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Asymptotic average of the local time for one-dimensional diffusion processes

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Abstract. In a variety of stochastic models it is necessary to have statistical information on the first-passage-time moments and on the asymptotic behavior of the average of the local time. This paper presents a survey of results on certain aspects of these problems and provides the basics for further study in this area.

1. Introduction and background

One-dimensional diffusion processes with one or two boundaries appear in many applications about economics, finance, queueing systems, engineering and mathematical biology (see, for instance, [3], [4], [11], [14], [16]). A relevant effort has been paid towards first passage time (FPT) problems in the presence of absorbing and/or reflecting boundaries, as well as to determination of the corresponding moments (cf., for instance, [5], [8] and references therein). In particular, in [15], [17] and [18] recurrence relations for the moments of the first passage time for time-homogeneous diffusion processes in the presence of a single boundary and in the presence of pairs of boundaries (both not reflecting) are given. For time-homogeneous diffusion processes an important role is played by the random variable $L(t, x)$, denoting the time spend by the considered process in the neighborhood of an interior state x between time 0 and time t (cf., for instance, [2], [9], [10], [12], [13]).

In this paper, a detailed study of the asymptotic behavior of the expected local time is carried out for one-dimensional time homogeneous diffusion processes in the presence of absorbing and/or reflecting boundaries (both not reflecting). Finally, the obtained results are applied to Wiener and Ornstein-Uhlenbeck (OU) diffusion processes.

Let $\{X(t), t \geq 0\}$ be a regular, one-dimensional time-homogeneous diffusion process with space-state $I = (r_1, r_2)$, drift $A_1(x)$ and infinitesimal variance $A_2(x)$, that we assume to satisfy Feller conditions (cf. [6]), such that $P\{X(0) = y\} = 1$.

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For all $t > 0$ and $x, y \in I$ we consider the transition distribution and the transition probability density function (pdf) of $X(t)$:

$$F(x,t|y) = P\{X(t) < x|X(0) = y\} \quad , \quad f(x,t|y) = \frac{\partial}{\partial x} F(x,t|y) \; . \tag{1}$$

A description of $X(t)$ can be obtained via the diffusion equations approach in which it is assumed that the transition pdf $f(x,t|y)$ satisfies the Fokker-Planck and the Kolmogorov equations

$$\frac{\partial f(x,t|y)}{\partial t} = -\frac{\partial}{\partial x}\,[A_1(x)\,f(x,t|y)] + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\,[A_2(x)\,f(x,t|y)] \ , \tag{2}$$

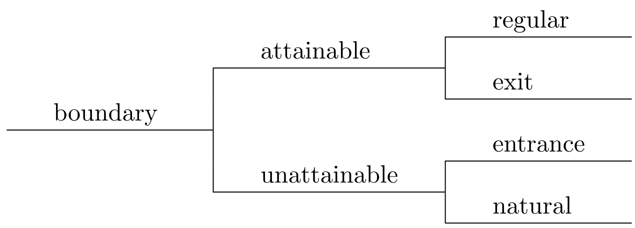
$$\frac{\partial f(x,t|y)}{\partial t} = A_1(y)\,\frac{\partial f(x,t|y)}{\partial y} + \frac{1}{2}\,A_2(y)\,\frac{\partial^2 f(x,t|y)}{\partial y^2} \ , \tag{3}$$

with the associated initial delta condition

$$\lim_{t \downarrow 0} f(x,t|y) = \delta(x-y) \; . \tag{4}$$

Condition (4) expresses the circumstance that initially the whole probability mass is concentrated at the initial state y. An interesting and frequently met situation arises when both boundaries are natural. Then, the initial condition (4) alone uniquely determines the transition pdf as a solution of the diffusion equations (2) and (3). However, (4) is not always sufficient to determine uniquely the transition pdf, so that suitable boundary conditions may have to be imposed (cf. [6], [7]).

