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On some applications of a generalization of Laguerre polynomials in statistics

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Abstract. We introduce a-generalized Laguerre polynomials, as a suitable generalization of (classical) Laguerre polynomials, and give new closed form formulae for their representation and computation by means of the symbolic method of moments. Applications to the computation of moments of a real non-central Wishart distribution as well as some special sub-classes as central and non-central chi-square random variables are provided. Connections with Hermite polynomials and polynomial processes are also given.

1. Introduction

Laguerre polynomials were introduced by Edmond Laguerre more than 150 years ago. Since they are orthogonal polynomials, many applications can be found as approximation of a smooth function in terms of their series expansion, which is the basis of spectral methods of solution of differential equations [10]. Laguerre polynomials $L_n(x)$ and generalized Laguerre polynomials $L_n^{(\nu)}(x)$ (see [15], Section 3.1) both are complete orthogonal sets of functions on the semi-infinite interval $[0,\infty)$.

In this paper, we introduce the notion of a-generalized Laguerre polynomial by inserting one more parameter $a \in \mathbb{R}$ into the generating function (g.f.) of $L_n^{(\nu)}(x)$.

Definition 1.1. a-Generalized Laguerre polynomials $\{L_n^{(a,\nu)}(x)\}$ have g.f.

$$\sum_{n\geq 0} L_n^{(a,\nu)}(x)\frac{z^n}{n!} = \frac{1}{(1-a\,z)^{\nu+1}} \exp\left\{x\frac{a\,z}{a\,z-1}\right\} \quad , \quad a,\,\nu\in\mathbb{R}\,. \tag{1.1}$$

The sequence $\{L_n^{(1,\nu)}(x)\}$ gives classical generalized Laguerre polynomials $\{L_n^{(\nu)}(x)\}$ and

$$L_n^{(a,\nu)}(x) = a^n L_n^{(\nu)}(x). \tag{1.2}$$

The aim of this paper is to use the symbolic method of moments to give a closed form formula which facilitates the computation of $\{L_n^{(a,\nu)}(x)\}$ and provides simple proofs of some main properties. Some of these properties are already known, as the

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connections between Laguerre polynomials and Hermite polynomials, some others are new, as an expansion of a-generalized Laguerre polynomials in terms of Hermite polynomials or their Abel representation.

Although the term "symbolic method" is also used in different context (see for example [9]), here we refer to a set of manipulation techniques aiming to simplify calculations. The main idea is to find closed form formulae easily implementable in any symbolic language, a field usually called "symbolic computation".

The basic device of the symbolic method of moments is to represent a unital sequence of numbers by a symbol α, named umbra, i.e., to associate the sequence $1, a_1, a_2, \cdots$ to the sequence $1, \alpha, \alpha^2, \cdots$ of powers of α through an operator $\mathbb{E}$ that looks like the expectation of r.v.'s. Therefore statements involving r.v.'s can be proved by using umbrae and then replacing these umbrae with r.v.'s. As instance in point, we refer to some classical family of r.v.'s as central and non-central chi-square distributions. More complicated is the computation of moments of Wishart distributions, which are the matrix analog of chi-squared distributions. This distribution characterizes the sample variance-covariance matrix of a multivariate Gaussian model, and play a fundamental role in multivariate statistics. In particular, its moments are needed to approximate the distribution of many statistics tests [12]. The symbolic method of moments has provided an efficient tool to compute moments of complex non-central Wishart distributions by using a special class of Sheffer polynomial sequences [4]. In this paper we prove that a-generalized Laguerre polynomials are Sheffer polynomial sequences too and provide a way to compute moments of real non-central Wishart distributions.

An application involving stochastic processes ends the paper. Indeed Laguerre polynomials are special time-space-harmonic polynomials, a family of polynomials such that when the indeterminate is replaced by a Lévy process, the resulting stochastic process is a martingale. These processes are employed together with the reduction-variance method for the pricing and the hedging of some bounded measurable European claims [2]. For matrix-valued polynomial processes, the computation of their coefficients requires the computation of a matrix exponential and efficient algorithms to deal with are not yet available. Some open problems end the paper.

2. Backgrounds on the symbolic method of moments

In the symbolic method of moments, an alphabet $\mathcal{A} = \{\alpha, \beta, \gamma, \cdots\}$ of indeterminates, named *umbrae*, is considered and any umbra is related to a sequence $\{a_n\}$ of real numbers by a suitable linear functional $\mathbb{E}$. The functional $\mathbb{E} : \mathbb{R}[\mathcal{A}] \to \mathbb{R}$ is defined on the polynomial ring $\mathbb{R}[\mathcal{A}]$, and such that $\mathbb{E}[\alpha^n] = a_n$ for all non-negative integers $n \geq 1$. We assume $\mathbb{E}[1] = 1$ that means $a_0 = 1$. The sequence $\{a_n\}$ is said to be the sequence of *moments* of α and we say that $\{a_n\}$ is umbrally represented by α. Two umbrae can represent the same sequence of moments, that is $\mathbb{E}[\alpha^n] = \mathbb{E}[\gamma^n]$ for all non-negative integers $n \geq 1$. In such a case we said that α is similar to γ, and write $\alpha \equiv \gamma$. A "weaker" equivalence involves umbral polynomials $p, q \in \mathbb{R}[\mathcal{A}]$. We say that p is umbrally equivalent to q, if $\mathbb{E}[p] = \mathbb{E}[q]$, in symbols $p \simeq q$. The operator $\mathbb{E}$ factorizes on distinct umbrae, that is $\mathbb{E}[\alpha^i \beta^j \cdots \gamma^k] = \mathbb{E}[\alpha^i]\mathbb{E}[\beta^j] \cdots \mathbb{E}[\gamma^k]$ (uncorrelation property). Special umbrae are:

a) the *unity umbra* u whose moments are $\{1\}$;

b) the *augumentation umbra* ε whose moments are $\mathbb{E}[\varepsilon^n] = \delta_{0,n}$, for all non-negative integers n and $\delta_{0,n}$ the Kronecker Delta;

c) the *singleton umbra* whose moments are $\mathbb{E}[\chi^n] = \delta_{1,n}$, for all non-negative integers n.

