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An overview on inverse first-passage-time problems for one-dimensional diffusion processes

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Abstract. We review a number of inverse first-passage-time problems for temporally homogeneous one-dimensional diffusion processes; we also consider diffusions with jumps and reflected diffusions. Several explicit examples are reported.

1. Introduction

First-passage time (FPT) problems for stochastic processes are an important issue with a variety of applications in several areas of sciences including Physics, Biology, Engineering and Mathematical Finance (see e.g.[61], [64]). In particular, over the years a number of papers was devoted to find the FPT density of a diffusion process through a moving boundary (see e.g. [9], [12], [13], [14], [15], [16], [25], [26], [29], [52], [67] and references therein), though few analytical results are known in closed form (they are available particularly for Brownian motion, and only for some special boundaries).

First, we focus on a temporally homogeneous, one-dimensional diffusion process $X(t)$ defined in an interval $I = (r_1, r_2)(-\infty \leq r_1 < r_2 \leq +\infty)$, and characterized by drift $b(x)$ and infinitesimal variance $\sigma^2(x)$, that is, $X(t)$ is the solution of the stochastic differential equation (SDE):

$$dX(t) = b(X(t))dt + \sigma(X(t))dB_t \quad , \quad X(0) = \eta \in I \tag{1.1}$$

where B_t is a standard Brownian motion (BM) and the functions b and σ are regular enough.

Subsequently, we admit that jumps may occur at random instants, so $X(t)$ becomes a jump-diffusion process (see e.g. [15]); such a kind of process is e.g. invoked for the description of stochastic neuronal activity (see [33]), and of random assets in Mathematical Finance (see [20]). Specifically, we consider diffusions with random jumps from a boundary (see [4], [8] and references therein). These have various applications in Queueing theory (see e.g. [28]), as well as in Mathematical Finance and in Biology (see [8]). Finally, we deal with reflected diffusion processes, characterized by the fact that they are forced to stay between two (reflecting) boundaries (see [3], [34]). Also reflected diffusions play an important role in a variety of applications. As for Queueing theory, they arise as heavy-traffic approximations of

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queueing systems (see e.g. [1], [2], [34] [71], [76], [77]); for applications in Finance and Insurance, see e.g. [19], [21], [27], [42], [59], [73], [74]; for applications in Biology, see e.g. [63].

If $S : [0, +\infty) \longrightarrow I$ is a continuous function with $S(0) \geq \eta$, let

$$\tau_S(\eta) = \inf\{t > 0 \, : \, X(t) \geq S(t) | X(0) = \eta\} \tag{1.2}$$

be the FPT of $X(t)$ over the boundary $S(t)$, with the condition that $X(0) = \eta$, and let $f_S(t|\eta)$ denote the FPT density, i.e. the probability density function (p.d.f.) of $\tau_S(\eta)$.

In many FPT problems, the initial position $\eta = x$ is supposed to be deterministic and fixed, so the direct problem consists in finding $f(t|\eta) = f_S(t|\eta)$, when the barrier S is assigned; on the contrary, various kinds of inverse FPT (IFPT) problems can be considered. In this paper, we examine two types of IFPT problems; in the first one (IFPT1), we focus on determining the boundary shape $S = S_f$, when the FPT density f and the starting point η are assigned; in the second one (IFPT2) we have to do with a FPT below a boundary: we assume that the boundary is known, while the initial position η is random. Thus, let $S(t)$ be an assigned continuous boundary with $X(0) = \eta$ independent of B_t and $P(\eta \geq S(0)) = 1$, then for a given distribution F, the IFPT2 problem consists in finding the density g of η (if it exists) for which it results $P(\tau_S \leq t) = F(t)$, where $\tau_S = \inf\{t > 0 \, : \, X(t) \leq S(t)\}$.

Similarly, the so called double-barrier IFPT2 problem consists of the following. Suppose that the initial position η is a random variable, independent of $X(t)$, let $a < b$ such that $P(a < \eta < b) = 1$, and denote by $\tau_{(a,b)}$ the first-exit time of $X(t)$ from the interval (a, b) i.e. $\tau_{(a,b)} = \inf\{t > 0 \, : \, X(t) \notin (a, b)\}$. Then, for a given distribution F, it is requested to find the density g of η (if it exists) for which it results $P(\tau_{(a,b)} \leq t) = F(t)$.

Notice that various properties of BM (e.g. scaling, time-inversion, independent increments) are not available for a diffusion process. Therefore, some success in the solution of IFPT problems can be achieved in the special case when the process $X(t)$ can be reduced to BM, via a variable change. A different approach, consisting in reducing the Kolmogorov's equation for a diffusion to the backward equation for BM was considered by Ricciardi in [62].

As far as the direct FPT problem is concerned, the value that the FPT density f_S takes at $t = 0$, in terms of the boundary S, is particularly interesting in various numerical methods found in the literature to estimate f_S, when S is given (see [11], [56], [58]).

The IFPT1 problem has important applications e.g. in Biology, when considering diffusion models for neural activity ([32], [66]). It was firstly formulated by A. Shiryaev in 1976 in the case when the assigned FPT density is exponential and the process X is BM. It is well-known ([30]) that to find the solution of the IFPT1 problem, one has to solve a non-linear integral equation, which is rather hard from the analytical point of view. The question of the existence and uniqueness of the solution to the IFPT1 problem is, to our knowledge, still open, even for BM (see [75]).

The IFPT2 problem has interesting applications in Mathematical Finance, in particular in credit risk modeling, where the FPT represents a default event of an obligor (see [38]) and in diffusion models for neural activity ([44]). Also for the

IFPT2 problem, the question of the existence of the solution is not a trivial matter (see [6], [7]).

The paper is organized as follows: Section 2 deals with the IFPT1 problem for simple diffusions, namely, without jumps; the IFPT2 problem is studied, for simple diffusions in Section 3, and for diffusions with jumps from a boundary in Section 4; finally, Section 5 is devoted to the IFPT2 problem for reflected diffusions. In all cases of IFPT problems, several explicit examples are reported.

2. The IFPT1 problem

Let $X(t)$ be the solution of the SDE (1.1), whose drift $b(x)$ and infinitesimal variance $\sigma^2(x)$ satisfy the following conditions:

- A1 $b, \sigma : I \to \mathbb{R}$ are continuous functions and a constant $K > 0$ exists, such that $|b(x) - b(y)| \leq K|x - y|$ and $b^2(x) + \sigma^2(x) \leq K(1 + x^2)$ for every $x, y \in I$;
- A2 σ is a non-negative, bounded function and it is differentiable for every x belonging to the interior of I. Moreover, there exists a strictly increasing function $H : \mathbb{R}^+ \to \mathbb{R}$ such that $H(0) = 0$, $\int_{0^+} H^{-2}(s)\, ds = +\infty$ and $|\sigma(x) - \sigma(y)| \leq H(|x - y|)$ for every $x, y \in I$.

The conditions A1 and A2 ensure that there exists a unique non-explosive solution of (1.1) with fixed initial point, which is a temporally homogeneous Markov process (see e.g. [31], [36]); A2 holds, for instance, if σ is Lipschitz-continuous, or Hölder-continuous of order $\geq 1/2$.

Suppose that the initial position $\eta = x$ is assigned; for a continuous boundary $S(t)$ with $X(0) = x \leq S(0)$, let $\tau_S(x) = \inf\{t > 0 : X(t) \geq S(t) | X(0) = x\}$ the FPT of $X(t)$ over the boundary $S(t)$, with the condition that $X(0) = x$, and let $f(t|x) = f_S(t|x)$ denote the FPT density, i.e. the probability density function (p.d.f.) of $\tau_S(x)$. For a given FPT density f and a starting point $X(t)$, the IFPT1 problem for X consists in determining the boundary shape S, such that $(d/dt)P(\tau_S(x) \leq t) = f(t)$. The IFPT1 problem for diffusions was studied deeply in [11] (see also [22], [69], [70], [75] for algorithms to get approximate solutions); see also [57].

As it is well-known (see e.g. [61]), the transition p.d.f. of the process $X(t)$:

$$p(x, t|y, s) = \frac{\partial}{\partial x} P\{X(t) \leq x | X(s) = y\} \quad , \quad 0 \leq s < t$$

verifies the Kolmogorov's equation:

$$\frac{\partial p}{\partial s} + b(y)\frac{\partial p}{\partial y} + \frac{1}{2}\sigma^2(y)\frac{\partial^2 p}{\partial y^2} = 0 \, ,$$

with the initial condition $\lim_{t \to 0^+} p(x, t|0, 0) = \delta(x)$. For a given boundary $S(t)$ and any $z \geq S(t)$, the FPT density $f_S(t)$ is the solution of the Fortet equation (see [30]):

$$p(z, t|0, 0) = \int_0^t f_S(u) p(z, t|S(u), u)\, du \, , \tag{2.1}$$

that is a Volterra integral equation of the first type. By integrating this equation on $[z, +\infty)$, one gets, for $z \geq S(t)$:

$$P(X(t) \geq z) = \int_0^t f_S(u) P(X(t) \geq z | X(u) = S(u))\, du \, . \tag{2.2}$$

In the IFPT1 problem, (2.1) and (2.2) have to be considered as equations in the unknown $S = S_f$, while $f = f_S$ and $x = X(0)$ are known. At least in principle, they can be solved to find S, i.e. the solution of the IFPT1 problem; however the analytical solution is hard to be obtained.

2.1. **The case of BM.** When X is BM the equation (2.2), written for $z = S(t)$, becomes:

$$\Psi\left(\frac{S(t)}{\sqrt{t}}\right) = \int_0^t f_S(u)\Psi\left(\frac{S(t)-S(u)}{\sqrt{t-u}}\right) du \ , \tag{2.3}$$

where $\Psi(x) = 1 - \Phi(x)$ and $\Phi(x) = \int_{-\infty}^x \exp(-t^2/2)\, dt/\sqrt{2\pi}$. Peskir et al. [56] have proposed two constructive methods to solve equation (2.3), that is to determine the shape of S_f when f is known. Successively, Sacerdote and Zucca [66] have applied these methods to Ornstein-Uhlenbeck (OU) process, by using a suitable space-time transformation which changes the OU process into a BM. The first (Montecarlo) method proposed in [56] is based on the approximation of the boundary by a piecewise linear function and therefore it makes use of well-known formulae for the first-crossing density of BM over such a type of boundary. Of course, this method is not applicable to a diffusion process, because analytical results about the crossing probabilities are generally unknown. The second method developed in [56] makes use of a numerical procedure; it consists in finding an approximation of the boundary $S = S_f$ at the points $t_i = ih$, for $i = 1, \cdots, N$, where $h = t/N$ is the time discretization step, and $t > 0$ is given and fixed. This is obtained by a first-order approximation of the equation (2.3), in the following way:

$$\Psi\left(\frac{S(t_i)}{\sqrt{t_i}}\right) = h\sum_{j=1}^{i} f(t_j)\Psi\left(\frac{S(t_i)-S(t_j)}{\sqrt{t_i-t_j}}\right) = \tag{2.4}$$

$$= h\left[\sum_{j=1}^{i-1} f(t_j)\Psi\left(\frac{S(t_i)-S(t_j)}{\sqrt{t_i-t_j}}\right) + \frac{1}{2}f(t_i)\right] \quad , \quad i = 1, \cdots, N \ .$$

Equations (2.4) form a non-linear system of N equations in the unknowns $S(t_1)$, $S(t_2), \ldots, S(t_N)$, whose solutions give, for a fixed f, the desired approximations of the boundary S at times t_i, $i = 1, \cdots, N$. These have been numerically found in [11] and [56]. Since the i-th equation of this system for $i > 1$ makes use only of the values $S(t_1), \cdots, S(t_{i-1})$ already found in the preceding step, the solution $S(t_i)$ of the i-th equation can be found recursively, by using the iterative secant method for solving non-linear equations (see [56]). Some care has to be used in the choice of the value $S(t_1)$, which starts the algorithm, in the dependence of f. A more precise estimation of the boundary S at the nodes t_i could be achieved by using a higher order approximation of (2.3), obtained e.g. calculating the integral in each interval (t_i, t_{i+1}) by the Simpson's rule which provides an error of order h^5; in this way one would obtain, in place of (2.4), a more complicated non-linear system in the unknowns $S(t_i)$. Actually, for certain FPT densities, an approximation of order greater than 1 is sufficient to obtain a sufficiently accurate estimation of the boundary which solves the IFPT1 problem.

In [75] the two methods described above for approximating the solution S_f of the IFPT1 problem for BM were studied in deep and their errors were found.

In [69] another approximation for the IFPT1 problem for BM was proposed; it is similar in spirit and method to the tangent approximation for the original FPT problem. The tangent approximation consists in approximating the boundary S by a straight line as in [24], [45], [72]; several refinements have been suggested, including the hazard-rate tangent approximation of Roberts and Shortland [65]. The method developed in [69] to solve approximately the IFPT1 problem can be viewed as a natural inverse of the tangent approximation; the authors showed that the technique is quite accurate in many cases, with errors of just a few per cent, while in certain cases of strongly bimodal or oscillating FPT densities f, the approximation performs poorly.

Let us denote by $g_\lambda(t)$ the exponential density with parameter λ, that is

$$g_\lambda(t) = \lambda e^{-\lambda t}\mathbf{1}_{[0,\infty)}(t) \ .$$

For $f(t) = g_1(t)$, the solution S_1^B of the IFPT1 problem for B_t, at discrete times $t_i = ih$, with time discretization step $h = 0.01$ was re-calculated in [11], by solving (2.4) (this was already done in [56]); its graph is shown in the Figure 1.

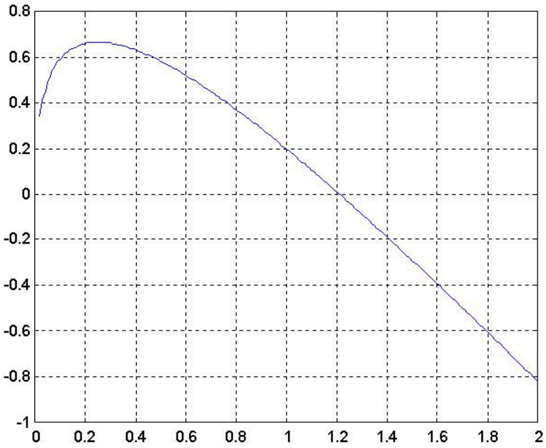

FIGURE 1. Approximate solution to the IFPT1 problem for B_t and the exponential density with mean 1.

As easily seen by using the scaling property of BM, the solution of the IFPT1 problem for B_t and g_λ is given by

$$S_\lambda^B(t) = \frac{S_1^B(\lambda t)}{\sqrt{\lambda}} \ . \tag{2.5}$$

Now, let us consider the case when X is the two-sided BM, i.e. $X(t) = |B_t|$; the transition probability density of $X(t)$ is given by (see e.g. [40], p. 123):

$$p(x,t|y,s) = \frac{1}{\sqrt{2\pi(t-s)}}\left[e^{-(x-y)^2/2(t-s)} + e^{-(x+y)^2/2(t-s)}\right] \ .$$

Then, for a given boundary S and $t > u \geq 0$, we get:

$$P\left[|B_t| \geq z | X(u) = S(u)\right] = \Psi\left(\frac{z - S(u)}{\sqrt{t-u}}\right) + \Psi\left(\frac{z + S(u)}{\sqrt{t-u}}\right) ,$$

where $\Psi(x) = \int_x^{+\infty} (1/\sqrt{2\pi}) e^{-s^2/2}\, ds$. Therefore, in this case equation (2.2), written for $z = S(t)$, becomes:

$$2\Psi\left(\frac{S(t)}{\sqrt{t}}\right) = \int_0^t f_S(u)\left[\Psi\left(\frac{S(t) - S(u)}{\sqrt{t-u}}\right) + \Psi\left(\frac{S(t) + S(u)}{\sqrt{t-u}}\right)\right] du .$$

A first-order approximation of this equation, analogous to that considered in (2.4) is:

$$2\Psi\left(\frac{S(t_i)}{\sqrt{t_i}}\right) = h \sum_{j=1}^{i-1} f(t_j)\left[\Psi\left(\frac{S(t_i) - S(t_j)}{\sqrt{t_i - t_j}}\right) + \Psi\left(\frac{S(t_i) + S(t_j)}{\sqrt{t_i - t_j}}\right)\right] +$$

$$+ h f(t_i)\left(\frac{1}{2} + \alpha\right) ,$$

where $t_i = ih$, $i = 1, \cdot, N$ and

$$\alpha = \begin{cases} 0 & \text{if } S(t_i) > 0 \\ \frac{1}{2} & \text{if } S(t_i) = 0 . \end{cases}$$

For $f(t) = g_1(t)$ and $h = 0.01$ the above equations was solved in [11] in the unknowns $S(t_i)$ by the secant method, thus finding the approximate solution, say $S_1^{|B|}$, of the IFPT1 problem for $|B_t|$ The shape of the boundary $S_1^{|B|}$ is shown in the Figure 2.

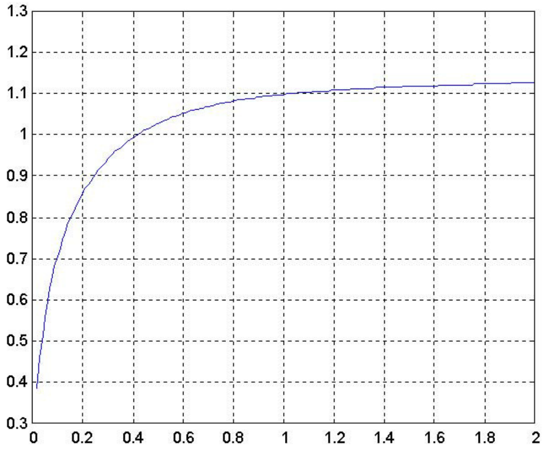

FIGURE 2. Approximate solution to the IFPT1 problem for $X(t) = |B_t|$ and the exponential density with mean 1.

2.2. **The case of a general diffusion.** For a general diffusion, in place of equation (2.4) one should consider the equation obtained by a first-order approximation of equation (2.1) with $z = S(t)$, i.e:

$$p(S(t_i), t_i|0, 0) = h \sum_{j=1}^{i} f(t_j) p(S(t_i), t_i|S(t_j), t_j) \quad , \quad i = 1, \cdots, N \ . \tag{2.6}$$

Unlike equation (2.4), the solution of the system (2.6) is quite difficult to find since the transition p.d.f. $p(\cdot, \cdot|\cdot, \cdot)$ is generally unknown, except for some special diffusions (see e.g. Example 2.2 in the following); thus, to approximate the solution of the IFPT1 problem by means of (2.6), one has to estimate $p(\cdot, \cdot|\cdot, \cdot)$ numerically, in advance. This can be obtained by simulating (long enough) trajectories of the process $X(t)$; however, the errors introduced and the complexity of the calculations make this approach unpractical.