1.1. **Boundary classification.** The integration problem for diffusion equations (2) and (3) has been solved by Feller (cf. [6], [7]), who introduced an original classification of the possible behavior of the process near the end points r_1 and r_2 of the diffusion interval. Denoting by r_i $(i = 1,2)$ the generic end point, it may occur that $X(t)$ never attains in finite time the state r_i with positive probability. In this case r_i is an *inaccessible* or *unattainable* boundary. Otherwise, r_i is said to be *accessible* or *attainable.* A finer classification partitions accessible end points into *regular* and *exit* boundaries, whereas inaccessible end points are *entrance* or *natural* boundaries. This boundary classification is depicted in the following diagram.



A time homogeneous diffusion process $X(t)$ can thus enter or leave a regular boundary. For a full characterization of the process, the behavior at the boundary must be specified. This can range from full absorption to full reflection. Instead, if r_i is an exit boundary, starting at r_i the process cannot reach any interior state $x \in I$ in finite time, no matter how near to r_i is x. Furthermore, an entrance boundary cannot be reached from the interior of the state space, while it is possible to consider a process that starts right there. Such a process quickly moves to the interior never to return to the entrance boundary. Finally, a time homogeneous diffusion process $X(t)$ can neither reach a natural boundary in finite mean time nor be originated there. The interest in this classification rests upon the fact, proved by Feller, that if one knows the nature of the end points of the diffusion interval one can decide what kind of boundary condition, if any, have to be associated with the diffusion equation in order to determine the transition pdf of the process. As proved by Feller, the classification of the boundaries depends solely upon certain integrability properties of the coefficient $A_1(x)$ and $A_2(x)$ of the diffusion equations. More specifically, for $x \in I$ let us introduce the scale function and the speed density

$$h(x) = \exp\left\{-2 \int^x \frac{A_1(\xi)}{A_2(\xi)}\, d\xi\right\} \quad , \quad s(x) = \frac{2}{A_2(x)\, h(x)} \ , \tag{5}$$

respectively. Then, for $x, y \in I$ $(x < y)$ the scale measure and the speed measure are:

$$H[x,y] = \int_x^y h(\xi)\, d\xi \quad , \quad S[x,y] = \int_x^y s(\xi)\, d\xi \ . \tag{6}$$

The relevant integrals for the boundaries classification are:

$$\begin{aligned}
&H(r_1) \equiv H(r_1,x] = \lim_{a\to r_1} H[a,x] \quad , \quad H(r_2) \equiv H[x,r_2) = \lim_{b\to r_2} H[x,b]\\
&S(r_1) \equiv S(r_1,x] = \lim_{a\to r_1} S[a,x] \quad , \quad S(r_2) \equiv S[x,r_2) = \lim_{b\to r_2} S[x,b]
\end{aligned}$$

$$\begin{aligned}
U(r_1) &\equiv U(r_1,x] = \lim_{a\to r_1} \int_a^x h(u)\, S[u,x]\, du \ ,\\
U(r_2) &\equiv U[x,r_2) = \lim_{b\to r_2} \int_x^b h(u)\, S[x,u]\, du \ ,\\
V(r_1) &\equiv V(r_1,x] = \lim_{a\to r_1} \int_a^x s(u)\, H[u,x]\, du \ ,\\
V(r_2) &\equiv V[x,r_2) = \lim_{b\to r_2} \int_x^b s(u)\, H[x,u]\, du \ .
\end{aligned} \tag{7}$$

The first classification of the end point r_i in *attainable* or *unattainable* boundary depends on finiteness or infinity of $U(r_i)$. Indeed, the boundary r_i is attainable if $U(r_i) < \infty$ and unattainable otherwise. Furthermore, r_i is attracting if $H(r_i) < \infty$ and nonattracting otherwise. Feller's criteria for such finer boundary classification are summarized in the following table (cf. [10]):

$H(r_i)$	$S(r_i)$	$U(r_i)$	$V(r_i)$	Classification of boundary r_1
$<\infty^*$	$<\infty^*$	$<\infty$	$<\infty$	regular boundary (attracting, attainable)
$<\infty$	$=\infty^*$	$<\infty^*$	$=\infty$	exit boundary (absorbing, attracting, attainable)
$<\infty^*$	$=\infty^*$	$=\infty^*$	$=\infty$	natural boundary (attracting, unattainable)
$=\infty^*$	$<\infty^*$	$=\infty$	$=\infty^*$	natural boundary (nonattracting, unattainable)
$=\infty^*$	$=\infty^*$	$=\infty$	$=\infty$	natural boundary (nonattracting, unattainable)
$=\infty^*$	$<\infty$	$=\infty$	$<\infty^*$	entrance boundary (nonattracting, unattainable)

In this table minimal sufficient conditions for establishing the nature of the boundary r_i are indicated by an asterisk.

