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Monotone convergence, SP polynomials, and Zeckendorf representations

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Abstract. Even when a solution to a difference equation has a limit, the solution usually does not approach its limit monotonically. We define the class of **SP** polynomials and show that for the associated difference equations, the ratios of positive solutions will *always* be monotone if the ratios are *initially* monotone. We demonstrate the usefulness of this result by showing that the proportion of 0's in the n bit k^{th} order Zeckendorf representation monotonically increases towards its limit.

1. ZECKENDORF REPRESENTATION

It is fairly well known that natural numbers can be represented in a Fibonacci base, that is, if x is any natural number,

$$x = \sum_{i=2} b_i f_i$$

where the f_i are the Fibonacci numbers, 1, 2, 3, 5, 8, 13, $\cdots$ and while the b_i's depend on x, each b_i is in $\{0, 1\}$. Further the n bit representation of x is unique when no two consecutive b_i's can both be 1 [11]. If instead of the usual Fibonacci numbers which are based on the recurrence $f_i = f_{i-1} + f_{i-2}$, one uses the k^{th} order Fibonacci numbers based on the recurrence $f_i = f_{i-1} + f_{i-2} + \cdots + f_{i-k}$, a representation still exists and is unique if no k consecutive b_i's are allowed to be 1. These representations are called the Zeckendorf representations [8, 2, 4]. One can think of the usual binary (base 2) representation as the limit of the Zeckendorf representations when $k = \infty$.

Zeckendorf representations have many application, for example in data structures [2], reliable data transmission [1], [3], and cryptography [9].

If we define the set of Zeckendorf, $\mathcal{Z}$, strings of length n as the binary strings of length n with no k consecutive 1's, then the number of such strings $\#_n$ is f_{n+2} where f_{n+2}, stands for the $(n+2)^{\text{nd}}$ Fibonacci number of order k. This fact follows from a simple argument. A $\mathcal{Z}$-string of length n starts with either 0 or 10 or 100, $\cdots$, or 11 $\cdots$ 10 ($k-1$ 1's), and for each of these prefixes the string after the prefix

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is a $\mathcal{Z}$ string of shorter length, which gives the recurrence

$$\#_n = \#_{n-1} + \cdots + \#_{n-k}$$

with the initial conditions $\#_1 = 2$, $\#_2 = 4$, $\cdots$, $\#_{k-1} = 2^{k-1}$, $\#_k = 2^k - 1$. (For consistency, one could put $\#_0 = 1$). But this recurrence is just the recurrence for the Fibonacci numbers of order k with the initial conditions off-set by 2.

Next we could count, W_n the number of bits in $\mathcal{Z}$ strings of length n, and obviously $W_n = nf_{n+2}$. Since the bits are only 0's and 1's, we could count, N_n, the number 0 bits in $\mathcal{Z}$ strings of length n. Obviously the number of 1's is $W_n - N_n$. A formula for N_n is a little harder to come by, but a difference equation for N_n is easy to state:

$$N_n = N_{n-1} + N_{n-2} + \cdots + N_{n-k} + f_{n+2} \ .$$

This equation follows from the above observation about the possible prefixes and the f_{n+2} term comes from the fact that each prefix has exactly one 0. Since half of the bits in each binary string are 0's, the initial conditions for N_n are

$$N_i = i\, 2^{i-1} \quad \text{for} \quad i \in [0, k] \ .$$

For $k = 2$, it's easy to obtain a formula for N_n:

$$N_n = \frac{1}{5}\left\{n(f_{n+4} + f_{n+2}) - 2f_n\right\}$$

and this gives

$$\frac{N_n}{W_n} = \frac{1}{5}\left\{\frac{f_{n+4} + f_{n+2}}{f_{n+2}} - \frac{2f_n}{nf_{n+2}}\right\} \ .$$

From this formula,

$$\lim_{n \longrightarrow \infty} \frac{N_n}{W_n} = \frac{\lambda_0^2 + 1}{5} \approx .7236 \ ,$$

and further since the first term converges exponentially while the second (negative) term converges like $1/n$, the ratio N_n/W_n will be approaching its asymptotic value from below and in an asymptotically monotone fashion.