An umbra is characterized either by its sequence of moments $\{a_n\}$ either by its generating function (g.f.), that is the formal power series

$$f(\alpha, z) = \sum_{n\geq 0} a_n \frac{z^n}{n!} \in \mathbb{R}[[z]] \, .$$

For example we have $f(\varepsilon, z) = 1, f(\chi, z) = 1 + z$ and $f(u, z) = e^z$. Two similar umbrae have the same g.f., that is $\alpha \equiv \gamma \Leftrightarrow f(\alpha, z) = f(\gamma, z)$. Operations among umbrae correspond to operations among g.f.'s, for example the g.f. of $\alpha + \gamma$ is $f(\alpha, z)f(\gamma, z)$. Formal power series allow us to work with g.f.'s which do not have a positive radius of convergence or have indeterminate coefficients [18]. For example, if we consider the geometric series $(1-z)^{-1} = 1+z+z^2+\cdots$, we can define an umbra with g.f. $(1-z)^{-1}$ without paying attention to the question of its convergence. The umbra having g.f. equals $(1 - z)^{-1}$ is the boolean unity $\bar{u}$, see [7] for more details. This umbra represents the sequence $\{k!\}$ and in the following will play a special role.

Taking a measure to a sequence of numbers is familiar in probability theory, when the n-th term of a sequence can be considered as the n-th moment of a r.v., under suitable hypotheses (the so-called Hamburger moment problem [19]). As Rota underlines in *Problem 1: the algebra of probability* [16], all of probability theory could be done in terms of r.v.'s alone by taking an ordered commutative algebra over the reals, and endowing it with a positive linear functional. This is why a_n is called the n-th moment of the umbra α and a r.v. X is said to be *represented* by an umbra α if its sequence of moments $\{a_n\}$ is umbrally represented by α. In order to avoid misunderstandings the expectation of a r.v. X will be denoted by E and its n-th moment by $E[X^n]$.

Auxiliary umbrae are introduced as special symbols representing operations among moments and used as they were symbols of the alphabet $\mathcal{A}$. For example, if $\{\alpha, \alpha', \cdots, \alpha''\}$ is a set of k uncorrelated umbrae, then in place of the summation $\alpha + \alpha' + \cdots + \alpha''$, a new symbol $k{\bullet}\alpha$ is introduced. The symbol $k{\bullet}\alpha$ is named *dot-product* of k and α and referred as *auxiliary umbra*. It is straightforward to prove that $f(k{\bullet}\alpha, z) = f(\alpha, z)^k$. The integer k may be replaced by any real $a \in \mathbb{R}$ as follows, without going into details, which are not necessary for the content of the paper. Let us remark that the n-th moment $E[(k{\bullet}\alpha)^n]$ is a polynomial $q_n(k)$ of degree n in k, that is $E[(k{\bullet}\alpha)^n] = q_n(k)$ (see [8] for an explicit expression of $q_n(k)$). If we replace k with $a \in \mathbb{R}$, then $q_n(a) \in \mathbb{R}$. We define the auxiliary umbra $a{\bullet}\alpha$ having the n-th moment equals $q_n(a)$, that is $\mathbb{E}[q_n(a)] = \mathbb{E}[(a{\bullet}\alpha)^n]$ for all non-negative integers n.

Remark 2.1. The employment of negative real numbers needs to be handled carefully. Indeed the *inverse umbra* of an umbra α is the auxiliary umbra $-1{\bullet}\alpha$ such that $\alpha+(-1{\bullet}\alpha) \equiv \varepsilon \equiv -1{\bullet}\alpha+\alpha$. In particular we have $f(-1{\bullet}\alpha, z) = [f(\alpha, z)]^{-1}$ and $f(-k{\bullet}\alpha, z) = [f(\alpha, z)]^{-k}$. The symbol $-1{\bullet}\alpha$ should not be confused with $-(1{\bullet}\alpha)$ whose g.f. is simply $f[-(1{\bullet}\alpha), z] = f(\alpha, -z)$. The same holds for $-(k{\bullet}\alpha)$ having g.f. $f[-(k.\alpha), z] = [f(\alpha, -z)]^k$. When no misunderstanding occurs, brackets are avoided.

The non-negative integer k in $k.\alpha$ may be replaced by an umbra γ. From a probabilistic point of view this replacement allows us to work with a suitable generalization of random sums, not necessarily indexed by a non-negative r.v. but with a more general device, as an umbra is. Without going into details, let us underline that the construction of $\gamma.\alpha$ is similar to $a.\alpha$. Starting from $E[(k.\alpha)^n] = q_n(k)$, k is replaced with the umbra γ, the umbral polynomial $q_n(\gamma)$ has evaluation $\mathbb{E}[q_n(\gamma)] \in \mathbb{R}$ and we assume $\mathbb{E}[q_n(\gamma)]$ is the n-th moment of the auxiliary umbra $\gamma.\alpha$, that is $\mathbb{E}[q_n(\gamma)] = \mathbb{E}[(\gamma.\alpha)^n]$. The umbra $\gamma.\alpha$ represents a symbolic summation γ times of the umbra α. Once established the rules, the main strength of the symbolic method is performing computations on auxiliary umbrae, just using polynomials, and replacing their indeterminates, as done previously, to simplify formulae and proofs.

The dot product of γ and α has many properties [8], for example $u.\alpha \equiv \alpha \equiv \alpha.u$ and the dot-product of two umbrae can be iterated as $\alpha.(\gamma.\delta) \equiv (\alpha.\gamma).\delta$. The reader is referred to the bibliography for more details. Here we just recall some special dot-products necessary in the following.

The umbra β such that $\beta.\chi \equiv u \equiv \chi.\beta$ is a special umbra, called the *Bell umbra*. Indeed its moments are the Bell numbers[2] and $f(\beta, z) = \exp(e^z - 1)$. The dot-product $\alpha.\beta.\gamma$, with β the Bell umbra, is said the composition umbra between α and γ. Indeed the g.f. of $\alpha.\beta.\gamma$ is the composition of $f(\alpha, z)$ and $f(\gamma, z)$, that is $f(\alpha.\beta.\gamma, z) = f[\alpha, f(\gamma, z) - 1]$.