All these considerations indicate that the IFPT1 problem can be satisfactorily solved for a diffusion only if one is able to reduce $X(t)$ to BM. In [11], for a given FPT density f, it is assumed that the solution S_f^B of the IFPT1 problem for B_t is known; then, an heuristic solution S_f to the IFPT1 problem for $X(t)$ is found, by restricting the class of boundaries among which to seek the solution. To this end, we recall that a continuous function $S : [0, \infty) \longrightarrow \mathbb{R}$ with $x \leq S(0)$, is said to be an upper function for $X(t)$ if $P(\tau_S(x) > 0) = 1$, otherwise S is said to be a lower function for $X(t)$. For $X(t) = B_t$ the Kolmogorov's test for upper boundaries (see [37], [58]) is the following:

let $S(t)$ *be continuous, increasing and such that*

$$\lim_{t \to 0^+} \frac{S(t)}{\sqrt{t}} = +\infty \ .$$

Then, S *is an upper function for* B_t *whenever*

$$\int_{0^+} \frac{S(t)}{t^{3/2}} \, e^{-S^2(t)/2t} \, dt < \infty \ .$$

When X is a diffusion, an analogous test for upper boundaries can be obtained by reducing to BM (see [11]).

In [11] for a given FPT density f, the solution to the IFPT1 problem was searched in the class of upper boundaries which are increasing (locally at $t = 0$).

Let $X(t)$ be the solution of (1.1) with $\eta = x$; we recall that X is said to be recurrent if for every $x \in I$ the probability of the process coming back to x infinitely often is one (see e.g. [35]). Let $w(x) \in C^2(I)$ be the *scale function* of $X(t)$, that is the solution of:

$$\mathcal{L}w(x) = 0 \ , \ x \in I \quad ; \quad w(0) = 0, \ w'(0) = 1 \ , \tag{2.7}$$

where $\mathcal{L}$ is the infinitesimal generator of X defined by:

$$\mathcal{L}h = \frac{1}{2} \, \sigma^2(x) \, \frac{d^2h}{dx^2} + b(x) \, \frac{dh}{dx} \quad , \quad h \in C^2(I) \ . \tag{2.8}$$

As easily seen, if the integral $\int_0^t \frac{2b(z)}{\sigma^2(z)} \, dz$ converges, the problem (2.7) has solution:

$$w(x) = \int_0^x \exp\left(- \int_0^t \frac{2b(z)}{\sigma^2(z)} \, dz \right) dt \ . \tag{2.9}$$

The scale function w is increasing, and it is used to establish the nature of boundaries and also the recurrence of X in terms of w. Indeed, if the boundaries r_1 and r_2 of I are unattainable (see e.g. [31], [35]), the recurrence of X is equivalent to the conditions (see [35]): $\lim_{x \to r_1} w(x) = -\infty$, $\lim_{x \to r_2} w(x) = +\infty$. For instance, BM is recurrent in $I = (-\infty, +\infty)$, being in this case $w(x) = x$.

By Itô's formula it follows that the process $\zeta(t) \doteq w(X(t))$ is given by:

$$\zeta(t) = w(x) + \int_0^t w'(w^{-1}(\zeta(s)))\sigma(w^{-1}(\zeta(s)))\, dB_s \ , \tag{2.10}$$

so $\zeta(t)$ is a local martingale. We denote by:

$$\rho(t) \doteq \langle \zeta \rangle_t = \int_0^t [w'(X(s))\sigma(X(s))]^2 \, ds \tag{2.11}$$

the quadratic variation of $\zeta(t)$. If $\rho(\infty) = \infty$, it can be shown (see e.g. [60]) that there exists a BM $\widetilde{B}$ such that $\zeta(t) = \widetilde{B}(\rho(t)) + w(x)$, $(X(0) = x)$. Thus, the solution $X(t)$ to (1.1) with $\eta = x$, is a function of a random-time-changed BM, namely it can be written in the form:

$$X(t) = w^{-1}(\widetilde{B}(\rho(t)) + w(x)) \tag{2.12}$$

and so, for $x \leq S(0)$, $\tau_S(x) = \inf\{t \geq 0 : X(t) \geq S(t) | X(0) = x\} = \inf\{t \geq 0 : \widetilde{B}(\rho(t)) + w(x) \geq w(S(t))\}$.

Definition 2.1. We say that $X(t)$ (with $X(0) = x$) is *conjugated to BM* if there exists an increasing differentiable function $v(x)$, with $v(0) = 0$, such that $X(t) = v^{-1}(B_t + v(x))$.

Notice that the case of $X(t)$ conjugated to BM is obtained as the special case when $\rho(t) = t$ and $\widetilde{B}_t = B_t$.

If X is conjugated to BM, the IFPT1 problem for X is easily reduced to that for B_t (see [11]); in fact, by using (2.12), some approximation results to the solution of the IFPT1 problem have been obtained in [11], also in the case when $\rho(t)$ is not deterministic. Furthermore, in [11] conditions were studied under which a boundary S can be the solution of the FPT1 problem relative to a given density f, and $E(X(\tau_S))$ results to be finite.

A class of diffusions conjugated to BM is given by processes $X(t)$ which are solutions of SDEs such as

$$dX(t) = \frac{1}{2}\, \sigma(X(t))\sigma'(X(t))dt + \sigma(X(t))dB_t \quad , \quad X(0) = x \tag{2.13}$$

with $\sigma(\cdot) \geq 0$. Indeed, if the integral $v(z) \doteq \int_x^z \frac{1}{\sigma(r)}\, dr$ is convergent for every z, by Itô's formula, we obtain that $X(t) = v^{-1}(B_t + v(x))$. Thus, examples of solutions to IFPT1 problems for X can be easily derived from those regarding BM, by replacing S with $v(S)$.

In the following we give some explicit examples of solutions to the IFPT1 problem, see [11].

Example 2.1 (Feller process or Cox-Ingersoll-Ross (CIR) model)**.** Let $X(t)$ be the solution of the SDE:

$$dX(t) = \frac{1}{4}dt + \sqrt{X(t) \vee 0}\, dB_t \quad , \quad X(0) = x \geq 0 \tag{2.14}$$

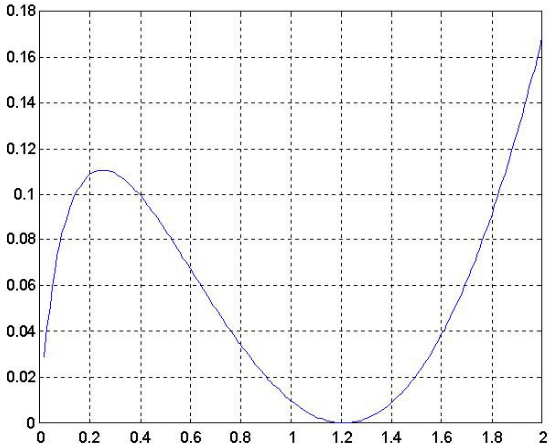

FIGURE 3. Approximate solution to the IFPT1 problem for the diffusion $X(t)$ which is solution of (2.14) with $X(0) = 0$, for the exponential density with mean 1.

(note that, although $\sqrt{x}$ is not Lipschitz-continuous, the solution is unique because $\sqrt{x}$ is Holder-continuous of order 1/2, see condition A2). Then, $X(t)$ turns out to be conjugated to BM via the function $v(x) = 2\sqrt{x}$, that is $X(t) = (1/4)(B_t + 2\sqrt{x})^2$, so $X(t) \geq 0$ for all $t \geq 0$. As easily seen, $S_{g_1}(t) = v^{-1} \circ S_1^B = S_1^B(t)^2/4$ is the solution of the IFPT1 problem for the exponential density with parameter 1, i.e. $g_1(t)$. Therefore, $S_{g_\lambda}(t) = v^{-1} \circ S_\lambda^B(t) = (1/4)(S_1^B(\lambda t))^2/\lambda$ solves the IFPT1 problem for the density g_λ. The shape of the boundary S_{g_1} is given in the Figure 3, in the case when $x = 0$.

Example 2.2 (Wright & Fisher-like process)**.** Let us consider the SDE:

$$(2.15) \quad dX(t) = (a + bX(t))dt + \sqrt{X(t)(1 - X(t)) \vee 0}\, dB_t \quad , \quad X(0) = x \in [0, 1]$$

with $a \geq 0$ and $a + b \leq 0$; under these conditions it can be shown (see [17]) that the process $X(t)$ does not exit from the interval $I \doteq [0, 1]$ for any time. This equation is used e.g. in the Wright-Fisher model for population genetics and in certain diffusion models for neural activity [43]. For $a = 1/4$ and $b = -1/2$, $X(t)$ turns out to be conjugated to BM via the function $v(x) = 2 \arcsin \sqrt{x}$, that is, $X(t) = \sin^2\left((1/2)B_t + \arcsin \sqrt{x}\right)$. Thus, the solution of the IFPT1 problem for $g_\lambda(t)$ is $S_{g_\lambda}(t) = v^{-1} \circ S_\lambda^B = \sin^2\left(S_\lambda^B(t)/2\right)$. For these values of a and b, and $x = 0$, the shape of the boundary $S_{g_1}(t)$ is shown in the Figure 4.

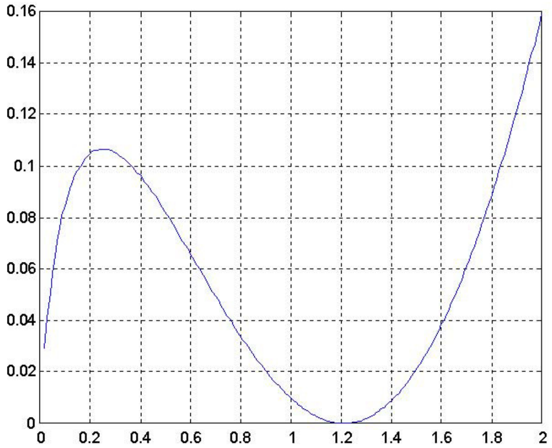

FIGURE 4. Approximate solution to the IFPT1 problem for the diffusion $X(t)$ which is solution of (2.15) with $a = 1/4$, $b = -1/2$, $X(0) = 0$, relative to the exponential density with mean 1.

Notice that, unless the case $a = 1/4$ and $b = -1/2$, $X(t)$ is not conjugated to BM; however, the transition probability density function of $X(t)$ can be explicitly written as an infinite series of hypergeometric functions (see e.g. [16], [43]). Thus, the solution of the IFPT1 problem for a given density f could be found, at least in principle, by solving equation (2.1), or its first-order approximation (2.6).

Example 2.3. Let us consider again $X(t)$ as in the equation (2.15) with $a = 1/4$, $b = -1/2$, $X(0) = 0$, and density $f(t) = (\alpha/\sqrt{2\pi t^3}\,)\, e^{-(\alpha+\beta t)^2/2t}$. Then, the solution of the IFPT1 problem is given by $S_f(t) = \sin^2\big((\alpha+\beta t)/2\big)$, because this boundary is transformed into the linear boundary $\alpha+\beta t$ for B_t (in fact, f is nothing but the Inverse-Gaussian density, i.e. the FPT density of BM over the straight line $\alpha + \beta t$).

Example 2.4. Let us consider the SDE $dX(t) = (1/3)X(t)^{1/3}dt + X(t)^{2/3}dB_t$, $X(0) = 0$, whose explicit solution is $X(t) = (1/27)B_t^3$. Then, $X(t)$ is conjugated to B_t via the function $v(x) = 3x^{1/3}$. We have $S_{g_\lambda} = (S^B_\lambda)^3/27$, where S^B_λ is given by (2.5).

Example 2.5. For $\mu \in \mathbb{R}$, let us consider the SDE:

$$dX(t) = \left(\mu\sqrt{1+X^2(t)} + \frac{1}{2}\,X(t)\right) dt + \sqrt{1+X^2(t)}\;dB_t \quad , \qquad X(0) = 0\ ,$$

whose solution is $X(t) = \sinh(\mu t + B_t)$. Then, the FPT of $X(t)$ over $S(t)$ coincides with the FPT of BM over the boundary $\hat{S}(t) = \sinh^{-1}(S(t)) - \mu t$. It can be shown (see [11]) that, if S is an upper boundary for $X(t)$, then $\hat{S}$ is an upper boundary for BM. Thus, the IFPT1 problem for $X(t)$ is reduced to that for BM.

Example 2.6. Let us consider the SDE:

$$dX(t) = \left(\lambda + \frac{b}{\sqrt{2}}\right) \frac{\sigma(X(t))}{\sqrt{2}}\, dt + \sigma(X(t))dB_t \quad , \quad X(0) = 0\ ,$$

with $\sigma(x) = a + bx,\ a, b, \lambda$ constants. Take $f(t) = (a/\sqrt{2\pi t^3}\,)\, e^{-(a+bt)^2/2t}$; then the solution of the IFPT1 problem is $S_f(t) = (b/a)(e^{a(a+bt)/\sqrt{2}} - 1)$, because this boundary is transformed into the linear boundary $a + bt$ for BM (see [14]).

Example 2.7. Let $G(t)$ be a deterministic differentiable function of time, vanishing at zero, and $\sigma > 0$. Let us consider the process $X(t) = G(t) + \sigma B_t$. If S is an upper function for $X(t)$, which is increasing (locally at $t = 0$), we have:

$$\tau_S = \inf\{t > 0\ :\ G(t) + \sigma B_t > S(t)\} = \inf\left\{t > 0\ :\ B_t > \frac{S(t) - G(t)}{\sigma}\right\}\ .$$

If we take e.g. $f = g_\lambda$, then it must be $(S(t) - G(t))/\sigma = S^B_\lambda(t)$, therefore $S_{g_\lambda}(t) \doteq \sigma S^B_\lambda(t) + G(t)$ solves the IFPT1 problem for the density g_λ.

Example 2.8 (Black-Scholes diffusion model)**.** Let us consider the SDE

$$dX(t) = rX(t)dt + \sigma X(t)dB_t \quad , \quad X(0) = x\ ,$$

where r and σ are positive constant. This is a well-known equation in the framework of Mathematical Finance, to describe the time evolution of a stock price $X(t)$. The explicit solution is $X(t) = xe^{(r-\sigma^2/2)t}e^{\sigma B_t}$. Thus, the solution of the IFPT1 problem for g_λ is given by

$$S_{g_\lambda}(t) = xe^{(r-\sigma^2/2)t}e^{\sigma S^B_\lambda(t)}\ ,$$

where S^B_λ is defined by (2.5).

3. THE IFPT2 PROBLEM FOR (SIMPLE) DIFFUSIONS

In this section, we review the results concerning the IFPT2 problem for a one-dimensional simple diffusion $X(t)$, namely, without jumps.

3.1. **The one-barrier case.**

We recall the single barrier IFPT2 problem for a diffusion $X(t)$ which is solution of equation (1.1), where the infinitesimal parameters satisfy conditions A1 and A2 and the initial position η is random. Let $S : [0, +\infty) \longrightarrow \mathbb{R}$ be a continuous boundary and $X(0) = \eta$ a random variable, independent of B_t and such that $P(S(0) \le \eta) = 1$; we consider the FPT τ_S of $X(t)$ below $S(t)$, that is $\tau_S = \inf\{t > 0\ :\ X(t) \le S(t)\}$ and we suppose that $\tau_S(x)$, i.e. the FPT conditional to $\eta = x$, is a.s. finite for every $x \in I$, and it possesses a density $f(t|x)$. Moreover, we suppose that the initial position η has a density $g(x)$ with support $[S(0), +\infty)$; then the density of τ_S is obtained as $f(t) = \int_{S(0)}^{\infty} f(t|x)g(x)\, dx$. Taking the Laplace transform on both sides we get:

$$\widehat{f}(\theta) = \int_{S(0)}^{\infty} \widehat{f}(\theta|x)g(x)\, dx\ . \tag{3.1}$$

Notice that, if $E(\tau_S(x)) = +\infty$ (this is the case if e.g. $X(t)$ is BM and $S(t) = a =$ const), then also $E(\tau_S) = +\infty$.

For a given FPT distribution function F (or equivalently for a given FPT density $f = F'$), the IFPT2 problem for one barrier consists in finding the density g of the random initial position η, if it exists, such that $P(\tau_S \le t) = F(t)$. The situation is sketched in the Figure 5.

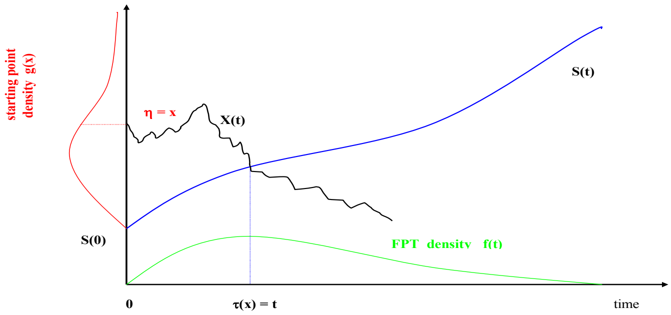

FIGURE 5. The single barrier IFPT2 problem for diffusions

3.1.1. *The case of BM and a linear boundary.* Assume $X(t)$ a BM, $S(t) = a + bt$, and set $\tau_{a,b} = \tau_{a+bt}$. The following results were obtained in [7] (see also [38]); we include their proofs for reader's convenience.

Proposition 3.1. *For $a, b \ge 0$, let us consider a BM starting from the random position $\eta > a$; suppose that the FPT below the boundary $S(t) = a + bt$ has an assigned probability density f and denote by $\widehat{f}(\theta) = \int_0^\infty e^{-\theta t} f(t)\, dt$, $\theta \ge 0$, the Laplace transform of f. Then, if there exists a solution g to the IFPT2 problem, its Laplace transform $\widehat{g}(\theta)$ must satisfy the equation:*

$$\widehat{g}(\theta) = e^{-\theta a} \widehat{f}\left(\frac{\theta(\theta + 2b)}{2}\right) . \tag{3.2}$$

Proof. For a fixed $x \ge a \ge 0$, since $-B_t$ has the same distribution as B_t, then $\tau_{a,b}(x) = \inf\{t > 0 : x + B_t \le a + bt\}$ is distributed as $\inf\{t > 0 : B_t \ge (x - a) - bt\}$, that is the first instant at which BM crosses the boundary $S(t) = x - a - bt$. As well-known, the conditions $x \ge a$ and $b \ge 0$ imply that $\tau_{a,b}(x)$ is finite with probability one. Thus, $\tau_{a,b}(x)$ has the inverse Gaussian distribution, namely its density is (see e.g. [41]) $f(t|x) = \big((x - a)/t^{3/2}\big)\, \phi\big((x - a - bt)/\sqrt{t}\,\big)$, where $\phi(y) = e^{-y^2/2}/\sqrt{2\pi}$, and its Laplace transform is:

$$\widehat{f}(\theta|x) = e^{-(x-a)(\sqrt{b^2+2\theta}-b)} . \tag{3.3}$$

From (3.1) we get

$$\widehat{f}(\theta) = \int_a^\infty \widehat{f}(\theta|x) g(x)\, dx = \int_a^\infty e^{-x(\sqrt{b^2+2\theta}-b)}\, e^{a(\sqrt{b^2+2\theta}-b)} g(x)\, dx =$$

$$= e^{a(\sqrt{b^2+2\theta}-b)} \widehat{g}(\sqrt{b^2 + 2\theta} - b) , \tag{3.4}$$

from which (3.2) easily follows.