1.2. **The local time of a diffusion process.** The *local time* $L(t,x)$ of $X(t)$ at an interior state $x \in I$ is a random variable defined as (cf., for instance, [2], [10]):

$$L(t,x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1\{|X(\tau) - x| \le \varepsilon\}\, d\tau \qquad (t > 0)\ , \tag{8}$$

where for $\varepsilon > 0$

$$1\{|X(\tau) - x| \le \varepsilon\} = \begin{cases} 1, & |X(\tau) - x| \le \varepsilon \\ 0, & \text{otherwise}\ . \end{cases}$$

Intuitively, $L(t,x)$ denotes a measure of how much time $X(t)$ has spent in the neighborhood of x up to time t, so that $L(t,x)$ is often called the *sojourn time*. In the literature, the local time (occupation density) is also defined as (cf. [12], [13])

$$L^*(t,x) = A_2(x)\, L(t,x)\ ,$$

where the factor $A_2(x)$ is included to agree with the general Meyer-Tanaka definition of local time of a semimartingale. An alternative definition of local time is (cf. [10]):

$$\widetilde{L}(t,x) = \lim_{\varepsilon \downarrow 0} \frac{\displaystyle\int_0^t 1\{|X(\tau) - x| \le \varepsilon\}\, d\tau}{\displaystyle\int_{x-\varepsilon}^{x+\varepsilon} s(z)\, dz} \qquad (t > 0)\ , \tag{9}$$

with $s(x)$ defined in (5). The denominator in (9) is a measure of how fast the diffusion process clock runs when $X(t)$ is located in $(x-\varepsilon, x+\varepsilon)$. We note that

$$\widetilde{L}(t,x) = \frac{L(t,x)}{s(x)} = \frac{h(x)}{2}\, A_2(x)\, L(t,x)\ .$$

The conditional average and the second order conditional moment of the local time (8) for the process $X(t)$ are related to its transition probability density $f(x,t|y)$ through the following relations (see, for instance, [2], [10]):

$$\begin{aligned} &E[L(t,x)|X(0)=y] = s(x)\, E[\widetilde{L}(t,x)|X(0)=y] = \\ &\quad = \int_0^t f(x,\tau|y)\, d\tau\ , \\ &E[L^2(t,x)|X(0)=y] = s^2(x) E[\widetilde{L}^2(t,x)|X(0)=y] = \\ &\quad = 2 \int_0^t d\tau \int_0^\tau d\vartheta\, f(x,\vartheta|y)\, f(x,\tau-\vartheta|x)\ . \end{aligned} \tag{10}$$

Therefore, the asymptotic conditional moments of the local time $L(t,x)$ are:

$$\begin{aligned} &L_1^\infty(x|y) = \lim_{t\to+\infty} E[L(t,x)|X(0)=y] = \int_0^{+\infty} f(x,\tau|y)\, d\tau\ , \\ &L_2^\infty(x|y) = \lim_{t\to+\infty} E[L^2(t,x)|X(0)=y] = 2\, L_1^\infty(x|y) L_1^\infty(x|x)\ , \end{aligned} \tag{11}$$

and the asymptotic conditional moments of $\widetilde{L}(t,x)$ are:

$$\widetilde{L}_1^\infty(x|y) = \frac{L_1^\infty(x|y)}{s(x)} \quad , \quad \widetilde{L}_2^\infty(x|y) = \frac{L_2^\infty(x|y)}{s^2(x)}\ . \tag{12}$$

Hence, $L_1^\infty(x|y)$ is the asymptotic average of the local time that the diffusion process $X(t)$ originating at y spends in the state x. The first of (11) shows that $L_1^\infty(x|y)$ is the time integral of the transition pdf, so that it can be also interpreted as the *average residence time density* of the process $X(t)$ at the point x (cf. [1]). We remark that if the process $X(t)$ is recurrent $L_1^\infty(x|y)$ and $\widetilde{L}_1^\infty(x|y)$ diverge, whereas they are finite if $X(t)$ is transient.