Since the composition of formal power series is an invertible function, there is a way to define an umbra whose g.f. is the compositional inverse of $f(\alpha, z)$. Let us denote by $\alpha^{<-1>}$ the umbra whose g.f. is $f(\alpha^{<-1>}, z) = f^{<-1>}(\alpha, z)$. Recall that for the compositional inverse of $f(\alpha, z)$ the following property holds:

$$f^{<-1>}[\alpha, f(\alpha, z) - 1] = f[\alpha, f^{<-1>}(\alpha, z) - 1] = 1 + z\,.$$

Moments of umbrae can be polynomials. In this case the functional $\mathbb{E}$ is defined on the polynomial ring $\mathbb{R}[x][\mathcal{A}]$ taking values on $\mathbb{R}[x]$. The uncorrelation property is updated as $\mathbb{E}[x^n \alpha^i \beta^j \cdots \gamma^k] = x^n \mathbb{E}[\alpha^i]\mathbb{E}[\beta^j] \cdots \cdots \mathbb{E}[\gamma^k]$. Umbrae representing polynomial sequences are said to be polynomial, those representing sequences of real numbers are said to be scalar. A special polynomial umbra is the auxiliary umbra $x.\alpha$. This umbra $x.\alpha$ represents the sequence of moments $\{q_k(x)\}$ obtained from $\{E[(n.\alpha)^k]\}$ replacing n with x. A compound Poisson r.v. is represented by the umbra $x.\beta.\alpha$ with g.f. $f(x.\beta.\alpha, z) = \exp[x\,(f(\alpha, z) - 1)]$. The umbra $x.\beta.\alpha$ is the composition umbra between $x.u$, with g.f. $f(x.u, z) = \exp(x\,z)$, and α. We can replace the indeterminate x with a real number $a \in \mathbb{R}$. Then $f(a.\beta.\alpha, z) = \exp[a\,(f(\alpha, z) - 1)]$ and the following property holds

$$(a + b).\beta.\alpha \equiv a.\beta.\alpha + b.\beta.\alpha\,. \tag{2.1}$$

Shifting the auxiliary umbra $x.\alpha$ with an umbra γ, we get the so-called Sheffer umbra. The name depends from its moments $\{\mathfrak{s}_k(x)\}$, that form a Sheffer polynomial

[2]The n-th Bell number is the number of partitions of a finite nonempty set with n elements.

sequence. Its generating function is

$$f(\gamma + x.\alpha, z) = f(\gamma, z) \exp\{x[f(\alpha, z) - 1]\} \,. \tag{2.2}$$

The sequence of polynomials $\{p_k(x)\}$, represented by the auxiliary umbra $x.\alpha$, is said the sequence associated to $\{\mathfrak{s}_k(x)\}$. This sequence is of binomial type, that is

$$p_n(x+y) = \sum_{i=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(x) \,. \tag{2.3}$$

Sheffer sequences have many properties. A good account of these properties in umbral terms are given in [5]. Let us underline that the main advantage of a symbolic representation of Sheffer polynomial sequences is the plainness of the overall setting which reduces their numerous properties to few fundamental statements.

3. a-Generalized Laguerre polynomials

The g.f. (1.1) of a-generalized Laguerre polynomials $\{L_n^{(a,\nu)}(x)\}$ fits (2.2). Therefore $\{L_n^{(a,\nu)}(x)\}$ is a Sheffer polynomial sequence umbrally represented by a polynomial umbra $\gamma + x.\alpha$, for a suitable choice of the scalar umbrae α and γ. Next theorem states which the umbrae α and γ are.

Theorem 3.1. $L_n^{(a,\nu)}(x) \simeq a^n \left[(\nu+1).\bar{u} - (x.\beta.\bar{u}^{<-1>})\right]^n.$

Proof. From (1.2) we have to prove that $L_n^{(\nu)}(x) \simeq \left[(\nu+1).\bar{u} - (x.\beta.\bar{u}^{<-1>})\right]^n$, that is equivalent to prove that the g.f. of the umbra $(\nu+1).\bar{u} - (x.\beta.\bar{u}^{<-1>})$ is the right hand side of (1.1) with $a = 1$. To this aim, since $(1-t)^{-1} = f(\bar{u}, t)$, note that

$$f\left((\nu+1).\bar{u}, t\right) = \frac{1}{(1-t)^{\nu+1}} \quad \text{and} \quad f(\bar{u}^{<-1>}, t) = \frac{1+2t}{1+t} \,.$$

The result follows since

$$\exp\left\{x \frac{t}{t-1}\right\} = \exp\left\{x\left[\frac{1-2t}{1-t} - 1\right]\right\} = f\left[-(x.\beta.\bar{u}^{<-1>}), t\right] \,.$$

□

Note that the sequence of a-generalized Laguerre polynomials $L_n^{(a,\nu)}(x)$ is of Sheffer type with $\alpha = (-a)\left(\beta.\bar{u}^{<-1>}\right)$ and $\gamma = (\nu+1).(a\bar{u})$.

Remark 3.2. The n-th polynomial associated to $\{L_n^{(a,\nu)}(x)\}$ is

$$L_n^{(a,-1)}(x) = L_{n,a}(x) = \mathbb{E}\left[(-a)^n \left(x.\beta.\bar{u}^{<-1>}\right)^n\right] \,,$$

obtained for $\nu = -1$. The sequence $\{L_{n,a}(x)\}$ is of binomial type and satisfies property (2.3). The polynomial $L_{n,1}(x) = L_n(x)$ is the n-th classical Laguerre polynomial, such that $L_{n,a}(x) = a^n L_n(x)$. This equality parallels equation (1.2).