□

Remark 3.2. Once $\widehat{g}$ has been found, such that $P(\tau_S \leq t) = F(t)$ (with $F' = f$), the function $\widehat{g}$ could not be the Laplace transform of the density function of a random variable. In that case the IFPT2 problem has no solution. This is why Proposition 3.1 is formulated in a conditional form.

For a non linear boundary S, even if one is able to calculate explicitly the Laplace transform $\widehat{f}(\theta|x)$ of the conditional FPT density $f(t|x)$, in general the expression (3.1) cannot be related to the Laplace transform of g calculated in some point (this happens for a linear boundary $S(t) = a + bt$, thanks to the particular exponential form of $\widehat{f}(\theta|x) = e^{-(x-a)(\sqrt{b^2+2\theta}-b)}$).

If $\widehat{f}(\theta)$ is analytic in a neighborhood of $\theta = 0$, then the moments of τ_S exist finite and they are obtained in terms of $\widehat{f}(\theta)$ by $E(\tau_S^k) = (-1)^k(\partial^k/\partial\theta^k)\widehat{f}(\theta)|_{\theta=0}$. The same holds for the moments of η, since by (3.2) also $\widehat{g}(\theta)$ turns out to be analytic. Then, by (3.2) one easily obtains that, for $b > 0$:

$$E(\eta) = a + bE(\tau_S) \quad , \quad \mathrm{Var}(\eta) = b^2\mathrm{Var}(\tau_S) - E(\tau_S)\ . \tag{3.5}$$

Since $\mathrm{Var}(\eta) \geq 0$, we get the compatibility condition:

$$b^2\mathrm{Var}(\tau_S) - E(\tau_S) \geq 0\ , \tag{3.6}$$

which is necessary so that the solution to the IFPT2 problem exists, in the case of analytic Laplace transforms $\widehat{f}$ and $\widehat{g}$. Note that, if e.g. $S(t) = a$ (i.e. $b = 0$), the moments of τ_S do not exist finite and the condition (3.6) loses meaning.

Remark 3.3. We have considered the FPT of $\eta + B_t$ below the linear boundary $S(t) = a + bt$, with $\eta \geq a$, $a, b \geq 0$. In an analogous way one could consider the FPT of $\overline{\eta} + B_t$ over the boundary $\overline{S}(t) = a - bt$, with $\overline{\eta} \leq a$, $a, b \geq 0$. In fact, since $-B_t$ has the same distribution as B_t, then $\inf\{t > 0 : \overline{\eta} + B_t \geq a - bt\}$ has the same distribution as $\inf\{t > 0 : B_t \leq \overline{\eta} - a + bt\} = \inf\{t > 0 : a - \overline{\eta} + B_t \leq bt\} \equiv \tau_{0,b}(a - \overline{\eta})$. Thus, the density of the starting point $\overline{\eta} \leq a$ in the IFPT2 problem over the boundary $a - bt$ is nothing but the density of the random variable $a - \overline{\eta}$ in the IFPT2 problem below the boundary bt. By using the same arguments of Proposition 3.1, we obtain that, if there exists the density $\overline{g}$ of $\overline{\eta}$, its Laplace transform, $\widehat{\overline{g}}(\theta)$, must satisfy the equation:

$$\widehat{\overline{g}}(\theta) = e^{-a\theta}\,\widehat{f}\left(\frac{\theta(\theta - 2b)}{2}\right)\ . \tag{3.7}$$

As easily seen, in this case one obtains $E(\overline{\eta}) = a - bE(\tau_S)$ and $\mathrm{Var}(\overline{\eta}) = \mathrm{Var}(\eta) = b^2\mathrm{Var}(\tau_s) - E(\tau_S)$.

Now, we go to consider the question of the existence of solutions to the IFPT2 problem (see also [7], [38]).

Proposition 3.4. *If the FPT density f is a Gamma density with parameters (γ, λ), then, the IFPT2 problem has solution, provided that $b \geq \sqrt{2\lambda}$, and the Laplace transform of the density g of the initial position η is given by:*

$$\widehat{g}(\theta) = \left[e^{-a\theta/2}\frac{(b - \sqrt{b^2 - 2\lambda})^\gamma}{(\theta + b - \sqrt{b^2 - 2\lambda})^\gamma}\right] \times \left[e^{-a\theta/2}\frac{(b + \sqrt{b^2 - 2\lambda})^\gamma}{(\theta + b + \sqrt{b^2 - 2\lambda})^\gamma}\right]\ . \tag{3.8}$$

The function $\widehat{g}(\theta)$ is the Laplace transform of the sum of two independent random variables, Z_1 and Z_2, such that $Z_i - a/2$ has distribution Gamma of parameters γ and λ_i $(i = 1, 2)$, where $\lambda_1 = b - \sqrt{b^2 - 2\lambda}$ and $\lambda_2 = b + \sqrt{b^2 - 2\lambda}$.

Proof. From the well-known expression of the Laplace transform of a Gamma density, we have $\widehat{f}(\theta) = \lambda^\gamma/(\lambda+\theta)^\gamma$. From (3.2) we get:

$$\widehat{g}(\theta) = e^{-\theta a} \frac{\lambda^\gamma}{(\lambda + \theta(\theta+2b)/2)^\gamma} = e^{-\theta a} \frac{(2\lambda)^\gamma}{[(\theta+b)^2 - (b^2 - 2\lambda)]^\gamma}$$

which can be easily put in the form (3.8).

□

Remark 3.5. If f is the Gamma density, the compatibility condition (3.6) becomes $b \geq \sqrt{\lambda}$, which is satisfied under the assumption $b \geq \sqrt{2\lambda}$ required by Proposition 3.4. In the special case when f is the exponential density with parameter λ, then η has the same distribution as $a + Z_1 + Z_2$, where Z_i are independent and exponential random variables with parameter λ_i, $i = 1, 2$

Proposition 3.6. *If the FPT density f has the Laplace transform:*

$$\widehat{f}(\theta) = \sum_{k=1}^{N} \frac{A_k}{(\theta + B_k)^{c_k}} \tag{3.9}$$

for some $c_k > 0$, $A_k, B_k > 0$, $k = 1, \cdots, N$, then, there exists a value $b^* > 0$ such that the solution to the IFPT2 problem exists, provided that $b \geq b^*$.

Proof. First, let us observe that, if the density g_k, $k = 1, \cdots N$ is the solution of the IFPT2 problem for the boundary $S(t) = a + bt$, with FPT density f_k, then $g(x) = \sum_{k=1}^N p_k g_k(x)$, with $p_k \geq 0$ and $\sum_{k=1}^N p_k = 1$, solves the IFPT2 problem for the FPT density $f(t) = \sum_k p_k f_k(t)$. Now, suppose that the Laplace transform of $f(t) = \sum_k p_k f_k(t)$ has the form(3.9). Then, from $\widehat{f}(\theta) = \sum_k p_k \widehat{f}_k(\theta)$ and the fact that $\widehat{f}(0) = 1$, we get $1 = \widehat{f}(0) = \sum_k p_k \widehat{f}_k(0) = \sum_k A_k / B_k^{c_k}$ so, we can take $p_k = A_k/B_k^{c_k} \geq 0$. Thus, the function f defined by (3.9) can be written as a convex combination $\widehat{f}(\theta) = \sum_{k=1}^N p_k \widehat{f}_k(\theta)$, where $\widehat{f}_k(\theta) = B_k^{c_k}/(\theta + B_k)^{c_k}$. But, the last function is the Laplace transform of a random variable having Gamma distribution with parameters (c_k, B_k). Then, by Proposition 3.4, if $b > b^* \doteq \max_{k=1,\cdots,N} \sqrt{2B_k}$, for each $k = 1, \cdots, N$ there exists g_k which solves the IFPT2 problem for the density f_k; precisely, g_k is the density of $a + Z_k^1 + Z_k^2$, where Z_k^i, $i = 1, 2$ have suitable Gamma distributions. As a consequence, the function $g(x) = \sum_{k=1}^N p_k g_k(x)$ solves the IFPT2 problem for the FPT density $f(t) = \sum_{k=1}^N p_k e^{-B_k t} \big(B_k^{c_k}/\Gamma(c_k)\big) t^{c_k - 1}$.

□

Remark 3.7. Similar results were obtained in [39], where the authors proved the existence of the solution to the IFPT2 problem, for a class of densities f which are infinite linear combinations of Gamma densities.

The following Proposition deals with the case when $b = 0$.

Proposition 3.8. *If $b = 0$ and the Laplace transform of the FPT density f is*

$$\widehat{f}(\theta) = \sum_{k=1}^{N} \frac{A_k}{(\sqrt{2\theta} + B_k)^{c_k}} \tag{3.10}$$

for some $c_k > 0$, $A_k, B_k > 0$, $k = 1, \cdots, N$, then the solution to the IFPT2 problem exists.

Proof. As in the proof of Proposition 3.6, the Laplace transform in (3.10) can be written as the convex combination $\widehat{f}(\theta) = \sum_{k=1}^{N} p_k \widehat{f}_k(\theta)$, where now $\widehat{f}_k(\theta) = B_k^{c_k}/(\sqrt{2\theta}+B_k)^{c_k}$. From (3.2), we get $\widehat{g}_k(\theta) = \widehat{f}_k(\theta^2/2) = B_k^{c_k}/(\theta+B_k)^{c_k}$ which is the Laplace transform of the density $g_k(x) = (B_k^{c_k}/\Gamma(c_k))\, e^{-B_k x} x^{c_k-1}$, that is the Gamma density with parameters (c_k, B_k). Then, $g(x) = \sum_{k=1}^{N} p_k g_k(x)$ solves the IFPT2 problem when $b = 0$.

□

Remark 3.9. The function $\widehat{f}(\theta)$ in (3.10) is not analytic in a neighbor of $\theta = 0$, so the moments of τ_S do not exist finite.

BM + large jumps. As example, let us consider now the piecewise-continuous process $\overline{X}(t)$, obtained by superimposing to the BM a jump process, that is $\overline{X}(t) = \eta + B_t$ for $t < T$, where T is an exponential distributed time with parameter $\mu > 0$; suppose that, for $t = T$, the process $\overline{X}(t)$ makes a downward jump crossing the boundary S, irrespective of its state before the occurrence of the jump. This kind of behavior is observed e.g. in the presence of a so called *catastrophes* (see e.g. [28]). Then, for $\eta \geq a$, the FPT of $\overline{X}(t)$ below the linear boundary $S(t) = a + bt$ is $\overline{\tau}_S = \inf\{t > 0 : \overline{X}(t) \leq a + bt\}$. The following proposition holds.

Proposition 3.10. *If there exists a solution $\overline{g}$ to the IFPT2 problem of $\overline{X}(t)$ below $S(t) = a + bt$ with $\overline{X}(0) = \eta \geq a$, $a, b \geq 0$, then its Laplace transform is*

$$\widehat{\overline{g}}(\theta) = e^{-a\theta}\left[\left(1 - \frac{2\mu}{\theta(\theta+2b)}\right)^{-1} \widehat{\overline{f}}\left(\frac{\theta(\theta+2b)}{2} - \mu\right) - \frac{2\mu}{\theta(\theta+2b) - 2\mu}\right] . \tag{3.11}$$

Proof. Conditionally to $\eta = x$, one has:

$$\begin{aligned} P(\overline{\tau}_S(x) \leq t) &= P(\overline{\tau}_S(x) \leq t | t < T) P(t < T) + 1 \times P(t \geq T) = \\ &= P(\tau_S(x) \leq t) e^{-\mu t} + (1 - e^{-\mu t}) \end{aligned} \tag{3.12}$$

Taking the derivative, one obtains the FPT density of $\overline{X}$, conditional to the starting position x:

$$\overline{f}(t|x) = e^{-\mu t} f(t|x) + \mu e^{-\mu t} \int_t^{+\infty} f(s|x)\, ds \, . \tag{3.13}$$

Straightforward calculation gives its Laplace transform:

$$\widehat{\overline{f}}(\theta|x) = \int_0^{\infty} e^{-\theta t} \overline{f}(t|x)\, dt = \frac{\theta}{\mu+\theta}\, \widehat{f}(\mu+\theta|x) + \frac{\mu}{\mu+\theta} \, . \tag{3.14}$$

Finally, from the equation $\widehat{\overline{f}}(\theta) = \int_a^{+\infty} \widehat{\overline{f}}(\theta|x) \overline{g}(x)\, dx$, where $\overline{f}$ is the FPT density of $\overline{X}(t)$, and by using (3.3), we have

$$\widehat{\overline{f}}(\theta) = \frac{1}{\mu+\theta}\left[\theta \times e^{a\left(\sqrt{b^2+2(\mu+\theta)}-b\right)}\, \widehat{\overline{g}}\left(\sqrt{b^2+2(\mu+\theta)} - b\right) + \mu\right] . \tag{3.15}$$

Equation (3.11) follows from (3.15) solving with respect to $\widehat{\overline{g}}(\theta)$, (notice that the corresponding formula in [7] contains an error that was corrected in a subsequent Erratum, see also [5]).

□

Remark 3.11. For $\mu = 0$, namely when no jump occurs, (3.11) reduces to (3.2).

3.1.2. *Transformation of the IFPT2 problem for diffusions in that for BM.* When $X(t)$ is not BM, the way to obtain results analogous to those of the previous propositions relies on the availability of an explicit formula for the Laplace transform $\widehat{f}(\theta|x)$ In the special case when $S(t) = a = \text{const}$, it is well known (see e.g. [37]) that $\widehat{f}(\theta|x)$ satisfies the equation:

$$\frac{1}{2} z''(x)\sigma^2(x) + z'(x)b(x) = \theta z(x) \ . \tag{3.16}$$

Precisely, when the FPT below the barrier a is considered (i.e. $x \geq a$), then $\widehat{f}(\theta|x)$ is the decreasing solution to (3.16) with the condition $\widehat{f}(\theta|a) = 1$; when the FPT over a is considered (i.e. $x \leq a$), then $\widehat{f}(\theta|x)$ is the increasing solution to (3.16) with the condition $\widehat{f}(\theta|a) = 1$. Note that (3.16) is a linear ODE with non constant coefficients, so, in general its solution cannot be obtained in closed form. In the case of a general diffusion, explicit solutions to the IFPT2 problem can be obtained if one is able to reduce to BM, as done in Section 2 for the IFPT1 problem.

(I) Let $X(t)$ be conjugated to B_t via the function v (see Definition 2.1 in Section 2), and consider the FPT of $X(t)$ below the constant boundary $S(t) = a \leq \eta$. Suppose that the FPT below the barrier a has density f, and that there exists the density g of η which solves the IFPT2 problem. For $\widetilde{\eta} = v(\eta)$, denote by $g_{\widetilde{\eta}}$ the density of $\widetilde{\eta}$. Then, from Proposition 3.1 with $b = 0$, η replaced by $\widetilde{\eta}$ and a replaced by $v(a)$, the Laplace transform of $g_{\widetilde{\eta}}$ satisfies:

$$\widehat{g}_{\widetilde{\eta}}(\theta) = e^{-v(a)\theta} \ \widehat{f}\left(\frac{\theta^2}{2}\right) \ . \tag{3.17}$$

The same expression holds in the case when the FPT of $X(t)$ over the barrier a is considered (see Remark 3.3).

(II) We recall from Section 2 that, under certain conditions, $X(t)$ can be written in the form $X(t) = w^{-1}(\widetilde{B}(\rho(t)) + w(\eta))$, where $\widetilde{B}$ is BM, w is the scale function, and ρ is the quadratic variation of $w(X(t))$, which is supposed to satisfy the condition $\rho(\infty) = \infty$. Moreover, if $\rho(t)$ is deterministic, we obtain:

$$\widetilde{\tau}_S := \rho(\tau_S) = \inf\{s \geq 0 \ : \ B_s + w(\eta) \leq w(S(\rho^{-1}s))\} \ .$$

Now, suppose that $w(S(\rho^{-1}s)) = a + bs$ and $w(\eta) \geq w(S(0)) = a$ (i.e. $\eta \geq S(0)$). Then, for a given FPT density f, by using the arguments of Subsection 3.1.1, one can find the density of the initial point $w(\eta)$, which solves the IFPT2 problem for the boundary $a + bs$, and the FPT density $\widetilde{f}(t) = f(\rho^{-1}(t))(\rho^{-1})'(t)$, that is the density of $\widetilde{\tau}_S$. Therefore, for the boundary $S(t) = w^{-1}(a + b\rho(t))$ with $S(0) \leq \eta$, the Laplace transform $\widehat{\widetilde{g}}(\theta)$ of the density of $w(\eta)$ is obtained solving the IFPT2 problem for $\widetilde{f}$. Finally, once obtained the density $\widetilde{g}$ of $w(\eta)$ by Laplace inversion, the density of η is obtained as $g(x) = \widetilde{g}(w(x))w'(x)$.

An interesting case is the integral process with deterministic integrand

$$X(t) = \eta + \int_0^t \sigma(s) \, dB_s \ ,$$

where $\sigma(\cdot) \geq 0$ is a deterministic function. Then, the scale function is $w(x) = x$, $\rho(t) = \int_0^t \sigma^2(s) \, ds$ and, if $\rho(\infty) = \infty$, then $X(t) = \eta + \widetilde{B}(\rho(t))$. Thus, for the

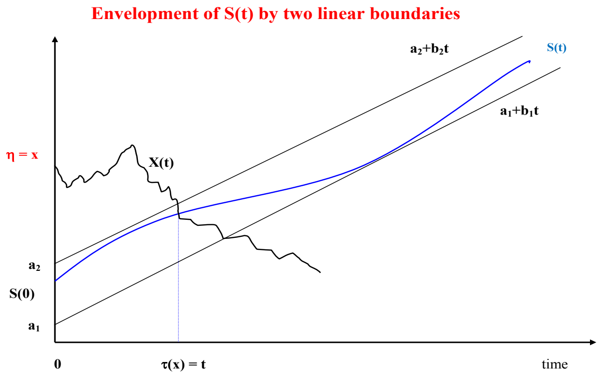

FIGURE 6. Approximating the solution of the IFPT2 problem for a non linear boundary

boundary $S(t) = a + b\rho(t$ with $S(0) \leq \eta$, we get the solution to the IFPT2 problem, following arguments similar to the previous ones.

When ρ is not deterministic, the solution to the IFPT2 problem is more difficult to recover (see [4], [5]).