In Sections 2, 3 and 4 we determine the asymptotic average of the local time for diffusions in the presence of pairs of boundaries, diffusion in the presence of single boundary and unrestricted diffusion, respectively. Finally, in Section 5 and 6 we analyze the asymptotic average of the local time for the Wiener and Ornstein-Uhlenbeck processes, respectively.

2. Diffusions in the presence of pairs of boundaries

In this Section we analyze diffusion processes restricted by two constant boundaries $a, b \in I$, with $a < b$. In particular, in Section 2.1 we study the problem for two absorbing boundaries, whereas in Section 2.2 we assume that one boundary is absorbing and the other one is reflecting.

2.1. **Two absorbing boundaries.** We assume that a and b are absorbing boundaries ($a < b$) and let $y \in (a, b)$. We define the following first passage time random variables

$$\begin{aligned}
\mathcal{T}^- &= \inf_{t\geq 0}\{t \,:\, X(t) < a;\; a < X(\tau) < b,\; \forall \tau \in (0,t)\} \quad , \quad X(0) = y\\
\mathcal{T}^+ &= \inf_{t\geq 0}\{t \,:\, X(t) > b;\; a < X(\tau) < b,\; \forall \tau \in (0,t)\} \quad , \quad X(0) = y\\
\mathcal{T} &= \inf_{t\geq 0}\{t \,:\, X(t) \notin [a,b]\} \equiv \inf\{\mathcal{T}^-, \mathcal{T}^+\} \quad , \qquad\qquad X(0) = y
\end{aligned}$$

and denote by $g^-(a, b, t|y)$ the first passage time pdf of $X(t)$ to a before crossing b, by $g^+(a, b, t|y)$ the FPT pdf to b before crossing a and by $g(a, b, t|y) = g^-(a, b, t|y)+ g^+(a, b, t|y)$ the first exit time pdf from the interval (a, b). The FPT probabilities $P^-(a, b|y)$ and $P^+(a, b|y)$ are:

$$\begin{aligned}
&P^-(a,b|y) = \int_0^{+\infty} g^-(a,b,t|y)\,dt = \frac{H[y,b]}{H[a,b]} \;, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a < y < b \;. \\
&P^+(a,b|y) = \int_0^{+\infty} g^+(a,b,t|y)\,dt = \frac{H[a,y]}{H[a,b]} \;,
\end{aligned} \tag{13}$$

We remark that the probability of the event "first-exit from the interval (a, b)" is unity, i.e. $P(a, b|y) = P^-(a, b|y) + P^+(a, b|y) = 1$.

We now denote by $\alpha^{(a,b)}(x, t|y)$ the transition pdf of $X(t)$ in the presence of two absorbing boundaries a and b ($a < y < b$)

$$\alpha^{(a,b)}(x,t|y) = \frac{\partial}{\partial x} P\left\{X(t) < x;\; a < X(\tau) < b,\; \forall \tau < t \mid X(0) = y\right\} \;. \tag{14}$$

By a direct enumeration of the paths going from y to x, one is led to:

$$\begin{aligned}
\alpha^{(a,b)}(x,t|y) = f(x,t|y) - \int_0^t \big[&g^-(a,b,\tau|y)\,f(x,t-\tau|a)+ \\
&+g^+(a,b,\tau|y)\,f(x,t-\tau|b)\big]\,d\tau \quad , \quad x,y \in (a,b) \;.
\end{aligned} \tag{15}$$

It is easily seen that the pdf $g(a,b,t|y)$ and $\alpha^{(a,b)}(x,t|y)$ are intimately related. Indeed,

$$\int_a^b \alpha^{(a,b)}(x,t|y)\,dx = 1 - \int_0^t g(a,b,\tau|y)\,d\tau \quad , \quad a<y<b\,. \tag{16}$$