Theorem 3.3 (Abel representation)**.** $L_{n,a}(x) \simeq (-a)^n \, x \, (x - (n.\chi))^{n-1}.$

Proof. From Remark 3.2, we need to prove that $L_n(x) \simeq (-1)^n \, x \, (x - (n.\chi))^{n-1}$. Let γ be an umbra with $\mathbb{E}[\gamma] = 1$. Denote by γ_D the derivative umbra of γ, that is an umbra with g.f. $f(\gamma_D, t) = 1 + t f(\gamma, t)$. As $\mathbb{E}[\gamma_D] \neq 0$, the umbra γ_D admits compositional inverse and

$$\left(x.\beta.\gamma_D^{<-1>}\right)^n \simeq x\,(x - n.\gamma)^{n-1} \tag{3.1}$$

from the Abel representation theorem of binomial sequences [7]. Assume to replace γ with $\bar{u}$. Then

$$\left(x.\beta.\bar{u}_D^{<-1>}\right)^n \simeq x\,(x - n.\bar{u})^{n-1} \simeq x\,(x + n.(-\chi))^{n-1} \tag{3.2}$$

as $\bar{u} \equiv -1.(-\chi)$. Moreover $\bar{u} \equiv \bar{u}_D$ and

$$L_n(x) \simeq [x.\beta.(-\bar{u}^{<-1>})]^n \simeq (-1)^n [x.\beta.\bar{u}^{<-1>}]^n \simeq (-1)^n [x.\beta.\bar{u}_D^{<-1>}]^n \,.$$

The result follows from (3.2) as $n.(-\chi) \equiv -(n.\chi)$.

□

a-Generalized Laguerre polynomials are obtained from $L_{n,a}(x)$ by shifting the indeterminate x with the singleton umbra χ.

Proposition 3.4. $L_n^{(a,\nu)}(x) \simeq L_{n,a}\,[x - ((\nu+1).\chi)]$.

Proof. We need to prove that $L_n^{(\nu)}(x) \simeq L_n\,[x - ((\nu+1).\chi)]$ from Remark 3.2. First observe that $\bar{u} \equiv (-\chi).\beta.(-\bar{u}^{<-1>})$ as

$$f\left((-\chi).\beta.(-\bar{u}^{<-1>}), t\right) = 1 - t|_{t \leftarrow \frac{1-2t}{1-t} - 1} = 1 + \frac{t}{1-t} = \frac{1}{1-t} \,.$$

Therefore $(\nu+1).\bar{u} \equiv (\nu+1).(-\chi).\beta.(-\bar{u}^{<-1>})$ and the result follows from Theorem 3.1 since $L_n^{(1,\nu)}(x) \simeq (-1)^n \left\{[(\nu+1).(-\chi) + x].\beta.\bar{u}^{<-1>}\right\}^n$, due to the distributive property of the dot-product.

□

A first corollary of Proposition 3.4 is the Abel representation of a-generalized Laguerre polynomials.

Corollary 3.5. $L_n^{(a,\nu)}(x) \simeq a^n [(\nu+1).\chi - x]\,\{[(\nu+1).\chi - x] + (n.\chi)\}^{n-1}$.

Proof. The Abel representation theorem (3.1) has been generalized [7], replacing the indeterminate x with an umbra α, that is

$$(\alpha.\beta.\gamma_D^{<-1>})^n \simeq \alpha(\alpha - n.\gamma)^{n-1} \,. \tag{3.3}$$

The result follows from (3.3) and (3.2), since we have $L_n^{(1,\nu)}(x) \simeq (-1)^n \times$ $\times \left\{[x - ((\nu+1).\chi)]\,.\beta.\bar{u}^{<-1>}\right\}^n$ from Proposition 3.4.

□

From a computational point of view and taking into account the applications we are going to propose, a more manageable expression for a-generalized Laguerre polynomials is given by the following theorem.

Theorem 3.6. $L_n^{(a,\nu)}(x) \simeq a^n\,\left(-x + (n+\nu).\chi\right)^n$.

Proof. From Remark 3.2, we need to prove that $L_n^{(1,\nu)}(x) \simeq (-x + (n+\nu).\chi)^n$. From Corollary 3.5 and by using the binomial expansion

$$L_n^{(1,\nu)}(x) \simeq [(\nu+1).\chi - x] \sum_{k=0}^{n-1} \binom{n-1}{k} (-x)^{n-k-1} [n.\chi + (\nu+1).\chi]^k \simeq$$

$$\simeq \sum_{k=0}^{n-1} \binom{n-1}{k} (-x)^{n-k-1} \left\{ \sum_{j=0}^{k} \binom{k}{j} (n.\chi)^j [(\nu+1).\chi]^{k-j+1} \right\} +$$

$$+\sum_{k=0}^{n-1}\binom{n-1}{k}(-x)^{n-k}\left\{\sum_{j=0}^{k}\binom{k}{j}(n.\chi)^j[(\nu+1).\chi]^{k-j}\right\}\simeq$$

$$\simeq(-x)^n+\sum_{k=1}^{n-1}(-x)^{n-k}\left[\binom{n-1}{k}\left\{\sum_{j=0}^{k}\binom{k}{j}(n.\chi)^j\times\right.\right. \tag{3.4}$$

$$\left.\times[(\nu+1).\chi]^{k-j}\right\}+\binom{n-1}{k-1}\left\{\sum_{j=0}^{k-1}\binom{k-1}{j}(n.\chi)^j\times\right.$$

$$\left.\left.\times[(\nu+1).\chi]^{k-j}\right\}\right]+\sum_{j=0}^{n-1}\binom{n-1}{j}(n.\chi)^j[(\nu+1).\chi]^{n-j}\ . \tag{3.5}$$