3.1.3. *Approximate solution of the IFPT2 problem for a non linear boundary.* Let us consider a non linear boundary $S(t)$ which is enveloped from above and below by two straight lines. Namely assume there exist $0 \leq a_1 \leq a_2$ and $0 \leq b_1 \leq b_2$, such that, for every $t \geq 0$:

$$a_1 + b_1 t \leq S(t) \leq a_2 + b_2 t \; . \tag{3.18}$$

The situation is sketched in the Figure 6. Then, the following proposition holds (for the proof, see [5]):

Proposition 3.12. *Let $S(t)$ be a boundary satisfying* (3.18) *and B_t a BM starting from the random position $\eta > S(0)$. If the FPT below the boundary $S(t)$ has assigned density f and there exists a density g with support $(S(0), +\infty)$, which is solution of the IFPT2 problem for B_t and the boundary $S(t)$, then:*

(*i*) *if $a_2 > a_1$ and the function $g \in L^p(S(0), a_2)$ for some $p > 1$, its Laplace transform $\widehat{g}(\theta)$ must satisfy $g_1(\theta) \leq \widehat{g}(\theta) \leq g_2(\theta)$ with*

$$\begin{aligned} g_1(\theta) &= e^{-a_2(\theta + 2(b_2 - b_1))} \left[\widehat{f}\left(\frac{\theta(\theta + 2b_2)}{2} \right) - \right. \\ &\qquad \left. - (a_2 - S(0))^{(p-1)/p} \left(\int_{S(0)}^{a_2} g^p(x)\, dx \right)^{1/p} \right] ; \\ g_2(\theta) &= e^{-a_1 \theta} \widehat{f}\left(\tfrac{\theta(\theta + 2b_1)}{2} \right) \end{aligned} \tag{3.19}$$

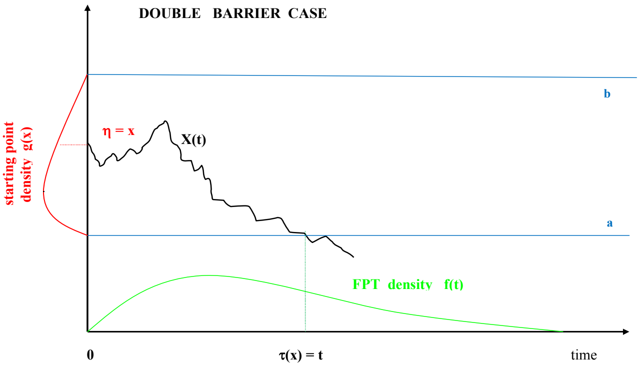

Figure 7. The double-barrier IFPT2 problem for diffusions

(ii) *if* $a_1 = a_2 = S(0)$*, then* $g_1(\theta) \leq \widehat{g}(\theta) \leq g_2(\theta)$*, with* $g_1(\theta)$ *and* $g_2(\theta)$ *given in* (3.19)*, holds without any further assumption on* g*, that is the term* $(a_2 - S(0))^{(p-1)/p} \left(\int_{S(0)}^{a_2} g^p(x)\, dx \right)^{1/p}$ *vanishes.*

Remark 3.13.

I. If g is bounded, then the term $(a_2 - S(0))^{(p-1)/p} \left(\int_{S(0)}^{a_2} g^p(x)\, dx \right)^{1/p}$ can be replaced with $(a_2 - S(0))\|g\|_\infty$.

II. The smaller $a_2 - a_1$ and $b_2 - b_1$ are, the better the approximation to the Laplace transform of g is. For instance, if $S(t) = a + b(t + \epsilon \sin t)$ $(a, b, \epsilon \geq 0)$ we can take $a_1 = a_2 = a$, $b_1 = b - \epsilon$, $b_2 = b + \epsilon$; if $S(t) = a + bt + \epsilon(\sin t + \cos t)$ $(a, b, \epsilon \geq 0)$ we can take $a_1 = a - \epsilon\sqrt{2}$, $a_2 = a + \epsilon\sqrt{2}$, $b_1 = b_2 = b$.

3.2. **The double-barrier case.** In this subsection, we deal with the double-barrier IFPT2 problem for a Wiener process $X(t)$. Let us suppose that the initial position $\eta = X(0)$ is a random variable, independent of $X(t)$, such that $P(a < \eta < b) = 1$, for some $a < b$, and F a given distribution. We recall that the double-barrier IFPT2 problem consists in finding the density g of η (if existing) such that $P(\tau_{(a,b)} \leq t) = F(t)$, where $\tau_{(a,b)} = \inf\{t > 0 : X(t) \notin (a, b)\}$. The situation is sketched in the Figure 7.

First, let us assume $X(t)$ a BM; subsequently, we will consider diffusion conjugated to BM and BM with drift.

3.2.1. *The case of BM.* Let $X(t)$ be a BM; for an interval (a, b), $b > a$, and $x \in (a, b)$, let $\tau_{(a,b)}(x) = \inf\{t > 0 : X(t) \notin (a, b) | \eta = x\}$ be the first-exit time of $X(t)$ from (a, b), with the condition that $\eta = x$. We denote by $f(t|x)$ the probability density of $\tau_{(a,b)}(x)$, by $g(x)$ the density of $\eta \in (a, b)$, and by $\widehat{g}(\theta) = \int_a^b e^{-\theta x} g(x)\, dx$ $(\theta \geq 0)$ the (possibly bilateral) Laplace transform of g. The following proposition holds (for the proof see [6]).

Proposition 3.14. *For* $a < b$*, let us consider a BM* B_t *starting from the random position* $\eta \in (a, b)$*; denote by* $f_{a,b}$ *the density of its first-exit time from the interval*

(a,b) and by $\widehat{f}_{a,b}(\theta) = \int_0^\infty e^{-\theta t} f_{a,b}(t)\,dt$, $\theta \geq 0$, its Laplace transform. Then, if there exists the solution g to the double-barrier IFPT2 problem for B_t and the barriers a and b, its Laplace transform $\widehat{g}(\theta)$ must satisfy the equation:

$$\widehat{g}(\theta)e^{(\theta/2)(a+b)} + \widehat{g}(-\theta)e^{-(\theta/2)(a+b)} = 2\cosh\left(\frac{\theta}{2}(b-a)\right)\widehat{f}\left(\frac{\theta^2}{2}\right) \ , \ \theta \geq 0 \ . \tag{3.20}$$

If the density g is symmetric with respect to $(1/2)(a+b)$, then:

$$\widehat{g}(\theta) = \frac{1}{2}\left(e^{-a\theta} + e^{-b\theta}\right)\widehat{f}_{a,b}\left(\frac{\theta^2}{2}\right) \quad , \quad \theta \geq 0 \ . \tag{3.21}$$

In particular, if $a = -\alpha$ and $b = \alpha > 0$ then

$$\widehat{g}(\theta) = \cosh(\alpha\theta)\,\widehat{f}_{-\alpha,\alpha}\left(\frac{\theta^2}{2}\right) \quad , \quad \theta \geq 0 \ . \tag{3.22}$$

Remark 3.15. As it is well-known, we have $E(\tau_{(a,b)}(x)) = (x-a)(b-x)$. Since $(x-a)(b-x) \leq (b-a)^2/4$ from $E(\tau_{(a,b)}) = \int_a^b E(\tau_{a,b}(x))g(x)\,dx$, we get the compatibility condition:

$$E(\tau_{(a,b)}) \leq \frac{(b-a)^2}{4} \tag{3.23}$$

necessary for the existence of the solution to the double barrier IFPT2 problem. From inequality (3.23), the first moment of $\tau_{(a,b)}$ is finite.

Remark 3.16. As in the single-barrier case, once $\widehat{g}$ has been found, such that (3.20) holds, it may be that $\widehat{g}$ is not the Laplace transform of the density function of a random variable. In this case, a solution to the double-barrier IFPT2 problem does not exist.

Taking into account the previous remark, it is possible to prove the existence of the density g of the initial position η, for a class of FPT densities f. For the sake of simplicity, we limit ourselves to the case when the barriers are -1 and 1, and g is required to be an even function with support in $(-1,1)$. For any integer $k \geq 0$, set $I_k(\theta) = \int_{-1}^1 e^{-\theta x} x^k\,dx$; as easily seen, $I_0(\theta) = 2\sinh(\theta)/\theta$ and the recursive relation $I_k(\theta) = \big((-1)^k e^\theta - e^{-\theta}\big)/\theta + (k/\theta)I_{k-1}(\theta)$ allows to calculate $I_k(\theta)$, for every k. Next Proposition gives a sufficient condition for the existence of the solution of the double-barrier IFPT2 problem for BM (for the proof, see [6]).

Proposition 3.17. *If the Laplace transform of $f(t)$ is*

$$\widehat{f}(\theta) = \sum_{k=1}^N p_k \widehat{f}_{2k}(\theta) \ , \tag{3.24}$$

with weights $p_k \geq 0$ such that $\sum_{k=1}^N p_k = 1$, and

$$\widehat{f}_{2k}(\theta) = \frac{2k+1}{4k\cosh(\sqrt{2\theta})}\left(\frac{2\sinh\sqrt{2\theta}}{\sqrt{2\theta}} - I_{2k}(\sqrt{2\theta})\right) \quad , \quad k = 1, \cdots, N$$

then the solution g of the IFPT2 problem for BM and the barriers -1 and 1 is

$$g(x) = \sum_{k=1}^N p_k g_{2k}(x) \ . \tag{3.25}$$

with $g_{2k}(x) = \big((2k+1)/4k\big)(1 - x^{2k})$, $k \geq 0$.

BM + large jumps. As an application of the results for BM (for the single-barrier case, see Proposition 3.10), we consider the piecewise-continuous process $\overline{X}(t)$, obtained by superimposing to BM a jump process, namely, for $\eta \in (a,b)$ and $t < T$, set $\overline{X}(t) = \eta + B_t$, where T is an exponential distributed time with parameter $\mu > 0$. We suppose that for $t = T$ the process $\overline{X}(t)$ makes a downward or upward jump and it crosses one of barriers a and b, irrespective of its state before the occurrence of the jump. Now, we are in the presence of a two-sided catastrophes. Then, for $\eta \in (a,b)$, the first-exit time of $\overline{X}(t)$ from the interval (a,b) is $\overline{\tau}_{(a,b)} = \inf\{t > 0 : \overline{X}(t) \notin (a,b)\}$. By using arguments similar to those used in the single-barrier case (see Proposition 3.10), the following result follows (for the proof, see [6]).

Proposition 3.18. *Let us consider the FPT density $\overline{f}$ for the process $\overline{X}(t)$. If there exists an even function $\overline{g}$ which is the solution to the double-barrier IFPT2 problem of $\overline{X}(t)$ for $-\alpha$ and α, then its Laplace transform is given by:*

$$\widehat{\overline{g}}(\theta) = \left[\frac{\theta^2 \widehat{\overline{f}}\left((\theta^2/2) - \mu\right) - 2\mu}{\theta^2 - 2\mu} \right] \cosh(\alpha\theta) \, . \tag{3.26}$$

Remark 3.19. For $\mu = 0$, namely when no jump occurs, (3.26) reduces to (3.22).

3.2.2. *Double-barrier IFPT2 problem for diffusions conjugated to BM.* Suppose that $X(t)$ is conjugated to BM (see Definition 2.1 in Section 2). The double-barrier IFPT2 problem for $X(t)$ and the barriers a and b is nothing but the double-barrier IFPT2 problem for B_t starting from $v(\eta)$ and the barriers $v(a)$ and $v(b)$. Suppose that the first-exit time of $X(t)$ from (a,b) has density f, and that there exists the density g of η which solves the double-barrier IFPT2 problem; if $\widetilde{\eta} = v(\eta)$, denote by $g_{\widetilde{\eta}}$ the density of $\widetilde{\eta}$. Then, from Proposition 3.14, with a and b replaced with $v(a)$ and $v(b)$, and η replaced with $v(\eta)$, we obtain that the Laplace transform of $\widehat{g}_{\widetilde{\eta}}$ must satisfy:

$$\begin{aligned} &\widehat{g}_{\widetilde{\eta}}(\theta) e^{(\theta/2)(v(a)+v(b))} + \widehat{g}_{\widetilde{\eta}}(-\theta) e^{-(\theta/2)(v(a)+v(b))} = \\ &= 2\cosh\left(\frac{\theta}{2}(v(b)-v(a))\right) \widehat{f}_{v(a),v(b)}\left(\frac{\theta^2}{2}\right) \quad , \quad \theta \geq 0 \, . \end{aligned} \tag{3.27}$$

Once obtained the density $g_{\widetilde{\eta}}$ of $\widetilde{\eta}$ by Laplace inversion, the solution g to the IFPT problem for $X(t)$ and the barriers a and b is $g(x) = g_{\widetilde{\eta}}(v(x))v'(x)$.

3.2.3. *Double-barrier IFPT2 problem for BM with drift.* Let $X(t)$ be BM with drift μ, that is, $X(t) = \eta + \mu t + B_t$. By arguments similar to those used in the proof of Proposition 3.14, the following result holds (for the proof, see [6]):

Proposition 3.20. *For $a < b$, let us consider the process $X(t) = \eta + \mu t + B_t$, starting from the random position $\eta \in (a,b)$. Denote by $\tau = \inf\{t > 0 : X(t) \notin (a,b)\}$ the first-exit time from the interval (a,b) with density f and by $\widehat{f}(\theta) = \int_0^\infty e^{-\theta t} f(t)\, dt$, $\theta \geq 0$ its Laplace transform. Then, if there exists a solution g to the IFPT2 problem of $X(t)$ for the barriers a and b, its Laplace transform $\widehat{g}(\theta)$ must satisfy the equation:*

$$\begin{aligned} &\widehat{g}\left(\mu - \sqrt{\mu^2 + 2\theta}\right)\left(e^{\mu b - a\sqrt{\mu^2+2\theta}} - e^{\mu a - b\sqrt{\mu^2+2\theta}}\right) + \\ &+ \widehat{g}\left(\mu + \sqrt{\mu^2 + 2\theta}\right)\left(e^{\mu a + b\sqrt{\mu^2+2\theta}} - e^{\mu b + a\sqrt{\mu^2+2\theta}}\right) = \end{aligned} \tag{3.28}$$

$$= 2\sinh\left((b-a)\sqrt{\mu^2+2\theta}\right)\widehat{f}(\theta) \quad , \quad \theta \geq 0\,.$$

In particular, if $a = -\alpha$, $b = \alpha > 0$, *we have:*

$$\widehat{g}\left(\mu - \sqrt{\mu^2+2\theta}\right)\sinh\left(\alpha(\mu + \sqrt{\mu^2+2\theta}\right) - \tag{3.29}$$

$$-\widehat{g}\left(\mu + \sqrt{\mu^2+2\theta}\right)\sinh\left(\alpha(\mu - \sqrt{\mu^2+2\theta}\right) =$$

$$= \widehat{f}(\theta)\sinh\left(2\alpha\sqrt{\mu^2+2\theta}\,)\right) \quad , \quad \theta \geq 0\,.$$

Remark 3.21. For $\mu = 0$, equation (3.28) reduces to equation (3.20), while equation (3.29) reduces to equation (3.22) if $g(-x) = g(x)$.

3.2.4. *Approximate solution of the IFPT2 problem for two time-varying barriers.* In the following we deal with the IFPT2 problem for Wiener process, for two boundaries varying with time t.

Let $S_1(t) \leq S_2(t)$, $t \geq 0$, be time dependent, continuous boundaries, and consider the process $X(t) = \eta + B_t + \mu t$, starting from η, which is randomly distributed over the interval $(S_1(0), S_2(0))$. We focus on the first time at which $X(t)$ reaches one of the boundaries $S_1(t)$ and $S_2(t)$, that is, $\tau_{S_1,S_2} = \inf\{t > 0 : X(t) = S_1(t) \text{ or } X(t) = S_2(t)\}$, while, as usual, we denote by $\tau_{S_1,S_2}(x)$ the first-exit time with the condition that $\eta = x \in (S_1(0), S_2(0))$. For a given FPT density $f(t)$, we aim to find an approximation to the Laplace transform of the density g of η, if it exists, which solves the IFPT2 problem for $X(t)$ and the boundaries $S_1(t), S_2(t)$. For the sake of simplicity, we will consider BM without drift (i.e. $\mu = 0$); an analogous result could be obtained for non zero drift, with heavier calculations.

To recover the result of next Proposition (for the proof, see [6]) a number of hypothesis is necessary. We assume there exist constants a, b, A, B such that $a \leq S_1(t) \leq A < b \leq S_2(t) \leq B$, $\forall t \geq 0$, with $S_1(0)$ equidistant from a and A, and $S_2(0)$ equidistant from b and B. Let B_t be BM starting from the random position η, randomly distributed over the interval $(S_1(0), S_2(0))$. Assume the first-passage time τ_{S_1,S_2} has density f and that there exists a density g with support $(S_1(0), S_2(0))$, which is solution to the IFPT2 problem for B_t and the boundaries $S_1(t), S_2(t)$. We also assume that there exist the solutions to the IFPT2 problems for B_t and the couples of barriers A, b and a, B.

Proposition 3.22. *If g is symmetric with respect to the middle point $(S_1(0) + S_2(0))/2$ of the interval $(S_1(0), S_2(0))$ and $g \in L^p(S_1(0), S_2(0))$ for some $p > 1$, then, its Laplace transform $\widehat{g}(\theta)$ is such that:*

$$\frac{1}{2}\left(e^{-A\theta} + e^{-S_2(0)\theta}\right)\left\{\widehat{f}\left(\frac{\theta^2}{2}\right) - C\|g\|_p\right\} \leq \tag{3.30}$$

$$\leq \widehat{g}(\theta) \leq \frac{1}{2}\left(e^{-a\theta} + e^{-S_2(0)\theta}\right)\left\{\widehat{f}\left(\frac{\theta^2}{2}\right) + C\|g\|_p\right\}\,,$$

where $C = (A - S_1(0))^{(p-1)/p} + (S_2(0) - b)^{(p-1)/p}$.

Remark 3.23. Of course, if $a \equiv S_1(t) \equiv A$ and $b \equiv S_2(t) \equiv B$, (3.30) reduces to (3.21). The constant C is small, for $S_1(0)$ close to A and b close to $S_2(0)$; if g is bounded, then $C = (A - S_1(0)) + (S_2(0) - b)$. Note that, the smaller $A - a$ and $B - b$ are, the better the approximation to the Laplace transform of g

is. Thus, the approximation given by Proposition 3.22 is particularly meaningful in the case when $S_1(t), S_2(t)$ have small variations compared to time t, namely, $\sup_{t\geq 0} S_i(t) - \inf_{t\geq 0} S_i(t)$ is small, for $i = 1, 2$. For instance, let $S_1(t) = s_1 + \epsilon \sin t$ and $S_2(t) = s_2 + \epsilon \sin t$, with $S_1(t) \leq S_2(t)$; if $a = s_1 - \epsilon$, $A = s_1 + \epsilon$, $b = s_2 - \epsilon$, $B = s_2 + \epsilon$, then $C = 2\epsilon^{(p-1)/p}$ and the pre-factors of $\{\ \}$ in (3.30) differ each other of a quantity of order 2ϵ.