When we calculated a few values (see Table 1) [4], it appeared that convergence was *always* monotone and not just asymptotically monotone. We also calculated some of these ratios for other values of k, and still found that the convergence seemed to be always monotone. We were then faced with the problem of proving monotone convergence. We hoped for a proof procedure that would apply to a reasonably large class of difference equations which included the Fibonacci equations as a special case. In this paper we will show that the proportion of 0's in the Zeckendorf representation is monotone increasing. We will do so by introducing a class of polynomials, which we call **SP** polynomials, and show that if a difference equation has one of these **SP** polynomials as its characteristic polynomial, then if the ratios of positive solutions to this difference equation are *initially* increasing, then these ratios are *always* increasing.

2. **SP** AND MONOTONE RATIOS

The behavior of solutions of difference equations can often be explained in terms of the properties of the associated characteristic polynomial. For example, if the coefficients of a polynomial have the sign pattern $(+ - - \cdots -)$ then the associated linear operator has a positive eigenvalue, λ_0, and if a solution to the difference is *initially* positive, then the solution is $\Theta(\lambda_0^n)$ and is always positive [7]. The

n	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
1	.5000	.5000	.5000	.5000	.5000	.5000	.5000
2	.6666	.5000	.5000	.5000	.5000	.5000	.5000
3	.6666	.5713	.5000	.5000	.5000	.5000	.5000
4	.6875	.5769	.5333	.5000	.5000	.5000	.5000
5	.6923	.5833	.5378	.5161	.5000	.5000	.5000
6	.6984	.5909	.5417	.5191	.5079	.5000	.5000
7	.7017	.5944	.5450	.5214	.5097	.5039	.5000
8	.7045	.5973	.5481	.5233	.5111	.5049	.5020
9	.7066	.5998	.5500	.5249	.5122	.5057	.5025
10	.7083	.6016	.5516	.5263	.5131	.5064	.5030
∞	.7236	.6184	.5663	.5379	.5218	.5125	.5071

TABLE 4. The proportion of 0's in the n bit Zeckendorf representation based on the k^{th} order Fibonacci numbers.

k^{th} order Fibonacci numbers $f_n^{(k)}$ are a positive (or if you will nonnegative) solution to such a difference equation, and for each k there is an α and a λ_0 so that $f_n^{(k)} = \alpha\,\lambda_0^n + O(1)$ [5, 6, 7]. We want to define a sign pattern which will help us understand the solutions to the difference equation associated with the squared Fibonacci polynomials.

Definition 1. A polynomial $x^k + c_1x^{k-1} + c_2x^{k-2} + \cdots + c_k$ is **SP** if $c_2 \geq 0$, $c_3 \geq 0$, $\cdots$, and $c_k > 0$.

Notice that c_1 may have either sign, but in our applications c_1 will be negative, i.e. our polynomials will have the sign pattern $+ - + + \cdots +$. The name **SP** is a sort of joke, it stands for *sorta positive.*

Theorem 1 (**SP** Theorem)**.** *Let x_n and y_n be two positive solutions to $L[x_n] = 0 = L[y_n]$ where the characteristic polynomial of L has degree k and is* **SP**. *If*

$$\frac{x_k}{y_k} \geq \frac{x_{k-1}}{y_{k-1}} \geq \cdots \geq \frac{x_1}{y_1}$$

then $x_{n+1}/y_{n+1} \geq x_n/y_n$ for $n \geq 1$ and if at least one of the above inequalities is strict (i.e. $>$), then $x_{n+1}/y_{n+1} > x_n/y_n$ for $n \geq k$.

Proof. Consider

$$D_{i,j} = \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} = x_iy_j - x_jy_i = y_iy_j\left(\frac{x_i}{y_i} - \frac{x_j}{y_j}\right) .$$

Clearly, $D_{i,j}$ is anti-symmetric, i.e., $D_{i,j} = -D_{j,i}$ and $D_{i,i} = 0$. Now consider $D_{n+1,n} = y_{n+1}\,y_n\big((x_{n+1}/y_{n+1}) - (x_n/y_n)\big)$. If x_n/y_n is increasing ($\geq$) or strictly increasing ($>$) then $D_{n+1,n} \geq 0$ or $D_{n+1,n} > 0$. Then assuming that x_n and y_n satisfy the **SP** operator $L[\ ]$,

$$\begin{aligned} D_{n+1,n} &= -c_1D_{n,n} - c_2D_{n-1,n} \cdots - c_kD_{n,n+1-k} = \\ &= c_2D_{n,n-1} \cdots + c_kD_{n,n+1-k} \ . \end{aligned}$$