The summation in (3.5) is such that

$$\sum_{j=0}^{n-1}\binom{n-1}{j}(n.\chi)^j[(\nu+1).\chi]^{n-j}\simeq \tag{3.6}$$

$$\simeq[(n-1).\chi+(\nu+1).\chi]^n\simeq[(n+\nu).\chi]^n\ .$$

The generic term of the summation starting in (3.4) and ending in (3.5) is

$$\left[\binom{n-1}{n-k}\left\{\sum_{j=0}^{n-k}\binom{n-k}{j}(n.\chi)^j[(\nu+1).\chi]^{n-k-j}\right\}+\right.$$

$$\left.+\binom{n-1}{n-k-1}\left\{\sum_{j=0}^{n-k-1}\binom{n-k-1}{j}(n.\chi)^j[(\nu+1).\chi]^{n-k-j}\right\}\right]\simeq \tag{3.7}$$

$$\simeq\binom{n}{k}\left[\sum_{j=0}^{n-k}\frac{k(n-k)!(n-1)\cdots(n-j+1)}{j!(n-k-j)!}\ [(\nu+1).\chi]^{n-k-j}+\right.$$

$$\left.+\sum_{j=0}^{n-k-1}\frac{(n-k)(n-k-1)!(n-1)\cdots(n-j+1)}{j!(n-k-j-1)!}\ [(\nu+1).\chi]^{n-k-j}\right]\simeq$$

$$\simeq\binom{n}{k}\left[\sum_{j=0}^{n-k-1}\frac{(n-k)!}{j!(n-k-j)!}\ (k+n-k-j)(n-1)\cdots(n-j+1)\times\right.$$

$$\left.\times\ [(\nu+1).\chi]^{n-k-j}\right]+(n-1)(n-2)\cdots(k+1)k\simeq$$

$$\simeq\binom{n}{k}\left[\sum_{j=0}^{n-k-1}\binom{n-k}{j}[(n-1).\chi]^{n-j}[(\nu+1).\chi]^{n-k-j}\right]+[(n-1).\chi]^{n-k}\simeq$$

$$\simeq\binom{n}{k}[(n-1).\chi+(\nu+1).\chi]^{n-k}\ . \tag{3.8}$$

Coupling (3.5), written as (3.6), with (3.7), written as (3.8), we have

$$L_n^{(\nu)}(x) \simeq (-x)^n + \sum_{k=1}^{n-1} \binom{n}{k} (-x)^k [(n+\nu){\boldsymbol .}\chi]^{n-k} + [(n+\nu){\boldsymbol .}\chi]^n \, ,$$

and the result follows.

□

Theorem 3.6 allows us to give an expansion of a-generalized Laguerre polynomials.

Corollary 3.7. $L_n^{(a,\nu)}(x) = a^n \sum_{k=0}^n \mathfrak{l}_{n,k,\nu} \, (-x)^k$ *with* $\mathfrak{l}_{n,k,\nu} = \binom{n+\nu}{n-k} \, n!/k!$.

Proof. From Remark 3.2, we need to prove that $L_n^{(\nu)}(x) = \sum_{k=1}^n \mathfrak{l}_{n,k,\nu} \, (-x)^k$. This result follows from Theorem 3.6, since

$$L_n^{(\nu)}(x) \simeq \sum_{k=0}^{n} \binom{n}{k} (-x)^k [(n+\nu){\boldsymbol .}\chi]^{n-k} \simeq \sum_{k=0}^{n} \frac{(n+\nu)_{n-k}}{(n-k)!} \frac{n!}{k!} (-x)^k \, .$$

□

Note that $\{l_{n,k,1}\}$ are the Lah numbers [14].

4. Applications

Central chi-squared distribution. As it is well known, the chi-squared distribution with k degrees of freedom is the distribution of a sum of k squared standard normal r.v.'s $\{Z_1, Z_2, \cdots, Z_k\}$, with $Z_i \sim \mathcal{N}(0,1)$ for $i = 1, 2, \cdots, k$. Let us recall that the moment generating function (m.g.f.) of a chi-square r.v. $\mathfrak{c}_k^2$ with k degrees of freedom[3] is $M_{\mathfrak{c}_k^2}(t) = (1-2t)^{-k/2}$. Therefore $M_{\mathfrak{c}_k^2}(t)$ fits (1.1), with $x = 0$, $\nu = k/2 - 1$, $a = 2$ and the 2-generalized Laguerre polynomials $L_n^{(2,k/2-1)}(x)\Big|_{x=0}$ are the moments of $\mathfrak{c}_k^2$. From Theorem 3.6, we recover

$$E\left[\left(\mathfrak{c}_k^2\right)^n\right] = 2^n \mathbb{E}\left[\left(n + \frac{k-2}{2}\right){\boldsymbol .}\chi\right]^n = 2^n \left(n + \frac{k-2}{2}\right)_n = 2^n \, \frac{\Gamma(n+k/2)}{\Gamma(k/2)} \tag{4.1}$$

where $\Gamma(x)$ is the Gamma function [1]. From Theorem 3.1, the following result is proved.

Theorem 4.1. *The r.v.* $\mathfrak{c}_k^2$ *is umbrally represented by the umbra* $(k/2){\boldsymbol .}(2\,\bar{u})$.

Thanks to this representation, the connection between Laguerre polynomials and Hermite polynomials can be easily recovered, since they both are related to the moments of $\mathfrak{c}_k^2$.

Let us recall that if $\{H_n^{(v)}(x)\}$ denotes the sequence of Hermite polynomials, then their compositional inverses $\{H_n^{(-v)}(x)\}$ are called Hermite polynomials with negative variance [15]. Hermite polynomials with negative variance give moments of a normal r.v. $X \sim \mathcal{N}(m, s^2)$. In [6] the umbral counterpart of $Z \sim \mathcal{N}(0,1)$ has been proved to be $\beta{\boldsymbol .}\delta$, where δ is the umbra with generating function $f(\delta, t) = 1 + t^2$. Any

[3] Since the singleton umbra is denoted by the greek letter χ, we denote the chi-square r.v. with $\mathfrak{c}_k^2$ instead of the usual χ_k^2.

other normal r.v. is represented by an umbra obtained by a linear transformation of $\beta{\bf .}\delta$, that is $m + s(\beta{\bf .}\delta) \equiv m + \beta{\bf .}(s\delta)$,

$$E[X^n] = H_n^{(-s^2)}(m) \simeq (m + \beta{\bf .}(s\,\delta))^n\,. \tag{4.2}$$

The following result is the umbral version of $\mathfrak{c}_k^2 \stackrel{d}{=} Z_1^2 + Z_2^2 + \cdots + Z_k^2$.

Proposition 4.1. $k{\bf .}(\beta{\bf .}\delta)^2 \equiv (k/2){\bf .}\,(2\,\bar{u})$.