3.3. **A few examples.** In this subsection, we report some explicit examples of solutions to the IFPT2 problem for single and double barrier (more details can be found in [6] and [7]).

3.3.1. *Single barrier.*

Example 3.1.

(i) (Feller process, see Example 2.1 in Section 2) We have $X(t) = (1/4)(B_t + 2\sqrt{\eta})^2$, so that $\tau_S = \inf\{t > 0 \ : \ B_t + 2\sqrt{\eta} \leq 2\sqrt{S(t)}\}$, The IFPT2 problem for $X(t)$, starting from the initial position $\eta \geq a^2/4$ and the quadratic boundary $S(t) = (1/4)(a + bt)^2$ $(a, b \geq 0)$, is reduced to the IFPT2 problem for BM starting from $\widetilde{\eta} = 2\sqrt{\eta}$ and the linear boundary $a + bt$.

For instance, if $a = 0$ and f is the exponential density with parameter $\lambda \leq b^2/2$, then, by using Proposition 3.4, the solution to the IFPT2 problem for the boundary $S(t) = (1/4)b^2t^2$ is

$$g_\eta(y) = \begin{cases} \dfrac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1}\dfrac{1}{\sqrt{y}}\left(e^{-2\lambda_1\sqrt{y}} - e^{-2\lambda_2\sqrt{y}}\right), & \text{if } b > \sqrt{2\lambda} \\ 4\lambda\sqrt{y}\, e^{-2\sqrt{2\lambda y}}, & \text{if } b = \sqrt{2\lambda} \end{cases} \qquad y > 0$$

(ii) (Wright & Fisher-like process, see Example 2.2 in Section 2). We have $X(t) = \sin^2\left((1/2)B_t + \arcsin\sqrt{\eta}\right)$, so that $\tau_S = \inf\{t > 0 \ : \ B_t + 2\arcsin(\sqrt{\eta}) \leq 2\arcsin(\sqrt{S(t)})\}$. The IFPT2 problem for $X(t)$ starting from $\eta \in [\sin^2(a/2), 1]$ and the boundary $S(t) = \sin^2\left((a/2) + (b/2)t\right)$ $(a, b \geq 0)$, is reduced to the IFPT2 problem for BM starting from $\widetilde{\eta} = 2\arcsin\sqrt{\eta}$ and the linear boundary $a + bt$.

For instance, if $S(t) = 0$ (i.e. $a = b = 0$) and f is the density with Laplace transform

$$\widehat{f}(\theta) = \frac{1 + e^{-\pi\sqrt{2\theta}}}{2(2\theta + 1)}$$

(note that, from the expression above it follows the moments of τ_s do not exist finite), then the solution to the IFPT2 problem is the uniform density in $[0, 1]$.

Example 3.2 (Geometric BM)**.** Let $X(t)$ be the solution of the SDE:

$$dX(t) = rX(t)dt + \sigma X(t)dB_t \quad , \quad X(0) = \eta > 0 \ ,$$

where r and σ are positive constant. This is a well-known equation in the framework of Mathematical Finance, when $X(t)$ represents the time evolution of a stock price. The explicit solution is $X(t) = \eta e^{\mu t} e^{\sigma B_t}$, where$\mu = r - \sigma^2/2$; since $\ln X(t) = \ln \eta + \mu t + \sigma B_t$, then $X(t)$ has a log-normal distribution. Now, let us consider the boundary $S(t) = e^{a+bt}$, for some $a \geq 0$, $b \geq \mu$, and suppose that $\eta \geq e^a$. Then, the IFPT2 problem for $X(t)$ with boundary S and FPT distribution F is reduced to the IFPT2 problem for BM, starting from $\ln(\eta/\sigma)$, with the linear boundary $(a/\sigma) + \left((b - \mu)/\sigma\right)t$, and the same FPT distribution F. Thus, by means of the results of Subsection 3.1, the density of $\ln(\eta/\sigma)$ is recovered and so that of η.

Example 3.3 (BM + large jumps)**.**
(i) For $a, \mu > 0$, let be

$$\widehat{f}(\theta) = \frac{a\mu\sqrt{2(\theta+\mu)} + \theta - \theta e^{-a\sqrt{2(\theta+\mu)}}}{a(\theta+\mu)\sqrt{2(\theta+\mu)})} \ .$$

Let $\overline{X}(t)$ be the process considered in the Proposition 3.10 and f the FPT density whose Laplace transform is given by the expression above; then, the solution to the IFPT2 problem, for $\overline{X}(t)$ and the boundary $S(t) = a$, is the uniform density g on the interval $[a, 2a]$.
(ii) For $c, \mu > 0$, let be

$$\widehat{f}(\theta) = \frac{c(\theta+\mu) + \mu\sqrt{2(\theta+\mu)}}{(\theta+\mu)(c+\sqrt{2(\theta+\mu)})} \ .$$

Consider the corresponding FPT density f; then, the solution to the IFPT2 problem for $\overline{X}(t)$ and the boundary $S(t) = a$, is $g(x) = ce^{-c(x-a)}$, $x \geq a$, i.e. $\eta = a + Z$, where Z is exponentially distributed with parameter c.

3.3.2. *Double barrier.*

Example 3.4 (BM)**.** Let us consider the FPT density:

$$f(t) = \frac{4}{(b-a)^2}\sum_{k=0}^{\infty} \exp\left[-2\left(k+\frac{1}{2}\right)^2 \pi^2 \frac{t}{(b-a)^2}\right] \ , \tag{3.31}$$

or the corresponding FPT Laplace transform:

$$\widehat{f}(\theta) = \frac{2}{(b-a)\sqrt{2\theta}} \frac{e^{-a\sqrt{2\theta}} - e^{-b\sqrt{2\theta}}}{e^{-a\sqrt{2\theta}} + e^{-b\sqrt{2\theta}}} \ . \tag{3.32}$$

Then, the solution g to the IFPT2 problem for B_t and the barriers a and b, is the uniform density in (a, b) (see [6] for details). If for instance $a = -\alpha = -b$, (3.31) and (3.32) become:

$$f(t) = \frac{1}{\alpha^2}\sum_{k=0}^{\infty} e^{-(k+1/2)^2\pi^2 t/2\alpha^2} \quad , \quad \widehat{f}(\theta) = \frac{\tanh(\alpha\sqrt{2\theta})}{\alpha\sqrt{2\theta}}$$

The graph of $f(t)$ is shown in the Figure 8, for $\alpha = 1$; notice that f is not an exponential density, because $f(0^+) = +\infty$. For the above FPT density f, the solution to the IFPT2 problem for BM and the barriers $-\alpha, \alpha$, is the uniform density in $(-\alpha, \alpha)$.

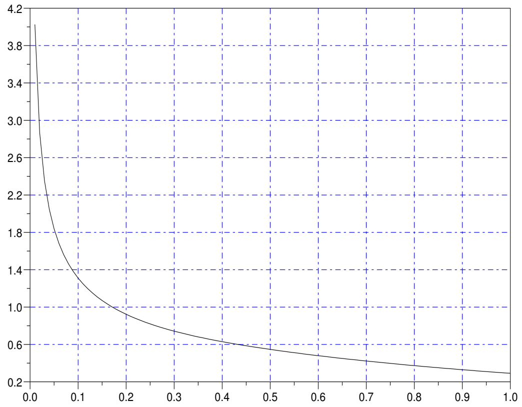

FIGURE 8. FPT density $f(t)$ of Example 3.4 for $a = -1$, $b = 1$.

Example 3.5 (BM)**.** Let us consider the FPT density $f(t)$ whose Laplace transform is

$$\widehat{f}(\theta) = \frac{3}{2\theta}\left(1 - \frac{\tanh\sqrt{2\theta}}{\sqrt{2\theta}}\right) .$$

Then, the solution g to the IFPT2 problem for B_t and the barriers -1 and 1, is the Beta-like density $g(x) = (3/4)(1 - x^2) \times \mathbf{1}_{[-1,1]}(x)$. Note that $g(x)$ is equal to $g_{2k}(x)$ of Proposition 3.17, for $k = 1$.

Remark 3.24. The FPT density given in Example 3.4 appears firstly in connection with the Wiener-Einstein process and it dates back to the papers of Bachelier ([18]) and Levy ([46]), successively considered again by Darling and Siegert ([26]).

Example 3.6. ($X(t)$ coniugated to BM). Let $X(t)$ be conjugated to BM via the function $v(x)$, and consider the FPT density

$$f(t) = \frac{4}{(v(b) - v(a))^2} \sum_{k=0}^{\infty} \exp\left[-2\left(k + \frac{1}{2}\right)^2 \pi^2 t/(v(b) - v(a))^2\right] . \tag{3.33}$$

Then, the solution to the IFPT2 problem for $X(t)$ and the barriers a and b is:

$$g(x) = \frac{v'(x)}{v(b) - v(a)} \times \mathbf{1}_{(a,b)}(x) . \tag{3.34}$$

Choosing some special processes for $X(t)$ we have the following results.

- Assume $X(t)$ be the Feller process starting from $\eta \geq 0$, which is conjugated to BM via the function $v(x) = 2\sqrt{x}$ (see Example 2.1 in Section 2). Consider the FPT density given by (3.33). Then, for $0 \leq a < b$, the solution g to the IFPT2 problem for $X(t)$ and the barriers a and b is:

$$g(x) = \frac{1}{2(\sqrt{b} - \sqrt{a})\sqrt{x}} \times \mathbf{1}_{(a,b)}(x) .$$

- Assume $X(t)$ be the Wright & Fisher-like process starting from $\eta \in [0,1]$, which is conjugated to BM via the function $v(x) = 2\arcsin\sqrt{x}$ (see Example 2.2 in Section 2). Consider again the FPT density given by (3.33). Then, for $0 \leq a < b \leq 1$, the solution g to the IFPT2 problem for X, and the barriers a and b is:
$$g(x) = \frac{1}{2(\arcsin\sqrt{b} - \arcsin\sqrt{a}\,)\sqrt{x(1-x)}} \times \mathbf{1}_{(a,b)}(x)\ .$$

Example 3.7 (BM + large jumps)**.** If
$$\widehat{\overline{f}}(\theta) = \frac{1}{\mu+\theta}\left[\frac{\theta\cdot\tanh(\alpha\sqrt{2(\mu+\theta)})}{\alpha\sqrt{2(\mu+\theta)}} + \mu\right]\ ,$$
then by Laplace inversion:
$$\overline{f}(t) = e^{-\mu t}\left[\sum_{k=0}^{\infty}\left(\frac{1}{\alpha^2} + \frac{2\mu}{(k+1/2)^2\pi^2}\right)\exp\left(-\frac{(k+1/2)^2\pi^2 t}{2\alpha^2}\right)\right]\ .$$
Assume $\overline{f}(t)$ as FPT density. Then, if $\overline{X}(t)$ is the process of Proposition 3.18, the even solution to the IFPT2 problem for $\overline{X}(t)$ and the barriers $-\alpha,\ \alpha$, is the uniform density $\overline{g}$ in $(-\alpha,\alpha)$ (see [6] for details).

Example 3.8 (BM with drift)**.** Let us consider the FPT density:
$$f(t) = \frac{2\pi^2}{(b-a)^2}\sum_{k=1}^{\infty}\frac{k^2\left(1+(-1)^{k+1}\cosh(\mu(b-a))\right)}{\mu^2(b-a)^2+k^2\pi^2}\cdot \cdot\exp\left(-\frac{k^2\pi^2 t}{2(b-a)^2} - \frac{\mu^2 t}{2}\right)\ , \tag{3.35}$$
whose Laplace transform is
$$\widehat{f}(\theta) = = \frac{1}{(b-a)\theta}\cdot\frac{\sqrt{\mu^2+2\theta}\left(\cosh((b-a)\sqrt{\mu^2+2\theta}\,) - \cosh(\mu(b-a))\right)}{\sinh((b-a)\sqrt{\mu^2+2\theta})}\quad ,\quad \theta \geq 0\ . \tag{3.36}$$
Then, the solution to the IFPT2 problem for $\eta + \mu t + B_t$ and the barriers a, b, is the uniform density in the interval (a,b) (see [6] for details).

4. THE IFPT2 PROBLEM FOR DIFFUSIONS WITH JUMPS FROM A BOUNDARY

In this section, we denote by $\widetilde{X}(t)$ the solution to a SDE such as (1.1). In particular, we assume $\widetilde{X}$ a non-negative diffusion driven by the SDE:
$$d\widetilde{X}(t) = b(\widetilde{X}(t))dt + \sigma(\widetilde{X}(t))dB_t\quad ,\quad \widetilde{X}(0) = x \geq 0 \tag{4.1}$$
with coefficients b and σ regular enough, as in Section 2. Let S be a barrier such that $S > x$; as usual, we denote by $\widetilde{\tau}_S(x) = \inf\{t > 0 : \widetilde{X}(t) > S|\widetilde{X}(0) = x\}$ the FPT of $\widetilde{X}(t)$ over S, with the condition that $\widetilde{X}(0) = x$.

We construct a new stochastic process $X(t)$ as follows. Starting from $x \in I = [0, +\infty)$ at time $t = 0$, we set $X(t) = \widetilde{X}(t)$ until the (random) time at which the boundary $c \geq 0$ is reached. Then the process $\widetilde{X}(t)$ is killed and a new process $X(t)$ is considered, making a random jump from c. Then, the new process $X(t)$ starts afresh (independently of the past history) from a point U which is a random variable with distribution ν, and then evolves according to the SDE (4.1), until the boundary c is reached again, and so on. This means that, if $X(t^-) = c$, then $X(t) = U$, where $P(U \leq u) = \nu(u)$. In the following, we take $c = 0$ and $U \geq 0$, for the sake of simplicity. Hitting the boundary $x = 0$ by the process $X(t)$ can be interpreted as the occurrence of a catastrophe, which consists in resetting the state of $X(t)$ to a new value, and this has various applications in Queueing theory (see e.g. [28]), as well as in Mathematical Finance and in Biology (see [8]). As an example, $X(t)$ can represent the number of individuals of a population at time t (for instance fishes in a little lake); whenever the population goes extinct (i.e. it reaches the level 0) a restocking is performed by a random amount.

Assume $S > 0$, with $U \in [0, S]$ and the jump distribution $\nu(u) = P(U \leq u)$ assigned; then, the FPT of $X(t)$ over S, when starting at $0 < x < S$, is:

$$\tau_S(x) = \tau_{S,\nu}(x) = \inf\{t > 0 \,:\, X(t) > S | X(0) = \widetilde{X}(0) = x\} \ . \tag{4.2}$$

The direct FPT problem for $X(t)$ was studied in [8]; here, we deal with the IFPT2 problem, analogous to that studied in Section 3 for diffusions without jumps:

for a given barrier $S > 0$ and $0 < x < S$, let F be a distribution on $[0, +\infty)$, then the IFPT2 problem consists in finding the jump distribution ν or its density g, if it exists, so that $\tau_S(x)$ has distribution F.

Notice that now the starting point $X(t)$ is fixed, while the jump distribution ν has to be found. We limit ourselves to consider the case when $\widetilde{X}(t)$ is a Wiener process, i.e. $\widetilde{X}(t) = x + \mu t + B_t$; then, we show some extensions to more general diffusions. In particular, if $\mu = 0$, $x = 0$, and F is a suitable exponential distribution, it turns out that there exists the solution g to the IFPT2 problem for $X(t)$ and the barrier S, and it is symmetric with respect to $S/2$ (see next Examples 4.1 and 4.2).

Let be $I = [0, +\infty)$ and $S > 0$ a constant, positive barrier and let us consider the process $X(t)$ in I with jumps from 0, obtained by $\widetilde{X}(t)$ as described above, where $\widetilde{X}(t)$ is the diffusion driven by the SDE (4.1), and killed at the boundary $\{x = 0\}$. We denote by $\mathcal{L}$, as always, the infinitesimal generator of $\widetilde{X}(t)$, and we assume that 0 and S are attainable boundaries for $\widetilde{X}(t)$.

The expectation and the moment generating function of the FPT of $X(t)$ over S were obtained explicitly in [8]. In particular, the following result holds.

Proposition 4.1. *For $x \in (0, S)$, let $f(t|x)$ be the FPT density of $X(t)$ over S, and for $\theta > 0$ denote by*

$$\widehat{f}(\theta|x) = \int_0^\infty f(t|x) e^{-\theta t}\, dt$$

its Laplace transform. Then:

$$\widehat{f}(\theta|x) = M_{x,S}(\theta) + M_{x,0}(\theta)\, \frac{\int_0^S M_{u,S}(\theta) g(u)\, du}{1 - \int_0^S M_{u,0}(\theta) g(u)\, du} \tag{4.3}$$

where g is the density of U; $M_{x,S}(\theta)$, as a function of x, is the solution of the problem

$$\begin{cases} \mathcal{L} z(x) = \theta\, z(x) & , \quad x \in (0, S) \\ z(0) = 0 & , \quad z(S) = 1 \end{cases} \tag{4.4}$$

$M_{x,0}(\theta)$, as a function of x, is the solution of the problem

$$\begin{cases} \mathcal{L}z(x) = \lambda z(x) & , \quad x \in (0, S) \\ z(0) = 1 & , \quad z(S) = 0 \end{cases} \tag{4.5}$$

□

The n-th moment of $\tau_S(x)$, if it exists finite, is obtained in terms of derivatives of $\widehat{f}(\theta|x)$ with respect to θ, i.e.:

$$E\left[(\tau_S(x))^n\right] = (-1)^n \frac{\partial}{\partial\theta} \widehat{f}(\theta|x)\Big|_{\theta=0} .$$

Remark 4.2. A suitable modification of the process $X(t)$ yields an asymmetric diffusion (see [10]). Note the similarity between (4.3) and formula (2.51) of [10].