If

$$\frac{x_k}{y_k} \geq \frac{x_{k-1}}{y_{k-1}} \geq \cdots \geq \frac{x_1}{y_1} ,$$

then $D_{k,k-i} \geq 0$ for $i \in [1, k-1]$, and if at least one of these inequalities is strict, $D_{k,1} > 0$. So $D_{k+1,k} > 0$ and by induction $D_{n,n-1} > 0$ for all $n > k$, and the corresponding ratios are strictly increasing.

$\square$

As an immediate application of this theorem, consider the generalized Fibonacci polynomial $x^k - x^{k-1} - \cdots - 1$, which is not **SP** but when multiplied by $(x-1)$, the product polynomial is $x^{k+1} - 2x^k + 1$, which is **SP**. Let $f_n = 1, 1, 2, 4, 7, 13, 24, \cdots$, the 3rd order Fibonacci numbers. Clearly the ratio

$$\frac{f_{n+1}}{f_n} = \frac{1}{1}, \frac{2}{1}, \frac{4}{2}, \frac{7}{4}, \frac{13}{7}, \cdots$$

is converging to λ_0 the positive root of $x^3 - x^2 - x - 1$ but this convergence is not monotone. On the other hand, $3 + f_n$ is a solution to the difference equation associated with $x^4 - 2x^3 + 1$, and, of course, so is f_{n+1}. Now consider

$$\frac{f_{n+1}}{3 + f_n} = \frac{1}{4}, \frac{2}{4}, \frac{4}{5}, \frac{7}{7}, \frac{13}{10}, \cdots$$

this sequence is also converging to λ_0, but now, because the characteristic polynomial $x^4 - 2x^3 + 1$ is **SP** the convergence is monotone.

3. SQUARED FIBONACCI POLYNOMIALS ARE **SP**

Here, we show that the Fibonacci polynomials, $ch(x) = x^k - x^{k-1} - \cdots - 1$, when squared and multiplied by an appropriate degree 2 polynomial, $r(x)$, are **SP**, that is $[ch(x)]^2\, r(x)$, has the sign pattern $(+ - + + \cdots +)$. The coefficients of $ch(x)$ are

$$1 \quad -1 \quad -1 \quad \cdots \quad -1 \,.$$

An easy calculation shows that the coefficients of $[ch(x)]^2$ are

$$(1) \qquad 1,\ -2,\ 7,\ -1,\ 0,\ 1,\ \cdots\ k-3,\ k,\ k-1,\ \cdots\ 2,\ 1.$$

Somewhat surprisingly the same $r(x)$ can be used as the multiplier for all these Fibonacci polynomials with $k \geq 2$. We use

$$r(x) = 5x^2 - 2x + 1 = 4x^2 + (x-1)^2 \,.$$

Then

$$[ch(x)]^2\, r(x) = 4x^2\, [ch(x)]^2 + [(x-1)ch(x)]^2 \ ,$$

but, by a well known formula [10], $(x-1)ch(x) = x^{k+1} - 2x^k + 1$, and $[(x-1)ch(x)]^2 = x^{2k+2} - 4x^{2k+1} + 4x^{2k} + 2x^{k+1} - 4x^k + 1$.

Using (1), for $k \geq 4$, we get the sequence of coefficients

$$5,\ -12,\ 0,\ 0,\ 4,\ \cdots\ 4(k-3),\ 4k+2,\ \cdots\ 8,\ 4,\ 0,\ 1 \,.$$

For $k = 2$, the coefficients are:

$$5, \quad -12, \quad 0, \quad 10, \quad 0, \quad 0, \quad 1 \,.$$

For $k = 3$, the coefficients are:

$$5, \quad -12, \quad 0, \quad 0, \quad 14, \quad 4, \quad 4, \quad 0, \quad 1 \,.$$

So, for all cases, the desired sign pattern holds.

Although a single $r(x)$ can be used on all Fibonacci polynomials, for specific values of k other $r(x)$'s can be used. For example, for $k = 2$, $r(x) = x^2 + 1$ can be used, and for $k = 3$, $r(x) = 3x^2 - 2x - 1$ can be used.