Proof. From (4.1) with $k = 1$ we have

$$\mathbb{E}\left(2^n\left[\left(n-\frac{1}{2}\right){\bf .}\chi\right]^n\right) = 2^n\left(n-\frac{1}{2}\right)_n = (2n-1)!! = \frac{(2n)!}{n!2^n} = \mathbb{E}[(\beta{\bf .}\delta)^{2n}]\,, \tag{4.3}$$

for the last equality see [6]. From (4.1) and Theorem 4.1, we have

$$(\beta{\bf .}\delta)^2 \equiv \frac{1}{2}{\bf .}\,(2\,\bar{u})\,. \tag{4.4}$$

If $\alpha, \gamma \in \mathcal{A}$ and $\alpha \equiv \gamma$, then $k{\bf .}\alpha \equiv k{\bf .}\gamma$ for all non-negative integers k, so the result follows.

□

For $k = 1$, 2-generalized Laguerre polynomials $L_n^{(2,-1/2)}(x)\Big|_{x=0}$ are moments of $\mathfrak{c}_1^2$. From (4.4), (4.1) and (4.2) we get $H_{2\,n}^{(-1)}(0) = L_n^{(2,-1/2)}(0)$. In dealing with non-central chi-squared distribution, this equality may be generalized replacing 0 with a suitable indeterminate, as shown in the next paragraph.

Non-central chi-squared distribution. The non-central chi-squared distribution appears in many applications, for example, in radar communications when computing the detection of signals in noise using a square-law detector [13]. Different algorithms have been proposed for computing their moments by using series expansion or recurrence relations. In the following, we show how to use the umbral representation of generalized Laguerre polynomials to simplify their computation.

If k independent r.v.'s $\{X_1, \cdots, X_k\}$ are considered, having normal distribution $X_i \sim \mathcal{N}(0, s_i)$ for $i = 1, \cdots, k$, the distribution of $\sum_{i=1}^k (X_i + m_i)^2/s_i^2$ is said a non-central chi-squared distribution with k degrees of freedom and non-centrality parameter $l^2 = \sum_{i=1}^k l_i^2$ with $l_i = m_i/s_i$ for $i = 1, \cdots, k$. Let us recall that the m.g.f. of a non-central chi-square r.v. $\mathfrak{c}_{l,k}^2$ is the power series given in (1.1) with $x = -l^2/2$, $\nu = k/2 - 1$, $a = 2$ and the 2-generalized Laguerre polynomials $L_n^{(2,k/2-1)}\left(-l^2/2\right)$ are the moments of $\mathfrak{c}_{l,k}^2$. From Theorem 3.6 and Corollary 3.7, we have

$$E\left[\left(\mathfrak{c}_{l,k}^2\right)^n\right] = 2^n\mathbb{E}\left[\left(\frac{l^2}{2} + \left(n-1+\frac{k}{2}\right){\bf .}\chi\right)^n\right] = \tag{4.5}$$

$$= \sum_{k=0}^n \binom{n-1+k/2}{n-k}\frac{n!}{k!}\,\frac{l^{2k}}{2^{k-n}}\,.$$

Theorem 4.3. *The r.v. $\mathfrak{c}_{l,k}^2$ is umbrally represented by* $\left[(k/2) + (l^2/2){\bf .}\beta\right]{\bf .}(2\,\bar{u})$.

Proof. The result follows from Proposition 3.4, by observing $-(x{\bf .}\beta{\bf .}\bar{u}^{<-1>}) \equiv -x{\bf .}\beta{\bf .}\bar{u}$ and by replacing x with $-l^2/2$ and ν with $k/2 - 1$.

□

Comparing Theorem 4.3 with Theorem 4.1, the umbral counterpart of $\mathfrak{c}^2_{l,k}$ is obtained from the umbral counterpart of $\mathfrak{c}^2_k$ by a suitable shifting with l. As observed at the end of the last paragraph, there is a deeper connection between the 2-generalized Laguerre polynomials and the Hermite polynomials. First assume $k = 1$, so that $\mathfrak{c}^2_{l,1} \stackrel{d}{=} (X + m)^2/s^2$, with $X \sim \mathcal{N}(0, s^2)$ and $l = m^2/s^2$. The following proposition holds.

Theorem 4.4. $L_n^{(2,-1/2)}\left(-l^2/2\right) = H_{2n}^{(-1)}(l).$

Proof. From the right hand side of (4.5) we have

$$\begin{aligned} L_n^{(2,-1/2)}\left(-\frac{l^2}{2}\right) &= 2^n \sum_{k=0}^{n} \frac{(n-1/2)!}{(n-k)!t(k-1/2)!} \frac{n!}{k!} \frac{l^{2k}}{2^k} = \\ &= 2^n \sum_{k=0}^{n} \binom{n}{k} \frac{(n-1/2)!}{(k-1/2)!} \frac{l^{2k}}{2^k} = \\ &= \sum_{k=0}^{n} \frac{(2n)!}{(n-k)! 2^{n-k}} \frac{l^{2k}}{(2k)!} = \sum_{k=0}^{n} \binom{2n}{2k} \mathbb{E}[(\beta{\boldsymbol .}\delta)^{2k}] l^{2(n-k)} \end{aligned} \tag{4.6}$$

where the first equality in (4.6) follows as $(k-1/2)! = (2k)!/(k!\,4^k)$ for $k = 1, 2, \cdots, n$ and the latter equality follows from (4.3). Since $\mathbb{E}[(\beta{\boldsymbol .}\delta)^{2k+1}] = 0$ for all non-negative integers k, the result follows by comparing (4.6) with (4.2).

□

From (4.2) and Theorem 4.3, next Corollary gives the umbral version of Theorem 4.4.

Corollary 4.5. $\left[1/2 + l^2/2{\boldsymbol .}\beta\right]{\boldsymbol .}(2\,\bar{u}) \equiv (l + \beta{\boldsymbol .}\delta)^2.$

For $l = 0$ equivalence (4.4) is recovered. From Corollary 4.5 and Theorem 4.3, the generalization to $k > 1$ is given in the following theorem.