4.1. **The case of the Wiener process.** By using equation (4.3), it is possible to study the IFPT2 problem for $X(t)$; for the sake of simplicity, we suppose that $\widetilde{X}(t) = x + \mu t + B_t$, with $\mu > 0$ and $S > x > 0$. For a given FPT distribution $F(t)$, denote by $f(t) = F'(t)$ the corresponding FPT density; thus, we aim to find the density g (if it exists) of the resetting initial value $U \in (0, S)$, after $X(t)$ hits 0, in such a way that $P(\tau_S(x) \leq t) = F(t)$. The following result holds (for the proof see [4]):

Theorem 4.3. *For $S > 0$, $\mu \geq 0$ and $S > x > 0$, let $\widetilde{X}(t) = x + \mu t + B_t$, and consider the associated process $X(t)$ with jumps from 0. Suppose that the FPT of $X(t)$ over S has an assigned probability density $f_\mu(t|x)$ and denote by $\widehat{f_\mu}(\theta|x) = \int_0^\infty e^{-\theta t} f_\mu(t|x)\, dt$, $\theta \geq 0$, the Laplace transform of f_μ. Then, if there exists a solution g to the IFPT2 problem for $X(t)$, its Laplace transform $\widehat{g}(\theta)$ must satisfy the equation:*

$$\widehat{f_\mu}(\theta|x) = \frac{e^{\mu(S-x)} \sinh(x\sqrt{\mu^2 + 2\theta}\,)}{\sinh(S\sqrt{\mu^2+2\theta}\,)} + \tag{4.6}$$

$$+ \frac{e^{-\mu x} \sinh((S-x)\sqrt{\mu^2+2\theta}\,)}{\sinh(S\sqrt{\mu^2+2\theta}\,)}\, \widehat{f_\mu}(\theta|0) ,$$

where

$$\widehat{f_\mu}(\theta|0) = \tag{4.7}$$

$$= \frac{e^{\mu S}[\widehat{g}(\mu - \sqrt{\mu^2+2\theta}) - \widehat{g}(\mu + \sqrt{\mu^2+2\theta})]}{2\sinh(S\sqrt{\mu^2+2\theta}) - e^{S\sqrt{\mu^2+2\theta}}\widehat{g}(\mu + \sqrt{\mu^2+2\theta}) + e^{-S\sqrt{\mu^2+2\theta}}\widehat{g}(\mu - \sqrt{\mu^2+2\theta}\,)}$$

is the Laplace transform of $f_\mu(t|x)$ for $x = 0$. Moreover, if $x = 0$ and $\mu = 0$, and we require that the density g is symmetric with respect to $S/2$, then formula (4.7) can be explicited and gives:

$$\widehat{g}(\theta|0) = \frac{\widehat{f_0}\left(\theta^2/2\,|\,0\right)\left(1 + e^{-S\theta}\right)}{1 + \widehat{f_0}\left(\theta^2/2\,|\,0\right)} . \tag{4.8}$$

Remark 4.4. As in the IFPT2 problem of Section 3, once $\widehat{g}$ has been found, such that (4.6) and (4.7) hold, the function $\widehat{g}$ could not be the Laplace transform of the density function of a random variable U with support $(0, S)$. In this case, a solution to the IFPT2 problem for $X(t)$ does not exist. In this case, solving the

IFPT2 problem is far stronger than the one proposed in Section 3, because the relation between $\widehat{g}$ and $\widehat{f}$ is strongly non linear.

By using the results of [8], compatibility conditions can be obtained involving the existence of a solution to the IFPT2 problem for drifted BM with jumps from 0 (see [4]).

Actually, it is possible to prove the existence of the density g of U for a class of FPT densities f. For the sake of simplicity, we limit ourselves to the case when $\mu = 0$, $x = 0$, $S = 1$, and g is required to be a function with support in $(0,1)$, which is symmetric with respect to the middle point $1/2$; in fact, for $\mu > 0$ the calculations involved are far more complicated.

For an integer $k \geq 0$, set $I_k(\theta) = \int_{-1}^{1} e^{-\theta x} x^k \, dx$; as easily seen, $I_0(\theta) = 2\sinh(\theta)/\theta$ and the recursive relation $I_k(\theta) = \big((-1)^k e^{\theta} - e^{-\theta}\big)/\theta + kI_{k-1}(\theta)/\theta$ allows us to calculate $I_k(\theta)$, for every k.

The following Proposition, proved in [4], gives a sufficient condition, for the existence of a solution g to the IFPT2 problem for the diffusion $X(t)$ with jumps from zero, associated to $\widetilde{X}(t) = B_t$ and the barrier $S = 1$.

Proposition 4.5. *Let $\widetilde{X}(t) = B_t$ and assume the Laplace transform of $f(t)$ is*

$$\widehat{f}(\theta) = \widehat{f}_{2k}(\theta) = \tag{4.9}$$

$$= \frac{(1 + 1/2k)e^{-\sqrt{\theta/2}} \left[\sqrt{2/\theta}\ \sinh(\sqrt{\theta/2}\,) - I_{2k}(\sqrt{\theta/2}\,) \right]}{1 + e^{\sqrt{2\theta}} - (1 + 1/2k)e^{-\sqrt{\theta/2}} \left[\sqrt{2/\theta}\ \sinh(\sqrt{\theta/2}\,) - I_{2k}(\sqrt{\theta/2}\,) \right]} ,$$

for some non-negative integer k. Then, the solution $g = g_{2k}$ of the IFPT2 problem for $X(t)$ exists for the barrier $S = 1$ and the FPT density f and is:

$$g_{2k}(u) = \left(1 + \frac{1}{2k}\right)\left(1 - (2u-1)^{2k}\right) \quad , \quad k \geq 0,\ u \in (0,1) . \tag{4.10}$$

Remark 4.6. A straightforward calculation shows that, if $U \in (0,1)$ has density g_{2k}, then $E(U^2) = (4k+5)/6(2k+3)$. By calculating the derivative of $\widehat{f}_{2k}(\theta)$ in (4.9) at $\theta = 0$, the FPT distribution corresponding to $\widehat{f}_{2k}$ has mean $E(\tau_1(0)) = 2(k+2)/3(2k+3)$.

4.2. **Diffusions conjugated to BM.** Assume that $\widetilde{X}(t)$ is conjugated to BM via the function v (see Definition 2.1 in Section 2), and consider the associated process $X(t)$ with jumps from 0. Notice that a random reflection of $\widetilde{X}(t)$ at zero corresponds to a random reflection of $B_t + v(x)$ at zero; moreover the first passage of $\widetilde{X}(t)$ through the barrier S, with $0 < S < a$, corresponds to the first passage of $B_t + v(x)$ through $v(S)$, and $\widetilde{\tau}_S(x) = \widetilde{\tau}_S^B(v(x))$, where the superscript B refers to BM. Furthermore, the density $g(u)$ of the position $U \in (0,S)$ from which $\widetilde{X}(t)$ starts afresh, after hitting zero, is related to the corresponding density $h(y)$ of the position $V = v(U) \in (0, v(S))$ from which $B_t + v(x)$ starts afresh, after hitting zero, by the equation $h(y) = g(v^{-1}(y))(v^{-1})'(y)$, $y \in (0, v(S))$. Therefore, the solution g to the IFPT2 problem for the process $X(t)$ associated to $\widetilde{X}$, relative to the FPT density f and the barrier S, can be written in terms of the solution h to the IFPT2 problem for the process associated to $v(x) + B_t$, relative to the FPT density f and the barrier $v(S)$, by using that $g(x) = h(v(x))v'(x)$.

Remark 4.7. More generally, a diffusion process $X(t)$ can be associated to $\widetilde{X}(t)$ with holding and jumps from 0, namely, every time $X(t)$ reaches the boundary 0, an exponentially distributed time holds, after which a random jump U occurs; the FPT problem for such a diffusion was studied in [55]. By using the results of [55] and the technique employed in [4], the IFPT2 problem can be studied also for a diffusion with holding and jumps from a boundary. We aim to study this problem in a future work.

4.3. **A few examples.** We report from [4] some explicit examples.

Example 4.1. Let $\widetilde{X}(t) = B_t$ and $S > 0$, and let $f(t)$ be the exponential density with parameter $\pi^2/4S^2$. The solution to the IFPT2 problem for $X(t)$ and S is $g(u) = (\pi/2S)\sin\big((\pi/S)u\big)$, $u \in (0, S)$.

By using the same calculations of Example 4.1, we get also:

Example 4.2. Let $\widetilde{X}(t) = B_t$ and let $f(t)$ be the exponential density with parameter λ; then the solution to the IFPT2 problem for $X(t)$ and $S = \pi/2\sqrt{\lambda}$ is $g(u) = \sqrt{\lambda}\,\sin(2u\sqrt{\lambda}\,)$, $u \in (0, S)$.

Example 4.3. The following example refers to the circumstance that the solution g is the uniform density over $(0, S)$. Let $\widetilde{X}(t) = B_t$ and $S > 0$, and let

$$\widehat{f}(\theta) = \frac{1 - e^{-S\sqrt{2\theta}}}{e^{-S\sqrt{2\theta}}\left(S\sqrt{2\theta} + 1\right) + S\sqrt{2\theta} - 1}\,.$$

Then, the solution to the IFPT2 problem for $X(t)$ and S is $g(u) = (1/S) \times \mathbf{1}_{(0,S)}(u)$, i.e. the uniform density over $(0, S)$. In the case $S = 1$, we can obtain the same result taking k goes to infinity in $\widehat{f}_{2k}$ and g_{2k} in Proposition 4.5. Moreover, the mean of the FPT distribution corresponding to $\widehat{f}$ is $1/3$, as follows by calculating $\lim_{k\to\infty} 2(k + 2)/3(2k + 3)$. (see Remark 4.6).

Example 4.4. The following example refers to the circumstance that the solution g is a Beta density. Let $\widetilde{X}(t) = B_t$ and $S > 0$, and let

$$\widehat{f}(\theta) = \frac{6\left(e^{-S\sqrt{2\theta}}\left(S\sqrt{2\theta} + 2\right) + S\sqrt{2\theta} - 2\right)}{S^3\theta^3\left(1 + e^{-S\sqrt{2\theta}}\right) - 6\left(e^{-S\sqrt{2\theta}}\left(S\sqrt{2\theta} + 2\right) + S\sqrt{2\theta} - 2\right)}\,.$$

Then, the solution to the IFPT2 problem for $X(t)$ and S is $g(u) = (6/S^3)\,u(S - u)$, $u \in (0, S)$. Notice that, for $S = 1$ the function g is the density g_{2k} of Proposition 4.5 for $k = 1$.

Example 4.5. The following example refers to the circumstance that the solution g is the Triangular Density in $(0, 1)$. Let $\widetilde{X}(t) = B_t$ and $S > 0$, and let

$$\widehat{f}(\theta) = \frac{2\left(1 - e^{\sqrt{\theta/2}}\right)^2}{\theta\left(1 + e^{\sqrt{2\theta}}\right) - 2\left(1 - e^{\sqrt{\theta/2}}\right)^2}$$

Then, the solution to the IFPT2 problem for $X(t)$ and S is the Triangular Density in $(0, 1)$, i.e. $g(u) = 4\,u \times \mathbf{1}_{(0,\,1/2]}(u) + 4\,(1 - u) \times \mathbf{1}_{(1/2,\,1)}(u)$.

Example 4.6. Let $\widetilde{X}(t) = B_t + \mu t$ and let

$$\widehat{f}(\theta) =$$

$$= \frac{\sqrt{\mu^2+2\theta}\, e^{\mu S} - \sqrt{\mu^2+2\theta}\, \cosh\left(S\sqrt{\mu^2+2\theta}\right) - \mu \sinh\left(S\sqrt{\mu^2+2\theta}\right)}{\sqrt{\mu^2+2\theta}\, \cosh\left(S\sqrt{\mu^2+2\theta}\right) - (2\theta S+\mu)\sinh\left(S\sqrt{\mu^2+2\theta}\right) - \sqrt{\mu^2+2\theta}\, e^{-S\mu}} \; .$$

Then, the solution to the IFPT2 problem for $X(t)$ and $S > 0$ is the uniform density in $(0, S)$. Notice that, for $\mu = 0$, $\widehat{f}(\theta)$ becomes that of Example 4.3.

Example 4.7. If $\widetilde{X}(t)$ is conjugated to BM (see Definition 2.1 in Section 2), examples of solutions to the IFPT2 problem for$X(t)$ with $x = 0$, can be easily derived from Examples 4.1 to 4.5. For instance, two diffusions $\widetilde{X}(t)$ of this kind are:

(i) the Feller process, with $b = 0$ and $a = 1/4$;
(ii) the Wright & Fisher-like process (see Section 2).

Example 4.8. The Ornstein-Uhlenbeck process (OU). Let $\widetilde{X}(t)$ be the solution of the SDE:

$$d\widetilde{X}(t) = -\mu \widetilde{X}(t)dt + \sigma dB_t \quad , \quad \widetilde{X}(0) = x \; ,$$

where μ, σ are positive constants.
The explicit solution is $\widetilde{X}(t) = e^{-\mu t}\left(x + \int_0^t \sigma e^{\mu s}\, dB_s\right)$. By using a time-change, we have $\widetilde{X}(t) = e^{-\mu t}\left(x + B(\rho(t))\right)$, where $\rho(t) = (\sigma^2/2\mu)\left(e^{2\mu t} - 1\right)$.
If $S(t)$ is a moving barrier, the FPT of $\widetilde{X}(t)$ over $S(t)$ is $\widetilde{\tau}_{S(t)} = \inf\{t > 0 : x + B(\rho(t)) \geq e^{\mu t} S(t)\}$ and so $\rho(\widetilde{\tau}_{S(t)}) = \inf\{u > 0 : x + B_u \geq \widetilde{S}(u)\}$, where $\widetilde{S}(u) = e^{\mu \rho^{-1}(u)} S(\rho^{-1}(u))$. Therefore, if $S(t) = S_0 e^{-\mu t}$, the IFPT2 problem for the associated process $X(t)$, moving barrier S and FPT distribution F, is reduced to the IFPT2 problem for BM starting from $X(t)$ and with constant barrier S_0 and FPT distribution $\widetilde{F} = F \circ \rho^{-1}$. Thus, if $x = 0$, explicit examples for the OU process and the exponential barrier $S(t) = S_0 e^{-\mu t}$ can be easily derived from Examples 4.1 to 4.5.

Example 4.9. The Geometric BM. Let $\widetilde{X}(t)$ be the Geometric BM starting from x (see Example 3.2 in Section 3), that is $\widetilde{X}(t) = x e^{\mu t} e^{\sigma B_t}$. Let us consider the moving barrier $S(t) = e^{\sigma S + \mu' t}$. Then, the IFPT2 problem for the associated process $X(t)$, for the boundary $S(t)$ and the FPT distribution F, is reduced to the IFPT2 problem for BM with drift $(\mu - \mu')/\sigma$, starting from $\ln x/\sigma$, for a constant boundary S, and the same FPT distribution F.

5. THE IFPT2 PROBLEM FOR REFLECTED DIFFUSIONS WITH TWO BOUNDARIES

Reflected diffusion processes with one or two boundaries play an important role in a variety of applications ranging from Economics, Finance, Queueing, and Mathematical Biology. As far as Economics and Finance are concerned, see e.g. [19], [21], [27], [42], [73]; for applications in Economics and Insurance, see e.g. [59], [74]. As for Queueing theory, diffusions with reflecting boundaries arise as heavy-traffic approximations of queueing systems (see e.g. [1], [2], [34] for reflected BM, and [71], [76], [77] for reflected Ornstein-Uhlenbeck (OU) process). Reflected OU processes appear also in certain models from Mathematical Biology (see e.g. [63]).

For further applications, see e.g. [48] and references therein. Although FPT problems have been studied mostly for ordinary diffusions, that is without reflecting, more recently (see e.g. [23], [47], [59]) some results appeared about the FPT of a one-dimensional reflected diffusion, through a threshold S.

In this section, we focus on the IFPT2 problem for a one-dimensional, temporally homogeneous reflected diffusion process $X(t)$ with boundaries a and b, which is the solution of the stochastic differential equation with reflecting boundaries (SDER):

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))dB_t + dL_t - dU_t \ , \\ X(0) = \eta \in [a,b] \ , \end{cases} \tag{5.1}$$

where B_t is standard BM, the initial position η is a random variable, independent of B_t, $L = \{L_t\}$ and $U = \{U_t\}$, $t \geq 0$, are the *regulators* of points a and b, respectively, namely the local times of $X(t)$ at a and b. The processes L and U are uniquely determined by the following properties (see e.g. [34]):

(i) both L_t and U_t are continuous nondecreasing processes with $L_0 = U_0 = 0$;
(ii) $X(t) \in [a,b]$ for every time $t \geq 0$;
(iii) L and U increase only when $X = a$ and $X = b$, respectively, that is, for $t \geq 0$, $\int_0^t \mathbf{1}_{\{X(s)=a\}}\, dL_s = L_t$ and $\int_0^t \mathbf{1}_{\{X(s)=b\}}\, dU_s = U_t$.

Under certain mild regularity conditions on the coefficients b and σ (see e.g. [51]), for fixed initial value, the SDER (5.1) has a unique strong solution $X(t)$ which remains in the interval $[a,b]$ for every time $t \geq 0$. For this reason, $X(t)$ is also called a *regulated diffusion* between a and b.

If $S \in [a,b]$ is a threshold such that $P(a \leq \eta \leq S) = 1$, we consider the FPT of $X(t)$ through S:

$$\tau_S = \inf\{t > 0 \, : \, X(t) = S\} \tag{5.2}$$

and, as before, we denote by $\tau_S(x) = \inf\{t > 0 \, : \, X(t) = S | \eta = x\}$ the FPT of $X(t)$ through S with the condition that $\eta = x$. We assume that $\forall x \in [a,S]$, $\tau_S(x)$ is finite with probability one and that it possesses a density $f(t|x)$. For a given distribution F, we aim to solve the following IFPT2 problem for X:

to find the density g of η (if it exists) for which it results $P(\tau_S \leq t) = F(t)$.

The IFPT2 problem for reflected diffusions has interesting applications in Mathematical Finance, in particular in credit risk modeling, where the FPT represents a default event of an obligor (see e.g. [38]), in Biology, specially in the framework of diffusion models for neural activity (see e.g. [44]), and in Queueing theory (see e.g. [1], [2], [34]).

As always, we suppose that drift $b(\cdot)$ and infinitesimal variance $\sigma^2(\cdot)$ in (5.1) are sufficiently regular functions (see [51]), in order to guarantee the existence and the uniqueness of the strong solution, for fixed initial condition. We denote by $\mathcal{L}$ the infinitesimal generator of the process X, acting on C^2-functions on (a,b) (see (2.8)).

Let $S \in [a,b]$ a given barrier; if $x \in [a,S]$, we denote by $\tau_{x\uparrow S}(x) = \inf\{t > 0 \, : \, X(t) = S | X(0) = x\}$ the first-hitting time of $X(t)$ to S, with the condition $X(0) = x$, namely the FPT of $X(t)$ through S, "from below". In analogous manner, if $x \in [S,b]$, we denote by $\tau_{x\downarrow S}(x) = \inf\{t > 0 \, : \, X(t) = S | X(0) = x\}$ the FPT of $X(t)$ through S, "from above". To simplify notations, from now on, we will denote $\tau_{x\uparrow S}(x)$ by $\tau_S(x)$ and $\tau_{x\downarrow S}(x)$ by $\widetilde{\tau}_S(x)$. We will suppose that $\tau_S(x)$ and $\widetilde{\tau}_S(x)$ are a.s. finite, for every fixed x.

In the following, we review some results concerning the direct first-hitting problem for X from [23].