4. INITIAL CONDITIONS

To show that the proportion of 0's in the Zeckendorf representation is an increasing function of n, we need a set of initial conditions to show that the ratios are initially increasing and then using our **SP** Theorem, we can conclude that the ratios are always increasing.

A minor difficulty arises in that for the k^{th} order Fibonacci numbers, the corresponding **SP** polynomial has degree $2k + 2$, hence to get the induction going we will need $2k + 2$ initial values. While we could try to find explicit formulas for N_n and W_n and then try to show that N_n/W_n is increasing, it may be simpler to use the recurrence "backwards" to find values for N_n and W_n when n is negative.

For W_n this is easy. Since $W_n = nf_{n+2}$,

$$W_0 = 0 \quad , \quad W_{-1} = -1 \cdot f_1 = -1 \quad , \quad W_{-2} = -2 \cdot f_0 = 0$$

and since $f_0, f_{-1}, \cdots, f_{-(k-2)}$ are all 0, $W_{-i} = 0$ for $i \in [1, k]$. Also, $W_n = n2^n$ for $n \in [1, k-1]$, $W_k = k(2^k - 1)$ and $W_{k+1} = (k+1)(2^{k+1} - 3)$. Hence we have $2k + 2$ consecutive values for W_n.

For N_n, we know that

$$N_n = n2^{n-1} \qquad \text{for } n \in [1, k-1] \; .$$

Also $N_{k+1} = (k+1)2^k - 2$ because $01 \cdots 11$ and $11 \cdots 10$ are the only length $k + 1$ binary strings with 0's which are not $\mathcal{Z}$ strings.

To obtain other values for N_n, we subtract the difference equation for N_{n-1} from the difference equation for N_n, and do the same subtraction for the difference equation for the Fibonacci numbers. This yields

$$N_n = 2N_{n-1} - N_{n-k-1} + f_{n+1} - f_{n+1-k} \; .$$

So,

$$\begin{aligned} N_0 &= -N_{k+1} - 2N_k + f_{k+2} - f_2 = \\ &= -\left[(k+1)2^k - 2\right] + 2k\,2^{k-1} + (2^k - 1) - 1 = \\ &= 0 \; . \end{aligned}$$

Also,

$$\begin{aligned} N_{-1} &= -N_k - 2N_{k-1} + f_{k+1} - f_1 = \\ &= -k\,2^{k-1} + 2(k-1)2^{k-2} + 2^{k-1} - 1 = \\ &= -1 \; . \end{aligned}$$

We claim that $N_{-i} = 0$ for $i \in [2, k]$.

$$N_{-i} = -N_{k-(i-1)} + 2N_{k-i} + f_{k-i+2} - f_{-(i-2)}$$

but for i in the specified range

$$f_{-(i-2)} = 0 \qquad\qquad f_{k-i+2} = 2^{k-i}$$

$$N_{k-(i-1)} = (k + 1 - i)2^{k-i}$$

$$N_{k-i} = (k - i)2^{k-i-1}$$

and the formula gives

$$N_{-i} = -(k+1-i)2^{k-i} + 2(k-i)2^{k-i-1} + 2^{k-i} - 0 = 0 \ .$$

Our strategy now is to show that

$$D_{i,j} \geq 0 \qquad \text{for } i \geq j \quad \text{and} \quad j \geq -k \ ,$$

and then use the **SP** difference equation for $D_{n+1,n}$ with these initial conditions coupled with the strict inequalities $D_{k,k-1} > 0$ and $D_{k+1,k} > 0$ (for $k \geq 3$) to show that N_n/W_n is strictly increasing for $n \geq k$ with $k \geq 3$, and for $n \geq 3$ for $k = 2$.

Claim 1.

$$D_{i,j} \geq 0 \quad \textit{for} \quad i \geq j \quad \textit{and} \quad j \geq -k \ .$$

Proof. We use $N_{-2} = N_{-3} = \cdots = N_{-k} = 0$ and $W_{-2} = W_{-3} = \cdots = W_{-k} = 0$ to get $D_{i,j} = 0$ for $j \in [-2, -k]$.

$$D_{i,-1} = \begin{vmatrix} N_i & W_i \\ N_{-1} & W_{-1} \end{vmatrix} = \begin{vmatrix} N_i & W_i \\ -1 & -1 \end{vmatrix} = (W_i - N_i) \geq 0$$

(If $i = 0$, $W_0 = N_0 = 0$ and $D_{0,-1} = 0$, but for $i > 0$, $W_i > N_i$ and $D_{j,-1} > 0$).