Theorem 4.6. *If $\{\delta_1, \cdots, \delta_k\}$ are k uncorrelated umbrae similar to δ, then* $\left[(k/2) + (l^2/2){\boldsymbol .}\beta\right]{\boldsymbol .}(2\,\bar{u}) \equiv \sum_{i=1}^{k}(l_i + \beta{\boldsymbol .}\delta_i)^2.$

From the well-known multinomial theorem and Theorem 4.6, the expansion of 2-generalized Laguerre polynomials in terms of Hermite polynomials follows.

Corollary 4.7. $L_n^{(2,k/2-1)}\left(-l^2/2\right) = \sum_{\substack{i_1,\cdots,i_k \in [n] \\ i_1+\cdots+i_k=n}} \binom{n}{i_1,\cdots,i_k} \prod_{j=1}^{k} H_{2i_j}^{(-1)}(l_j).$

Remark 4.8. A different way to write $\mathfrak{c}^2_{l,k}$ is $\mathrm{Tr}[(\boldsymbol{Z} + \boldsymbol{l})^T(\boldsymbol{Z} + \boldsymbol{l})]$ where $\boldsymbol{Z} = (Z_1, \cdots, Z_k) \sim \mathcal{N}(\boldsymbol{0}, I_k), \boldsymbol{l} = (l_1, \cdots, l_k)$. If $\boldsymbol{l} = \boldsymbol{0}$, then the central chi-squared distribution with k degrees of freedom is recovered. These representation opens the way to deal with the more general Wishart distribution, as shown in the next paragraph.

Non-central Wishart distribution. Let $\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n$ be row random vectors independently drawn from a k-variate normal distribution with $\boldsymbol{0}$ mean and full rank covariance matrix Σ. Let $\boldsymbol{m}_1, \cdots, \boldsymbol{m}_n$ be real row vectors of dimension k. The non-central Wishart random matrix of order k is

$$W_k(n, \Sigma, M) = W_k(n) = \sum_{i=1}^{n} (\boldsymbol{X}_i + \boldsymbol{m}_i)^T (\boldsymbol{X}_i + \boldsymbol{m}_i) \quad \text{with} \quad M = \sum_{i=1}^{n} \boldsymbol{m}_i^T \boldsymbol{m}_i \,.$$

The matrix $\Omega = \Sigma^{-1} M$ is called the non centrality matrix.
Since $\mathrm{Tr}(\Omega) = \sum_{i=1}^{n} \boldsymbol{m}_i \Sigma^{-1} \boldsymbol{m}_i^T$, the name parallels the non-centrality parameter of $\mathfrak{c}_{l,k}^2$. The m.g.f. of $\mathrm{Tr}[W_k(n)]$ is

$$M_{\mathrm{Tr}[W_k(n)]}(t) = \frac{1}{\det(I - 2t\Sigma)^{n/2}} \exp\left(\mathrm{Tr}[(I - 2t\Sigma)^{-1} M t]\right) . \tag{4.7}$$

Assume $\theta_1, \cdots, \theta_k$ are the eigenvalues of Σ, and Q the eigenvector matrix such that $\Sigma = Q \Lambda Q^T$ with $\Lambda = \mathrm{diag}(\theta_1, \cdots, \theta_k)$. Then the following theorem states the umbral counterpart of $\mathrm{Tr}[W_k(n)]$.

Theorem 4.9. *Set $\boldsymbol{v}_j = \boldsymbol{m}_j \Sigma^{-1/2} Q^T$ for $j = 1, 2, \cdots, n$ and $l_i^2 = \sum_{j=1}^{n} v_{ij}^2$ for $i = 1, 2, \cdots, k$ with v_{ij} the i-th component of the vector $\boldsymbol{v}_j$. Then $\mathrm{Tr}[W_k(n)]$ is umbrally represented by the umbra*

$$\sum_{i=1}^{k} \left\{ \left[\frac{n}{2} + \frac{l_i^2}{2} \boldsymbol{.} \beta \right] \boldsymbol{.} (2\theta_i \bar{u}_i) \right\} . \tag{4.8}$$

Proof. We will prove that the m.g.f. (4.7) is the g.f. of (4.8). First, let us observe that if $\theta_1, \cdots, \theta_k$ are the eigenvalues of Σ, then $\det(I - 2t\Sigma)^{-1} = \prod_{i=1}^{k} (1 - 2\,\theta_i\, t)^{-1}$. Moreover $\mathrm{Tr}[(I - 2\,t\,\Sigma)^{-1} M\, t] = \mathrm{Tr}[(I - 2\,t\,\Sigma)^{-1}\, M\, Q^T\, Q\, t] = \mathrm{Tr}[(I - 2\,t\,\Lambda)^{-1}\, Q\, M\, Q^T\, t] = \mathrm{Tr}[(I - 2\,t\,\Lambda)^{-1}\, (\Lambda\, t)\, Q\, \Sigma^{-1/2}\, M\, \Sigma^{-1/2}\, Q^T] = \sum_{j=1}^{n} \mathrm{Tr}[(I - 2\,t\,\Lambda)^{-1} (\Lambda\, t)\, \boldsymbol{v}_j^T\, \boldsymbol{v}_j]$ with $\boldsymbol{v}_j = \boldsymbol{m}_j \Sigma^{-1/2} Q^T$ and

$$\sum_{j=1}^{n} \mathrm{Tr}[(I - 2t\Lambda)^{-1} (\Lambda t)\, \boldsymbol{v}_j^T\, \boldsymbol{v}_j] = \sum_{j=1}^{n} \sum_{i=1}^{k} \frac{v_{ij}^2 \theta_i t}{1 - 2\,\theta_i\, t} = \sum_{i=1}^{k} l_i^2 \frac{\theta_i t}{1 - 2\,\theta_i\, t} .$$

Then the m.g.f. (4.7) may be rewritten as

$$M_{\mathrm{Tr}[W_k(n)]}(t) = \prod_{i=1}^{k} \left[\frac{1}{(1 - 2\,\theta_i\, t)^{n/2}} \exp\left\{ l_i^2 \frac{\theta_i t}{1 - 2\,\theta_i\, t} \right\} \right] \tag{4.9}$$

and therefore $M_{\mathrm{Tr}[W_k(n)]}(t) = \prod_{i=1}^{k} M_{\mathfrak{c}_{l_i,n}^2}(\theta_i t)$. The result follows from Theorem 4.3.

