-slice;
3) if $U = \cup_i U_i$, where U_i are open subsets of $\mathbf{R}^p$, then $f : U \to S$ is a p-slice iff $f/_{U_i}$ are p-slices.

Differentiable mappings, the notion of tangent space, and the calculus of differential forms can be extended to diffeological spaces. If differentiable manifolds are replaced by diffeological spaces in the definition of a Lie group, one gets the notion of a *diffeological Lie group*, and this is the generalization announced above. In analogy with the usual Lie groups, there exists a Lie algebra of a diffeological Lie group under some restrictive conditions for which we refer the reader to [3], [4]. We suppose that these conditions are always satisfied hereafter, whenever we speak of a diffeological Lie group. (In particular, one must require that the flows of invariant vector fields are 1-slices.)

If we come back to a Lie groupoid $(\Gamma, \alpha, \beta, \Gamma_0)$, and define a *bisection* as being a submanifold Σ of Γ such that $\alpha/_{\Sigma}, \beta/_{\Sigma}$ are diffeomorphisms, it turns out that the set of bisections forms a group with respect to the composition induced by that of Γ. Furthermore, if we denote by G_Γ the subgroup of bisections whose actions on Γ_0 by composition in Γ have compact support, G_Γ has a natural diffeology given by the p-slices $s : U \to G_\Gamma$ (U open in $\mathbf{R}^p$) where $s(u)(x_0)$ ($u \in U, x_0 \in \Gamma_0$) is C^∞ in both variables, and $s(u)$ acts with compact support in Γ_0 [4]. A corresponding Lie algebra structure is obtained on the space $\Gamma_c(E)$ of the compactly supported sections of the Lie algebroid of the groupoid.

Now, we can finally formulate the following "Lie's third theorem" due to Dazord [4]

Theorem 13 *Let (M, Λ, E) be a Jacobi manifold such that its Lie algebroid J^1M is integrable to a global contact Lie groupoid Γ with the Jacobi manifold of units M. Then the subgroup G^0_Γ of G_Γ consisting of the Legendrian bisections of Γ is a diffeological Lie group with a corresponding Lie algebra isomorphic to the Jacobi-Lie algebra $C^\infty(M)$.*

Remember that a Legendrian submanifold of a contact manifold (Γ, θ) is an integral submanifold of the equation $\theta = 0$ which has the maximal possible dimension.

Bibliography

[1] C. Albert, Un théorème de réalisation des variétés de Jacobi. C. R. Acad. Sci. Paris, Sér. I, 317 (1993), 77-80.

[2] C. Albert and P. Dazord, Théorie des groupoïdes symplectiques, I. Publ. Dept. Math. Lyon, 4/B (1988), 53-103, II - Ibid., Nouvelle Série, 1990, 27-99.

[3] P. Dazord, Lie groups and algebras in infinite dimensions: a new approach. In: Symplectic Geometry and Quantization, Contemporary Math., A. M. S., 179 (1994), 17-44.

[4] P. Dazord, Sur l'intégration des algèbres de Lie locales et la préquantification. Bull. Sci. math., 121 (1997), 423-462.

[5] P. Dazord, A. Lichnerowicz and Ch.-M. Marle, Structures locales des variétés de Jacobi. J. Math. pures et appl., 70 (1991), 101-152.

[6] I. Dorfman, Dirac structures and integrability of nonlinear evolution equations. J. Wiley & Sons, New York, 1993.

[7] I. M. Gelfand and I. Ya. Dorfman, The Schouten bracket and Hamiltonian operators. Funkt. Anal. Prilozhen. 14(3) (1980), 71-74.

[8] F. Guedira and A. Lichnerowicz, Géométrie des algèbres de Lie de Kirillov. J. Math. pures et appl., 63 (1984), 407-484.

[9] R. Ibáñez, M. de León, J. C. Marrero and E. Padrón, Nambu-Jacobi and generalized Jacobi manifolds. Preprint, 1997.

[10] Y. Kerbrat and Z. Souici-Benhammadi, Variétés de Jacobi et groupoïdes de contact. C. R. Acad. Sci. Paris, Sér. I, 317 (1993), 81-86.

[11] A. Kirillov, Local Lie algebras. Russian Math. Surveys, 31 (1976), 55-75.

[12] Y. Kosmann-Schwarzbach and F. Magri, Poisson-Nijenhuis structures. Ann. Inst. Henri Poincaré, 53 (1990), 35-81.

[13] M. de León, J. C. Marrero and E. Padrón, H-Chevalley-Eilenberg cohomology of a Jacobi manifold and Jacobi-Chern class. C. R. Acad. Sci. Paris, Sér. I, 325 (1997), 405-410.

[14] M. de León, J. C. Marrero and E. Padrón, On the geometric quantization of Jacobi manifolds. J. Math. Phys., 38 (1997), 6185-6213.

[15] P. Libermann, On contact and symplectic groupoids. In: Diff. Geom. and its applications. Proc. Conf. Opava 1992 (O. Kowalski and D. Krupka, eds.), Silesian University, Opava 1993, 29-45.

[16] P. Libermann, On contact groupoids and their symplectification. In: Analysis and Geometry in Foliated Manifolds. Proc. VII International Colloq. on Differential Geometry, Santiago de Compostela 1994. World Scientific, Singapore, 1995.

[17] A. Lichnerowicz, Les variétés de Jacobi et leurs algèbres de Lie associées. J. Math. pures et appl., 57 (1978), 453-488.

[18] A. Lichnerowicz, La géométrie des transformations canoniques. Bull. Soc. Math. de Belgique, 31 (1979), 105-135.

[19] K. MacKenzie and P. Xu, Lie bialgebroids and Poisson groupoids. Duke Math. J., 73(2) (1994), 415-452.

[20] J. M. Souriau, Groupes différentiels et Physique Mathématique. In: Feuilletages et Quantification Géometrique (P. Dazord and N. Desolneux-Moulis, eds.), Collection Travaux en Cours, Hermann, Paris, 1984, 73-79.

[21] I. Vaisman, Lectures on the geometry of Poisson manifolds. Progress in Math. Series, 118, Birkhäuser, Basel, 1994.

[22] I. Vaisman, Locally conformal symplectic manifolds. International J. of Math. and Math. Sci., 8 (1985), 521-536.

[23] A. Weinstein, The local structure of Poisson manifolds. J. Differential geometry, 18 (1983), 523-557.

[24] A. Weinstein, Symplectic groupoids and Poisson manifolds. Bull. American Math. Soc., 16 (1987), 101-103.