Let $t_n(a,b|y)$ be the n-th order moment of $\mathcal{T}$. By virtue of (16) one has:

$$t_n(a,b|y) = \int_0^{+\infty} t^n g(a,b,t|y)\,dt = n\int_0^{+\infty} dt\; t^{n-1}\int_a^b \alpha^{(a,b)}(x,t|y)\,dx \qquad (n=1,2,\cdots)\,. \tag{17}$$

The moments of $\mathcal{T}$ can be iteratively computed as (see, for instance, [10], [15]):

$$\begin{aligned} t_n(a,b|y) &= nP^+(a,b|y)\int_y^b s(z)H[z,b]t_{n-1}(a,b|z)\,dz+ \\ &+nP^-(a,b|y)\int_a^y s(z)H[a,z]t_{n-1}(a,b|z)\,dz \qquad (n=1,2,\cdots,\; a<y<b) \end{aligned} \tag{18}$$

with $t_0(a,b|z)=P(a,b|z)=1$ for $a<z<b$.

Proposition 2.1. *For $x,y\in(a,b)$, with a and b absorbing boundaries $(a<b)$, one has*

$$\int_0^{+\infty} \alpha^{(a,b)}(x,t|y)\,dt = \begin{cases} P^+(a,b|y)\,s(x)\,H[x,b] & , \quad a<y\le x<b \\ P^-(a,b|y)\,s(x)\,H[a,x] & , \quad a<x\le y<b\,. \end{cases} \tag{19}$$

Proof. Let $k(x)$ be a real function that represents a cost rate incurred whenever the process $X(t)$ is in the state x. The function

$$c(a,b|y) = \int_0^{+\infty} dt \int_a^b \alpha^{(a,b)}(x,t|y)\,k(x)\,dx \qquad (a<y<b) \tag{20}$$

satisfies the following differential equation

$$A_1(y)\,\frac{du(y)}{dy} + \frac{A_2(y)}{2}\,\frac{d^2u(y)}{dy^2} = -k(y) \tag{21}$$

to solve with the boundary conditions:

$$\lim_{y\downarrow a} u(y) = 0 \quad , \quad \lim_{y\uparrow b} u(y) = 0\,.$$

Equation (21) can be written as:

$$\frac{d}{dy}\left[\frac{1}{h(y)}\,\frac{du(y)}{dy}\right] = -k(y)\,s(y)\,,$$

so that, integrating in the interval (a,y), one obtains:

$$\frac{1}{h(y)}\,\frac{du(y)}{dy} = -\int_a^y k(\xi)s(\xi)\,d\xi + \beta\,, \tag{22}$$

where β is an arbitrary real number. By integrating again in the interval (a,y) and making use of the boundary conditions, it follows:

$$c(a,b|y) = -\int_a^y dz\;h(z)\int_a^z k(\xi)s(\xi)\,d\xi + \frac{\displaystyle\int_a^y h(z)\,dz}{\displaystyle\int_a^b h(z)\,dz}\int_a^b dz\;h(z)\int_a^z k(\xi)s(\xi)\,d\xi\,,$$

so that, by rearranging the terms, for $a < y < b$ one has:

$$c(a,b|y) = P^+(a,b|y) \int_y^b k(z)\, s(z)\, H[z,b]\, dz + \\ + P^-(a,b|y) \int_a^y k(z)\, s(z)\, H[a,z]\, dz \,. \tag{23}$$

By assuming that

$$k(z) = \begin{cases} 1 & , \quad x \le z \le x + \Delta \\ 0 & , \quad \text{otherwise} \,, \end{cases} \tag{24}$$

from (20) and (23) one obtains:

$$\int_x^{x+\Delta} dz \int_0^{+\infty} \alpha^{(a,b)}(z,t|y)\, dt =$$

$$= \begin{cases} P^+(a,b|y) \displaystyle\int_x^{x+\Delta} s(z)\, H[z,b]\, dz & , \quad a < y \le x < b \\ P^-(a,b|y) \displaystyle\int_x^{x+\Delta} s(z)\, H[a,z]\, dz & , \quad a < x \le y < b \,, \end{cases}$$

from which, by dividing both sides by Δ and taking the limit as $\Delta \to 0$, (19) follows.