$D_{i,0} = 0$ because $N_0 = W_0 = 0$. Also

$$D_{i,1} \geq 0 \quad , \quad D_{i,2} \geq 0 \quad , \cdots \quad , \quad D_{i,k-1} \geq 0$$

because for j in the range $[1, k-1]$, $W_j = 2N_j$ and so

$$D_{i,j} = \begin{vmatrix} N_i & W_i \\ N_j & 2N_j \end{vmatrix} = N_j \left(2N_i - W_i\right) \geq 0$$

since at least half of the bits are 0.

With these initial inequalities, the difference equation for $D_{i,j}$ can be used to show $D_{i,j} \geq 0$ for all larger values of i and all larger values of j.

□

Claim 2. *If $k \geq 3$, then $D_{n+1,n} > 0$ for $n \geq k-1$.*

Proof. From the coefficients of the **SP** polynomials and the assumption that $D_{i,j} \geq 0$ for $i \geq j$,

$$D_{n+1,n} \geq 4D_{n,n-3} \ ,$$

but

$$D_{k,k-1} > 0 \quad \text{and} \quad D_{k,k-2} > 0 \quad \text{and} \quad D_{k+1,k} > 0 \text{ (for } k \geq 3) \ .$$

So, this is a sufficient base to induct that $D_{n+1,n} > 0$ for $n \geq k-1$.

□

Claim 3. *For $k = 2$, $D_{n+1,n} > 0$ for $n \geq 3$.*

Proof. From the **SP** formula,

$$D_{n+1,n} = 10D_{n,n-2} + D_{n,n-5} \ .$$

$$D_{4,3} = 10D_{3,1} + D_{3,-2} > 0$$

$$\text{since} \quad D_{3,-2} = 0 \quad \text{and} \quad D_{3,1} = \begin{vmatrix} 10 & 15 \\ 1 & 2 \end{vmatrix} > 0 \ .$$

Moreover

$$D_{5,4} = 10D_{4,2} + D_{4,-1} > 0$$

since both the addends are positive. ($D_{4,2} > 0$ follows from $D_{4,3} > 0$). This is a sufficient base to induct that $D_{n+1,n} > 0$ for all larger values of n.

□

These three claims give the following theorem.

Theorem 2. *The proportion of* 0*'s in the* n *bit* k^{th} *order Zeckendorf representation is an increasing function of* n*, and for* $k \geq 3$ *it is a strictly monotone increasing function for* $n \geq k-1$*, and for* $k = 2$ *it is monotone increasing for* $n \geq 3$*.*

5. Discussion

By introducing the idea of **SP** polynomials and associated operators, we have shown that certain ratios of solutions to difference equations converge monotonically. It should be mentioned that in most usual examples, even if there is convergence, the convergence is NOT monotone. For example, the usual sequence of Fibonacci numbers is converging toward $\lambda_0^n/\sqrt{5}$ but this convergence is not monotone, the values always oscillate above and below the limiting function. Again, if we consider ratios of solutions to the Fibonacci difference equation, the ratios will approach a limit, but they will oscillate above and below this limit. In fact, we can show that if x_n is a positive solution to a nonnegative (aperiodic) difference equation, then x_n/λ_0^n will converge, but the convergence cannot be monotone [7].

The situation we investigated is different. Instead of a nonegative operator, we have the square of a nonnegative operator. For these squared operators, ratios of positive solutions will converge and we can show that this convergence will be asymptotically monotone. That is, if x_n and y_n are the two solutions then there exists an n_0 so that if $n \geq n_0$ then x_n/y_n converges monotonically to its limit [4]. But, the value of n_0 may depend on the values of x_n and y_n. We have shown a stronger result in that, for a Fibonacci operator, there exists a K independent of x_n and y_n so that if the x_n/y_n ratios are K in a row monotone, x_n/y_n is always thereafter monotone.

Further investigation may allow us to delineate or at least find sufficient conditions under which squared nonnegative operators are **SP**. For such operators, we could show that if ratios of solutions were initially (K in a row) monotone, then the ratios would always be monotone.

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