5.1. The Laplace transform and the moments of the first-hitting time of $X(t)$ to S.

Theorem 5.1 ([23]). *Let $X(t)$ be the solution of the SDER (5.1) with deterministic and fixed initial condition $X(0)=x$ and let $S \in [a,b]$. For $x \in [a,S]$ and $\theta \geq 0$, suppose that $u(x)=u_\theta(x)$ is the solution of the following equation:*

$$\begin{cases} \mathcal{L}u(x) = \theta u(x) \quad , \quad x \in (a,S) \; , \\ u'(a) = 0 \; . \end{cases} \tag{5.3}$$

Then, if $u(S) \neq 0$ for $S \in [x,b]$, the Laplace transform of $\tau_S(x)$ is explicitly given by:

$$E\left(e^{-\theta \tau_S(x)}\right) = \frac{u(x)}{u(S)} \; . \tag{5.4}$$

Analogously, for $x \in [S,b]$ and $\theta \geq 0$, let $v(x)=v_\theta(x)$ be the solution of the problem:

$$\begin{cases} \mathcal{L}v(x) = \theta v(x) \quad , \quad x \in (S,b) \; , \\ v'(b) = 0 \, . \end{cases} \tag{5.5}$$

Then, if $v(S) \neq 0$ for $S \in [a,x]$, the Laplace transform of $\widetilde{\tau}_S(x)$ is explicitly given by:

$$E\left(e^{-\theta \widetilde{\tau}_S(x)}\right) = \frac{v(x)}{v(S)} \; . \tag{5.6}$$

Let us consider e.g. $\tau_S(x)$, namely the FPT from below with the condition $X(0)=x \in [a,S]$, and denote by $M(\theta,x)=u(x)/u(S)$ the Laplace transform of $\tau_S(x)$ and by $T_n(x)=E[(\tau_S(x))^n]$ its moment of order n, $(n=1,2,\cdots)$; as well-known,

$$T_n(x) = (-1)^n \, \frac{\partial^n}{\partial \theta^n} M(\theta,x) \Big|_{\theta=0} \; ,$$

if it exists finite. For fixed $\theta \geq 0$, one has $u(x)=u(S)M(\theta,x)$; so the problem (5.3) can be written as:

$$\begin{cases} \mathcal{L}M(\theta,x) = \theta M(\theta,x) \quad , \quad x \in (a,S) \; , \\ \dfrac{\partial}{\partial x} M(\theta,x) \Big|_{x=a} = 0 \end{cases} \tag{5.7}$$

where the operator $\mathcal{L}$ acts on M only as a function of x. Therefore, by taking the n-th derivative of $M(\theta,x)$ with respect to θ in both members of equation (5.7), and evaluating this derivative at $\theta=0$, the following proposition holds.

Proposition 5.2. *For $n=1,2,\cdots$, the n-th order moments of $\tau_S(x)$, if existing, is the solution to the problem:*

$$\begin{cases} \mathcal{L}T_n(x) = -nT_{n-1}(x) \quad , \quad x \in (a,S) \; , \\ T_n(S) = 0 \; , \; T_n'(a) = 0 \end{cases} \tag{5.8}$$

where $T_0(x) \equiv 1$.

Analogous equations hold for the n-th order moments of $\widetilde{\tau}_S(x)$, say $\widetilde{T}_n(x)$, but the equations hold for $x \in (S, b)$ and the second boundary conditions have to be replaced with $\widetilde{T}'_n(b) = 0$.

It is possible to derive the explicit solutions of problems (5.3) and (5.8), for the Laplace transform and the moments of $\tau_S(x)$, in the case of reflected BM with drift μ, that is the diffusion $X^{(\mu)}$ with reflecting boundaries a and b, having infinitesimal generator $\mathcal{L}^{(\mu)}\phi(x) = \mu\phi'(x) + (1/2)\phi''(x)$, $\phi \in C^2$ (see [3]). For $x \in [a, S]$, denote by $\tau_S^{(\mu)}(x)$ the FPT of $X^{(\mu)}$ through S from below, and by $T_n^{(\mu)}(x)$ its n-th moment. The Laplace transform $M^{(\mu)}(\theta, x) := E\left(e^{-\theta\tau_S^{(\mu)}(x)}\right)$ of $\tau_S^{(\mu)}(x)$ is:

$$M^{(\mu)}(\theta, x) := e^{-(S-x)(\sqrt{\mu^2+2\theta}-\mu)} \times \tag{5.9}$$
$$\times \frac{\theta e^{-2(x-a)\sqrt{\mu^2+2\theta}} + \mu^2 + \theta + \mu\sqrt{\mu^2+2\theta}}{\theta e^{-2(S-a)\sqrt{\mu^2+2\theta}} + \mu^2 + \theta + \mu\sqrt{\mu^2+2\theta}} \, .$$

For $a \to -\infty$ the right-hand member of (5.9) tends to $e^{-(S-x)(\sqrt{\mu^2+2\theta}-\mu)}$, which is the well-known expression of the Laplace transform of the first-hitting time of ordinary BM with drift μ through S, when starting from $x < S$. Taking the limit as μ goes to zero in (5.9), we obtain:

$$M^{(0)}(\theta, x) = E\left(e^{-\theta\tau_S^{(0)}(x)}\right) = \frac{e^{-x\sqrt{2\theta}} + e^{-(2a-x)\sqrt{2\theta}}}{e^{-S\sqrt{2\theta}} + e^{-(2a-S)\sqrt{2\theta}}} \, . \tag{5.10}$$

In the special case $a = 0$, the expression above writes:

$$\frac{e^{-x\sqrt{2\theta}} + e^{x\sqrt{2\theta}}}{e^{-S\sqrt{2\theta}} + e^{S\sqrt{2\theta}}} = \frac{\cosh(x\sqrt{2\theta}\,)}{\cosh(S\sqrt{2\theta}\,)} \quad , \quad x \in [0, S] \, . \tag{5.11}$$

Then, the Laplace transform inversion yields

$$f^{(0)}(t|x) = \frac{\pi}{S^2} \sum_{k=0}^{\infty} (-1)^k \left(k + \frac{1}{2}\right) \cos\left[\left(k + \frac{1}{2}\right)\frac{\pi x}{S}\right] \cdot \tag{5.12}$$
$$\cdot \exp\left[-\left(k + \frac{1}{2}\right)^2 \frac{\pi^2 t}{2S^2}\right] \quad , \quad t \geq 0$$

which is the density of $\tau_S^{(0)}(x)$ for $a = 0$ and $x \in [0, S]$ (cf. [26], [59]).

By solving (5.8) with $n = 1$, we obtain the mean of $\tau_S^{(\mu)}(x)$:

$$T_1^{(\mu)}(x) = \frac{1}{2\mu^2}\left[e^{2\mu(a-S)} - e^{2\mu(a-x)}\right] + \frac{S-x}{\mu} \quad , \quad x \in [a, S] \, . \tag{5.13}$$

For μ going to zero, we obtain:

$$T_1^{(0)}(x) = -x^2 + 2ax + S(S - 2a) \quad , \quad x \in [a, S] \, . \tag{5.14}$$

As for the second order moment, by solving (5.8) with $n = 2$, we get:

$$T_2^{(\mu)}(x) = \frac{x^2}{\mu^2} - \frac{x}{\mu^3}\left(e^{2\mu(a-S)} + 1 + 2S\mu + e^{2\mu(a-x)}\right) + c_1 + c_2 e^{-2\mu x} \, , \tag{5.15}$$

where $c_2 = (e^{2\mu a}/2\mu^4)\left[4a\mu - e^{2\mu(a-S)} - 2 - 2S\mu\right]$ and $c_1 = -c_2 e^{-2\mu S} +$ $+ (S/\mu^3)\left[2e^{2\mu(a-S)} + 1 + S\mu\right]$.

For $\mu = 0$, we obtain $T_2^{(0)}(x) = (x^4/3) - (4/3)ax^3 - 2S(S-2a)x^2 + Ax + B$, $x \in [a,S]$, where $A = (8/3)a^3 + 4aS(S-2a)$, $B = (5/3)S^4 - (20/3)aS^3 + 8S^2a^2 - (8/3)a^3S$. In particular, for $a = 0$, we get $T_1^{(0)}(x) = -x^2 + S^2$, $T_2^{(0)}(x) = (x^4/3) - 2S^2x^2 + (5/3)S^4$, and so the variance of $\tau_S^{(0)}(x)$ is $\mathrm{Var}(\tau_S^{(0)}(x)) = (2/3)\left(S^4 - x^4\right)$, $x \in [0,S]$.

Explicit formulae for the Laplace transform of the first-hitting time to a barrier S are known also for reflected OU processes, reflected Bessel processes and some other processes (see [23], [47]), but they involve special functions. An explicit spectral representation of the hitting time density was found in [59] for reflected BM, and in [49], [50] for Cox-Ingersoll-Ross (CIR) and OU processes.

5.2. **The IFPT2 problem for reflected BM with drift.** For a given barrier $S \in [a,b]$, and $X(0) = \eta \in [a,S]$, suppose that $\tau_S(x)$ (i.e. the FPT conditional to $\eta = x$) is a.s. finite for every $x \in [a,S]$, and it possesses a density $f(t|x)$. Moreover, we suppose that the initial position η has a density $g(x)$ with support (a,S); for $\theta \geq 0$ we denote by $\widehat{f}(\theta|x) = \int_0^{+\infty} e^{-\theta x} f(t|x)\,dt$ the Laplace transform of $f(t|x)$ and by $\widehat{g}(\theta) = \int_a^S e^{-\theta x} g(x)\,dx$ the (possibly bilateral) Laplace transform of g. Then, the density of τ_S is obtained as $f(t) = \int_a^S f(t|x)g(x)\,dx$ and taking the Laplace transform on both sides we get, as in the preceding sections:

$$\widehat{f}(\theta) = \int_a^S \widehat{f}(\theta|x) g(x)\,dx \; . \tag{5.16}$$

Now, we go to solve the IFPT2 problem, in the case when $X(t) = X^{(\mu)}(t)$ is BM with drift μ, reflected between the boundaries a and b.

For a given FPT distribution function F (or equivalently for a given FPT density $f = F'$) our aim is to find the density g of the random initial position η, if it exists, such that $P(\tau_S \leq t) = F(t)$. The following results holds (see [3]):

Theorem 5.3. *For $S \in [a,b]$, let $X^{(\mu)}(t)$ be BM with drift μ, reflected between the boundaries a and b and starting from the random position $\eta \in [a,S]$. Assume that the FPT of $X^{(\mu)}$ through S from below has an assigned probability density f and denote by $\widehat{f}(\theta) = \int_0^\infty e^{-\theta t} f(t)\,dt$, $\theta \geq 0$, the Laplace transform of f. Then, if there exists a solution g to the IFPT2 problem for $X^{(\mu)}(t)$, its Laplace transform $\widehat{g}$ must satisfy the equation:*

$$\begin{aligned} \widehat{f}(\theta) = \Big[& \theta e^{2a\sqrt{\mu^2+2\theta}}\, \widehat{g}\left(\sqrt{\mu^2+2\theta} + \mu\right) + \\ & + \left(\mu^2 + \theta + \mu\sqrt{\mu^2+2\theta}\right) \widehat{g}\left(\mu - \sqrt{\mu^2+2\theta}\right)\Big] \Big[\theta e^{-S\left(\sqrt{\mu^2+2\theta}-\mu\right)} e^{2a\sqrt{\mu^2+2\theta}} + \\ & + \left(\mu^2 + \theta + \mu\sqrt{\mu^2+2\theta}\right) e^{S\left(\sqrt{\mu^2+2\theta}+\mu\right)}\Big]^{-1} \quad , \quad \theta \geq 0 \; . \end{aligned} \tag{5.17}$$

In particular, if $\mu = 0$, the above formula takes the form:

$$\widehat{f}(\theta) = \frac{\widehat{g}(\sqrt{2\theta}) + \widehat{g}(-\sqrt{2\theta})e^{-2a\sqrt{2\theta}}}{e^{-S\sqrt{2\theta}} + e^{(S-2a)\sqrt{2\theta}}} \quad , \quad \theta \geq 0 \; . \tag{5.18}$$

Furthermore, if $\mu = 0$ *and if the density* g *is required to be symmetric with respect to* $(a+S)/2$*, then:*

$$\widehat{g}(\theta) = \frac{e^{-S\theta} + e^{-(2a-S)\theta}}{1 + e^{(S-a)\theta}} \, \widehat{f}\left(\frac{\theta^2}{2}\right) \quad , \quad \theta \geq 0 \, . \tag{5.19}$$

Proof. By using (5.9) with $M^{(\mu)}(\theta, x) = \widehat{f}(\theta|x)$, we have:

$$\widehat{f}(\theta) = \int_a^S \widehat{f}(\theta|x) g(x)\, dx =$$

$$= e^{-S(\sqrt{\mu^2+2\theta}-\mu)} \left[\theta e^{-2(S-a)\sqrt{\mu^2+2\theta}} + \mu^2 + \theta + \mu\sqrt{\mu^2+2\theta}\right]^{-1} \times$$

$$\times \int_a^S e^{x(\sqrt{\mu^2+2\theta}-\mu)} \left[\theta e^{-2(x-a)\sqrt{\mu^2+2\theta}} + \mu^2 + \theta + \mu\sqrt{\mu^2+2\theta}\right] g(x)\, dx \, .$$

The integral can be written as

$$\int_a^S \left[\theta e^{-x(\sqrt{\mu^2+2\theta}+\mu)} e^{2a\sqrt{\mu^2+2\theta}} + e^{x(\sqrt{\mu^2+2\theta}-\mu)} \left(\mu^2 + \theta + \mu\sqrt{\mu^2+2\theta}\right)\right] g(x)\, dx =$$

$$= \theta e^{2a\sqrt{\mu^2+2\theta}} \widehat{g}(\sqrt{\mu^2+2\theta}+\mu) + \left(\mu^2 + \theta + \mu\sqrt{\mu^2+2\theta}\right) \widehat{g}(\sqrt{\mu^2+2\theta}-\mu) \, .$$

Thus, by replacing this last equality in the formula above, (5.17) follows, after some manipulations. Formula (5.18) is then obtained, by taking $\mu = 0$. Moreover, if one seeks that the density g is symmetric with respect to $(a+S)/2$, namely $\widehat{g}(-\theta) = e^{(a+S)\theta}\widehat{g}(\theta)$, formula (5.18) can be explicited with respect to the Laplace transform of g, and (5.19) follows.

□

Remark 5.4. Notice that, as in the IFPT2 problems of Sections 3 and 4, the function $\widehat{g}$ may not be the Laplace transform of some probability density function. In this case the IFPT2 problem has no solution. If we replace reflected drifted BM with a more general reflected diffusion, we still obtain (5.16). However, even if the explicit form of the Laplace transform $\widehat{f}(\theta|x)$ of the conditional FPT density $f(t|x)$ is available, in general the expression (5.16) cannot be related to the Laplace transform of g, calculated in some point (this happens for reflected drifted BM, thanks to the particular form of the Laplace transform of the FPT, which depends on $X(t)$ only by means of exponentials of linear functions of $X(t)$ (see (5.9)).

If $\widehat{f}(\theta)$ is analytic in a neighbor of $\theta = 0$, then the k-th order moment of τ_S exists finite and such that $E(\tau_S^k) = (-1)^k \, (\partial^k/\partial\theta^k)\widehat{f}(\theta)\Big|_{\theta=0}$. The same thing holds for the moments of η, if $\widehat{g}(\theta)$ is analytic. Compatibility conditions were found in [3] which are necessary for the existence of the solution to the IFPT2 problem, in the case of reflected drifted BM.

Remark 5.5. For regulated drifted BM $X^{(\mu)}(t)$, starting from $\eta \in [a, S]$, we have considered the FPT through S from below. In analogous way, if $\eta \in [S, b]$, one can consider $\widetilde{\tau}_S^{(\mu)}$, namely the FPT of $X^{(\mu)}$ through S from above. By using the same arguments of Theorem 5.3, and (5.6), it is possible to write the Laplace transform of $\widetilde{\tau}_S^{(\mu)}$ in terms of the Laplace transform of η, if it exists; for instance, if $\mu = 0$, and we suppose that the density of η is symmetric with respect to $(S+b)/2$, then the solution g to the IFPT2 problem from above has the following Laplace transform (cf. with (5.19)):

$$\widehat{g}(\theta) = \frac{e^{-S\theta} + e^{-(2b-S)\theta}}{1 + e^{(S-b)\theta}} \, \widehat{f}\left(\frac{\theta^2}{2}\right) \quad , \quad \theta \geq 0 \, . \tag{5.20}$$

Throughout the rest of this section, for IFPT2 problem we will mean the problem concerning the FPT from below, namely, τ_S.

Remark 5.6. Let $X(t)$ be a regulated BM. If τ_S has Gamma distribution, there is no hope that there exists the solution g to the IFPT2 problem, with g symmetric with respect to $(a+S)/2$. In fact, suppose that $f(t)$ is a Gamma density with some parameters $\alpha, \lambda > 0$, namely $\widehat{f}(\theta) = \big(\lambda/(\theta+\lambda)\big)^{\alpha}$ and take $a = 0$ and $S = 1$, for the sake of simplicity. Then, inserting this expression of $\widehat{f}(\theta)$ into (5.19), and calculating the second derivative at zero of the candidate $\widehat{g}$, we see that it is negative, so that $\widehat{g}$ is not the Laplace transform of any density g of η, since it should be $E(\eta^2) = \widehat{g}''(0) \leq 0$, which is impossible.

Now, we further investigate the question of the existence of solutions to the IFPT2 problem. Referring to regulated drifted BM, we will prove the existence of the density g of the initial position $\eta \in [a, S]$ for a class of FPT densities f. For the sake of simplicity, we limit ourselves to the case when $\mu = 0$, $a = 0$, $S = 1 < b$ and g is required to be symmetric with respect to $1/2$. In fact, for $\mu \neq 0$ the calculations involved are far more complicated.

For any integer $k \geq 0$, set $I_k(\theta) = \int_{-1}^{1} e^{-\theta x} x^k \, dx$; as easily seen, $I_0(\theta) = 2\sinh(\theta)/\theta$ and the recursive relation $I_k(\theta) = \big((-1)^k e^{\theta} - e^{-\theta}\big)/\theta + (k/\theta) I_{k-1}(\theta)$ allows to calculate $I_k(\theta)$, for every k.

The following Proposition gives a sufficient condition, in order that there exists the solution to the IFPT2 problem for regulated BM (cf. Proposition 4.5 in Section 4).

Proposition 5.7. *Let $X(t)$ be a regulated BM between the boundaries 0 and b, and let $S = 1 < b$; suppose that the Laplace transform of $f(t)$ has the form:*

$$\widehat{f}(\theta) = \widehat{f}_{2k}(\theta) := \frac{\cosh(\sqrt{\theta/2})}{\cosh(\sqrt{2\theta})} \left(1 + \frac{1}{2k}\right) \times \tag{5.21}$$

$$\times \left[\sqrt{\frac{2}{\theta}} \sinh\left(\sqrt{\frac{\theta}{2}}\right) - I_{2k}\left(\sqrt{\frac{\theta}{2}}\right) \right] \, ,$$

for some integer $k > 0$. Then, there exists the solution $g = g_{2k}$ of the IFPT problem for $X(t)$, for the FPT density f such that

$$g_{2k}(x) = \left(1 + \frac{1}{2k}\right)\left(1 - (2x-1)^{2k}\right) \quad , \quad x \in (0,1) \, . \tag{5.22}$$

Proof. An easy calculation shows that

$$\widehat{g}_{2k}(\theta) = \left(1 + \frac{1}{2k}\right) e^{-\theta/2} \left[\frac{2}{\theta} \, \sinh\left(\frac{\theta}{2}\right) - I_{2k}\left(\frac{\theta}{2}\right) \right] \, .$$

Since g_{2k} is symmetric with respect to $1/2$, the result follows by inserting $\widehat{g}_{2k}$ into (5.19).