□

Remark 4.10. Note that if $n = 1$ and $\Sigma = I_k$ the non-central chi-square r.v. is recovered as follows by comparing (4.8) with the result of Theorem 4.1, taking into account property (2.1).

As corollary and by using the well-known multinomial theorem, moments of $\mathrm{Tr}[W_k(n)]$ can be computed by using $2\theta_i$-generalized Laguerre polynomials

$$\mathbb{E}[(\mathrm{Tr}[W_k(n)])^r] = \sum_{\substack{i_1, \cdots, i_k \in [r] \\ i_1 + \cdots + i_k = r}} \binom{r}{i_1, \cdots, i_k} \prod_{j=1}^{k} L_{i_j}^{(2\theta_j,\, n/2-1)} \left(-\frac{l_j^2}{2} \right) .$$

Gamma processes. The employment of moments of Sheffer umbrae covers different fields. Among them, one of the most attractive is mathematical finance in connection with a special class of stochastic processes, that is Lévy processes. These processes well fit the main dynamics of a market being continuous processes interspersed with jump discontinuities of random size and at random times. In [3], the symbolic representation of a Lévy process has been proven to be $t \boldsymbol{.} \gamma$, where γ

represents the increment of order 1 of $\{X_t\}_{t\geq 0}$, that is X_1. In order to include the risk neutrality, a martingale pricing is required in dealing with options. But Lévy processes do not necessarily share the martingale property unless they are centred.

A different approach consist in using the so-called polynomial processes $\{P(x,t)\}_{t\geq 0}$ built by considering a suitable family of polynomials and by replacing the indeterminate x with a stochastic process $\{X_t\}_{t\geq 0}$.

Definition 4.11. The polynomial sequence $\{P(x,t)\}_{t\geq 0}$ is called *time-space harmonic* (*TSH*) with respect to a stochastic process $\{X_t\}_{t\geq 0}$ if

$$E[P(X_t,t) \mid \mathfrak{F}_s] = P(X_s,s)\;,$$

for all $0 \leq s \leq t$, where $\mathfrak{F}_s = \sigma\,(X_\tau \,:\, 0 \leq \tau \leq s)$ is the natural filtration[4] associated with $\{X_t\}_{t\geq 0}$.

Lévy-Sheffer polynomials [3] are a special class of TSH polynomials umbrally represented by $x{\boldsymbol .}\beta{\boldsymbol .}\alpha + t{\boldsymbol .}\gamma$. These polynomials are TSH with respect to Lévy processes umbrally represented by $\{-t{\boldsymbol .}\alpha{\boldsymbol .}\beta{\boldsymbol .}\gamma^{<-1>}\}$. Here the employment of the minus sign permits to simplify some algebraic manipulations. The following result generalizes the applications of Laguerre-type polynomials given in [17].

Theorem 4.12. *The a-generalized Laguerre polynomials are TSH polynomials with respect to a Gamma process of parameters (a^{-1},b).*

Proof. a-Generalized Laguerre polynomials are TSH polynomials taking into account Theorem 3.1, choosing as umbra α the umbra $-a\bar{u}^{<-1>}$, as umbra γ the umbra $a\bar{u}$, and replacing $\nu+1$ with t. To prove that the corresponding Lévy process is a Gamma process, let us observe that the g.f. of $-t{\boldsymbol .}\alpha{\boldsymbol .}\beta{\boldsymbol .}\gamma^{<-1>} \equiv (-a\bar{u}){\boldsymbol .}\beta{\boldsymbol .}(a\bar{u})^{<-1>}$ is

$$\left.\frac{1-2za}{1-za}\right|_{\frac{1+2za}{1+za}} = 1 - az\;.$$

Then we have $(-a\bar{u}){\boldsymbol .}\beta{\boldsymbol .}(a\bar{u})^{<-1>} \equiv -a\chi$ and the g.f. of the corresponding Lévy process is $(1-az)^{-t}$ which corresponds to a Gamma process of parameters $(a^{-1},1)$. The generalization to (a^{-1},b) follows by replacing t with bt.

□

5. Conclusions

In this paper, we have provided a very simple closed formula to represent a-generalized Laguerre polynomials. We have proved a few fundamental properties, but more can be given as Sheffer polynomial sequences. Two kind of applications have been proposed.

The first involves distribution theory and can be further developed within random matrices. As example, a-generalized Laguerre polynomials can be employed to characterize more general class of moments of real non-central Wishart distributions,

$$E\left\{\mathrm{Tr}[W_k(n)H_1]^{i_1}\,\mathrm{Tr}[W_k(n)H_2]^{i_2}\cdots\mathrm{Tr}[W_k(n)H_m]^{i_m}\right\} \quad , \quad H_1,\cdots,H_m \in \mathbb{R}^{k\times k}\;,$$

[4] A natural filtration $\mathfrak{F}_t$ is the σ-algebra generated by the pre-images $X_s^{-1}(B)$ for Borel subsets B of $\mathbb{R}$ and times s with $0 \leq s \leq t$.

similarly to what has been done in [4]. To reach this goal, the multivariate version of the symbolic moment method needs to be introduced, which is beyond the purposes and possibilities of this paper and so will be developed in a future work.

The second application is strictly related to stochastic processes, since Lévy-Sheffer polynomials provide a tool to define a more general class of polynomial processes useful in mathematical finance. A different in-depth analysis deserves TSH polynomials when one deals with matrix-valued stochastic processes. A first attempt in this direction involving Laguerre polynomials and Hermite polynomials can be found in [11]. In umbral terms, this means to replace g.f.'s with hypergeometric functions, which up to now has not yet been developed. Such a theory will have interesting applications within random matrices too, providing a symbolic representation for different matrix-valued polynomials whose computational handling is still an open problem, as for example zonal polynomials [3].

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