□

Remark 5.8. A straightforward calculation shows that, if $\eta \in (0,1)$ has density g_{2k}, then $E(\eta^2) = (4k+5)/6(2k+3)$. By recalling that $E(\tau_S^{(0)}) = -E(\eta^2) + 2aE(\eta) + S(S-2a)$ (see [3]), and inserting $E(\eta^2)$ into the last expression, with $a = 0$, $S = 1$, we obtain that the FPT-distribution corresponding to $\widehat{f}_{2k}$ has mean $E(\tau_1^{(0)}) = (8k+13)/6(2k+3)$.

Regulated BM + large jumps. In analogous manner, as done in Subsection 3.1, we consider now the piecewise-continuous process of Proposition 3.10, obtained by superimposing to a BM a jump process, with large jumps, and we denote by $\overline{X}(t)$ the reflected diffusion $\overline{X}(t)$ with boundaries a, b, associated to it. Then, for $\eta \in [a,S]$ the FPT of $\overline{X}(t)$ over S is $\overline{\tau}_S = \inf\{t > 0 : \overline{X}(t) \geq S\}$. By recalling formulae (3.12), (3.13) and (3.14), from the equation $\widehat{\overline{f}}(\theta) = \int_a^S \widehat{\overline{f}}(\theta|x)\overline{g}(x)\,dx$, where $\overline{f}$ is the FPT density of $\overline{X}(t)$, and $\overline{g}$ is the density of η, we get $\widehat{\overline{f}}$. Here, for the sake of simplicity, we limit ourselves to the case when $a = 0$ and the density $\overline{g}$ is symmetric with respect to $S/2$. Then, by using Theorem 5.3, we get:

$$\widehat{\overline{f}}(\theta) = \frac{1}{\mu+\theta}\left[\theta \times \frac{\widehat{\overline{g}}\left(\sqrt{2(\mu+\theta)}\right)\left(1+e^{S\sqrt{2(\mu+\theta)}}\right)}{2\cosh\left(S\sqrt{2(\mu+\theta)}\right)} + \mu\right].$$

Thus, by solving with respect to $\widehat{\overline{g}}(\theta)$, the following result holds:

Proposition 5.9. *For $a = 0 < S < b$, if there exists a function $\overline{g}$, symmetric with respect to $S/2$, which is the solution to the IFPT2 problem of $\overline{X}$, relative to S and the FPT density $\overline{f}$, then its Laplace transform is given by:*

$$\widehat{\overline{g}}(\theta) = \frac{2\cosh(S\theta)}{(\theta^2/2-\mu)(1+e^{S\theta})}\left[\frac{\theta^2}{2}\,\widehat{\overline{f}}\left(\frac{\theta^2}{2}-\mu\right)-\mu\right]. \tag{5.23}$$

Remark 5.10. For $\mu = 0$, namely when no jump occurs, (5.23) reduces to (5.19) with $a = 0$.

5.3. **Reduction of reflected diffusions to reflected BM.** By analogy with the definition given for ordinary diffusions (see Definition 2.1 in Section 2), we introduce the notion of process conjugated to a regulated BM.

Definition 5.11. Let $X(t)$ be a diffusion with reflecting boundaries a and b, which is driven by the SDER (5.1) with fixed $X(0) = x \in [a,b]$. We say that $X(t)$ is *conjugated to regulated BM* if there exists an increasing differentiable function $V(x)$, with $V(0) = 0$, such that $X(t) = V^{-1}\left(B_t + V(x) + \overline{L}_t - \overline{U}_t\right)$, for any $t \geq 0$, where $\overline{L}_t = V'(a)L_t$ and $\overline{U}_t = V'(b)U_t$ are regulators.

A class of reflected diffusions which are conjugated to regulated BM is given by processes which are solutions of SDERs such as:

$$dX(t) = \frac{1}{2}\sigma(X(t))\sigma'(X(t))dt + \sigma(X(t))dB_t + L_t - U_t \quad , \quad X(0) = x \tag{5.24}$$

with $\sigma(\cdot) \geq 0$. Indeed, if the integral $V(x) := \int^x \frac{1}{\sigma(r)}\,dr$ is convergent, by Itô's formula for reflected diffusions (see e.g. [34]), one gets $V(X(t)) = B_t + V(x) + V'(a)L_t - V'(b)U_t$.

Let us consider a diffusion $X(t)$, with reflecting boundaries a and b, which is conjugated to a regulated BM via the function V. Then, the process $Y(t) := V(X(t))$ is a regulated BM between the boundaries $V(a)$ and $V(b)$, starting from $V(x)$, that is:

$$Y(t) = V(x) + B_t + \overline{L}_t - \overline{U}_t \ ,$$

where $\overline{L}_t = V'(a)L_t$ and $\overline{U}_t = V'(b)U_t$ are the regulators of $Y(t)$, which increase only when $Y = V(a)$ and $Y = V(b)$, respectively. Thus, for $x \in [a, S]$:

$$\tau_S(x) = \inf\{t \geq 0 \, : \, X(t) = S | X(0) = x\} =$$

$$= \inf\{t \geq 0 \, : \, Y(t) = V(S)\} = \tau^Y_{S'}(V(x)) \ ,$$

where $S' = V(S)$ and the superscript refers to the process Y. Moreover, if the initial position $\eta = X(0) \in [a, S]$ is random, its density $g(x)$ is related to the corresponding density $\widetilde{g}$ of the initial position $\widetilde{\eta} = V(\eta) \in [V(a), V(S)]$ of $Y(t)$, by the equation $\widetilde{g}(y) = g(V^{-1}(y))(V^{-1})'(y)$, $y \in [V(a), V(S)]$. Furthermore, the density $f(t)$ of τ_S and its Laplace transform are:

$$f(t) = \int_{V(a)}^{V(S)} f^Y(t|V(y))\widetilde{g}(y)\, dy \quad \text{and} \quad \widehat{f}(\theta) = \int_{V(a)}^{V(S)} \widehat{f}^Y(\theta|V(y))\widetilde{g}(y)\, dy \ ,$$

where $f^Y(t|y)$ is the density of $\tau^Y_{S'}(y)$ and $\widehat{f}^Y(\theta|y)$ is its Laplace transform. Therefore, if $X(t)$ is conjugated to regulated BM via the function V, then the solution g to the IFPT2 problem for $X(t)$, for the FPT density f and the barrier S, can be written in terms of the solution $\widetilde{g}$ to the IFPT2 problem for a regulated BM $Y(t)$ with FPT density f and barrier $V(S)$, by using $g(x) = \widetilde{g}(V(x))V'(x)$. From Theorem 5.3 it follows that the Laplace transform of $\widetilde{g}$ satisfies (5.18), with $\widehat{g}$ replaced by $\widehat{\widetilde{g}}$, η replaced with $\widetilde{\eta}$, a replaced with $V(a)$, and S replaced with $V(S)$; in particular, if one seeks that $\widetilde{g}$ is symmetric with respect to $(V(a) + V(S))/2$, then (see (5.19)) the Laplace transform of $\widetilde{g}$ is explicitly given by:

$$\widehat{\widetilde{g}}(\theta) = \frac{e^{-V(S)\theta} + e^{-(2V(a)-V(S))\theta}}{1 + e^{(V(S)-V(a))\theta}} \, \widehat{f}\left(\frac{\theta^2}{2}\right) \quad , \quad \theta \geq 0 \, . \tag{5.25}$$

By inverting this Laplace transform, one can recover $\widetilde{g}$ and therefore g.

5.4. **A few examples.**

Example 5.1. Let $X(t)$ be regulated BM with boundaries a, b ($a < S < b$), starting from $\eta \in [a, S]$ and consider the FPT density:

$$f(t) = \frac{1}{(S-a)^2} \sum_{k=0}^{\infty} \exp\left[-\frac{(k+1/2)^2\, \pi^2 t}{2(S-a)^2}\right] \ , \tag{5.26}$$

or the corresponding FPT Laplace transform:

$$\widehat{f}(\theta) = \frac{\tanh((S-a)\sqrt{2\theta})}{(S-a)\sqrt{2\theta}} \ . \tag{5.27}$$

Then, the solution g to the IFPT2 problem for $X(t)$ is the uniform density in (a, S). In particular, for $a = 0$, $S = 1$, (5.26) and (5.27) become:

$$f(t) = \sum_{k=0}^{\infty} \exp\left[-\frac{1}{2}\left(k + \frac{1}{2}\right)^2 \pi^2 t\right] \quad \text{and} \quad \widehat{f}(\theta) = \frac{\tanh\sqrt{2\theta}}{\sqrt{2\theta}} \ . \tag{5.28}$$

The first three moments of this FPT distribution are, respectively, 2/3, 16/15, 272/105, and the solution g to the IFPT2 problem is the uniform density in $(0, 1)$.

Example 5.2. For $a = 0 < S < b$, let $X(t)$ be regulated BM starting from $\eta \in [a, S]$, and consider the FPT density whose Laplace transform is:

$$\widehat{f}(\theta) = \frac{\pi^2}{2} \, \frac{1 + \cosh\left(S\sqrt{2\theta}\right)}{\cosh\left(S\sqrt{2\theta}\right)(2\theta S^2 + \pi^2)}$$

(if e.g. $S = 1$, the first two moments of this distribution are, respectively, $1/2+2/\pi^2$ and $5/12 + 1/\pi^2 + 4/\pi^4$). Then, the solution to the IFPT2 problem for $X(t)$ is $g(x) = (\pi/2S)\sin(\pi x/S)$, $x \in (0, S)$.

Example 5.3. Take $a = 0$, $S = 1$, and let $X(t)$ be regulated BM starting from $\eta \in [0, 1]$; consider the FPT density whose Laplace transform is:

$$\widehat{f}(\theta) = \frac{\left(1 + e^{\sqrt{2\theta}}\right)\left(e^{\sqrt{2\theta}} - 2e^{\sqrt{\theta/2}} + 1\right)}{\theta \cosh(\sqrt{2\theta})\, e^{\sqrt{2\theta}}}$$

(the first two moments of this distribution are, respectively, 17/24 and 811/1440). Then, the solution to the IFPT2 problem for $X(t)$ is the triangular density in $(0, 1)$, i.e. $g(x) = 4x \times \mathbf{1}_{(0,\,1/2]}(x) + 4(1 - x) \times \mathbf{1}_{(1/2,\,1)}(x)$.

Example 5.4. Take $a = 0$, $S = 1$, and let $X(t)$ be regulated BM starting from $\eta \in [0, 1]$; consider the FPT density whose Laplace transform is:

$$\widehat{f}(\theta) = \frac{3\left(1 + e^{\sqrt{2\theta}}\right)\left(e^{-\sqrt{2\theta}}(\sqrt{2\theta} + 2) + \sqrt{2\theta} - 2\right)}{\theta\sqrt{2\theta}\left(e^{\sqrt{2\theta}} + e^{-\sqrt{2\theta}}\right)}$$

(the first two moments of this distribution are, respectively, 7/10 and 39/70). Then, the solution g to the IFPT2 problem for $X(t)$ is a Beta density in $[0, 1]$, i.e. $g(x) = 6x(1 - x)$. Notice that $\widehat{f}$ and g are obtained as special cases of $\widehat{f}_{2k}$ and g_{2k} of Proposition 5.7, for $k = 1$.

Example 5.5. For $0 = a < S < b$ let $\overline{X}(t)$ be a BM + large jumps, as considered in Proposition 5.9, and let:

$$\widehat{\overline{f}}(\theta) = \frac{1}{\mu + \theta}\left[\frac{\theta \cdot \tanh(S\sqrt{2(\mu + \theta)})}{S\sqrt{2(\mu + \theta)}} + \mu\right] .$$

By Laplace inversion, one obtains:

$$\overline{f}(t) = e^{-\mu t}\left[\sum_{k=0}^{\infty}\left(\frac{1}{S^2} + \frac{2\mu S^2}{(k + 1/2)^2\pi^2}\right)\exp\left(-\frac{(k + 1/2)^2\pi^2 t}{2S^2}\right)\right] ,$$

which can be written as

$$\overline{f}(t) = e^{-\mu t}\phi(t) + \mu e^{-\mu t}\int_t^{\infty}\phi(s)\, ds ,$$

where $\phi(t)$ is the FPT density considered in Example 5.1 with $a = 0$.

Then, the solution $\overline{g}$ to the IFPT2 problem for $\overline{X}(t)$, relative to $S \in (0, b)$ and $\overline{f}$, which is symmetric with respect to $S/2$, is the uniform density in $(0, S)$.

Example 5.6 (Reflected geometric BM)**.** Let $0 < a < S < b$, and $X(t)$ the solution of the SDER:

$$dX(t) = rX(t)dt + \sigma X(t)dB_t + dL_t - dU_t \quad , \quad X(0) = \eta \in [a, S] ,$$

where r and σ are positive constant (cf. the SDE of Example 3.2 in Section 3). As easily seen, $\ln X(t) = \ln \eta + \mu t + \sigma B_t + \overline{L}_t - \overline{U}_t$, where $\mu = r - \sigma^2/2$ and $\overline{L}_t, \overline{U}_t$ are regulators; thus, $\ln X(t)/\sigma$ is regulated BM with drift μ/σ, between the boundaries $\ln a/\sigma$, $\ln b/\sigma$. Then, the IFPT2 problem for $X(t)$ for S and the FPT density f is reduced to the IFPT2 problem for regulated drifted BM, starting from $\ln \eta/\sigma$, relative to $\ln S/\sigma$ and the same FPT density f. Explicit examples can be obtained from Examples 5.1-5.4.

Example 5.7. Let $X(t)$ be a reflected diffusion in $[a, b]$, which is conjugated to regulated BM via the function V. Then, examples of solutions to the IFPT2 problem for $X(t)$ can be obtained from Examples 5.1–5.4 involving regulated BM. For instance, let us consider the FPT density

$$f(t) = \frac{1}{(V(S) - V(a))^2} \sum_{k=0}^{\infty} \exp\left[- \frac{(k+1/2)^2 \pi^2 t}{2(V(S) - V(a))^2} \right] , \tag{5.29}$$

Then, the solution to the IFPT2 problem for $X(t)$ relative to the barrier S ($a < S < b$) is:

$$g(x) = \frac{V'(x)}{V(S) - V(a)} \times \mathbf{1}_{(a,S)}(x) \; . \tag{5.30}$$

In fact, from (5.25) and Example 5.1, $\widetilde{\eta} = V(\eta)$ turns out to be uniformly distributed over the interval $(V(a), V(S))$, and so the relation $g(x) = \widetilde{g}(V(x))V'(x)$ yields (5.30).

As explicit examples of reflected diffusions $X(t)$ which are conjugated to regulated BM, we mention the followings:

(i) $X(t)$ driven by

$$dX(t) = \frac{1}{3}\, X(t)^{1/3} dt + X(t)^{2/3}\, dB_t + dL_t - dU_t \quad , \quad X(0) = \eta \in [a, b] \; , \tag{5.31}$$

which is conjugated to regulated BM via the function $V(x) = 3x^{1/3}$, that is

$$X(t) = \left(\eta^{1/3} + \frac{1}{3}\, B_t + \overline{L}_t - \overline{U}_t \right)^3 \; .$$

Here, as well as in the next examples, $\overline{L}_t = V'(a) L_t$ and $\overline{U}_t = V'(b) U_t$.

(ii) $X(t)$ driven by

$$dX(t) = \frac{3c^2}{8}\, (X(t))^{1/2} dt + c (X(t))^{3/4}\, dB_t + dL_t - dU_t \; , \tag{5.32}$$

$$X(0) = \eta \in [a, b] \quad (a \geq 0) \; ,$$

for $c > 0$, which is conjugated to regulated BM via the function $V(x) = (4/c) x^{1/4}$, that is

$$X(t) = \left(\eta^{1/4} + \frac{c}{4}\, B_t + \overline{L}_t - \overline{U}_t \right)^4 \; .$$

(iii) (Reflected Feller process) For $b > a \geq 0$, the process $X(t)$ driven by

$$dX(t) = \frac{1}{4}\, dt + \sqrt{X(t)}\, dB_t + dL_t - dU_t \quad , \quad X(0) = \eta \in [a, b] \; , \tag{5.33}$$

which is conjugated to regulated BM via the function $V(x) = 2\sqrt{x}$, that is

$$X(t) = \frac{1}{4}(B_t + 2\sqrt{\eta} + \overline{L}_t - \overline{U}_t)^2 .$$

(iv) (Reflected Wright & Fisher-like process) $X(t)$ driven by

$$dX(t) = \left(\frac{1}{4} - \frac{1}{2}X(t)\right) dt + \sqrt{X(t)(1-X(t))}\, dB_t + dL_t - dU_t ,$$

$$X(0) = \eta \in [a, b] ,$$

for $0 \leq a < b \leq 1$, which is conjugated to regulated BM via the function $V(x) = 2 \arcsin \sqrt{x}$, that is

$$X(t) = \sin^2(B_t/2 + \arcsin \sqrt{\eta} + \overline{L}_t - \overline{U}_t) .$$

Notice that, if we take $a = 0$ and $b = 1$, both boundaries are attainable and, in order that $X(t)$ stays in $[0, 1]$ there is no need to impose reflection in 0 and 1, because the process without reflecting cannot exit the interval $[0, 1]$, for any time t (see e.g. [7], [17]).

If the FPT density is given by (5.29), from (5.30) we obtain that the solutions to the IFPT2 problems for the processes (i)-(iv) above, in the presence of barrier S, are explicitly given by:

(i) $g(x) = \left[3x^{2/3}\left(S^{1/3} - a^{1/3}\right)\right]^{-1} \times \mathbf{1}_{(a,S)}(x)$;

(ii) $g(x) = \frac{1}{4}\left[x^{3/4}\left(S^{1/4} - a^{1/4}\right)\right]^{-1} \times \mathbf{1}_{(a,S)}(x)$;

(iii) $g(x) = \frac{1}{2}\left[\sqrt{x}\left(\sqrt{S} - \sqrt{a}\right)\right]^{-1} \times \mathbf{1}_{(a,S)}(x)$;

(iv) $g(x) = \frac{1}{2}\left[\left(\arcsin \sqrt{S} - \arcsin \sqrt{a}\right) \sqrt{x(1-x)}\right]^{-1} \times \mathbf{1}_{(a,S)}(x)$.

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