□

Note that (19) represents the asymptotic conditional average $L_1^\infty(x|y)$ of the local time when $X(t)$ is restricted between two absorbing boundaries. Moreover, taking into account (17) for $n = 1$, we integrate both sides of (19) with respect to x in the interval (a, b). Proceeding in this way, we re-obtain $t_1(a, b|y)$ as in (18).

2.2. **One absorbing boundary and one reflecting boundary.** We consider separately the following cases: (i) an absorbing boundary in a and a reflecting boundary in b; *(ii)* a reflecting boundary in a and an absorbing boundary in b.

Case (i). Let $\beta^{(a,b)}(x,t|y)$ be the transition pdf of $X(t)$ in the presence of an absorbing boundary in a and a reflecting boundary in b. The function $\beta^{(a,b)}(x,t|y)$ is the unique solution of (2) with initial condition (4) and boundary conditions

$$\lim_{x \downarrow a} [A_2(x)\, h(x)\, \beta^{(a,b)}(x,t|y)] = 0 \,,$$

$$\lim_{x \uparrow b} \left\{ A_1(x)\, \beta^{(a,b)}(x,t|y) - \frac{1}{2} \frac{\partial}{\partial x} \left[A_2(x)\, \beta^{(a,b)}(x,t|y) \right] \right\} = 0 \,.$$

It is also the unique solution of (3) with initial condition (4) and boundary conditions

$$\lim_{y \downarrow a} \beta^{(a,b)}(x,t|y) = 0 \quad , \quad \lim_{y \uparrow b} \left[h^{-1}(y)\, \frac{\partial}{\partial y} \beta^{(a,b)}(x,t|y) \right] = 0 \,.$$

By a direct enumeration of the sample paths, one has:

$$\beta^{(a,b)}(x,t|y) = \alpha^{(a,b)}(x,t|y) + \\ + \int_0^t g^+(a,b,\tau|y)\, \beta^{(a,b)}(x,t-\tau|b)\, d\tau \quad , \quad x, y \in (a,b] \,, \tag{25}$$

where $\alpha^{(a,b)}(x,t|y)$ denotes the transition pdf in the presence of two absorbing boundaries a,b and $g^{+}(a,b,t|y)$ the FPT pdf to b before crossing a.

Furthermore, let $\gamma^{-}(a,b,t|y)$ be the FPT pdf to a in the presence of a reflecting boundary in b starting from $y \in (a,b]$. It is immediate to prove that $Q^{-}(a,b|y) = \int_0^{+\infty} \gamma^{-}(a,b,t|y)\,dt = 1$, so that the first passage to a occurs with certainty. The functions $\beta^{(a,b)}(x,t|y)$ and he FPT pdf $\gamma^{-}(a,b,t|y)$ are related as follows:

$$\int_a^b \beta^{(a,b)}(x,t|y)\,dx = 1 - \int_0^t \gamma^{-}(a,b,\tau|y)\,d\tau \quad , \quad a < y \le b\,. \tag{26}$$

Let $t_n^{-}(a,b|y)$ be the n-th order moment of FPT to a in the presence of a reflecting boundary in b. By virtue of (26) one has:

$$\begin{aligned} t_n^{-}(a,b|y) &= \int_0^{+\infty} t^n \gamma^{-}(a,b,t|y)\,dt = \\ &= n \int_0^{+\infty} dt\; t^{n-1} \int_a^b \beta^{(a,b)}(x,t|y)\,dx \qquad (n = 1,2,\cdots)\,. \end{aligned} \tag{27}$$

Alternatively, the moments $t_n^{-}(a,b|y)$ can be iteratively computed as:

$$\begin{aligned} t_n^{-}(a,b|y) &= n \int_a^y dz\; h(z) \int_z^b s(u)\, t_{n-1}^{-}(a,b|u)\,du \\ &\qquad\qquad (n = 1,2,\cdots\,;\; a < y \le b)\,, \end{aligned} \tag{28}$$

with $t_0^{-}(a,b|z) = Q^{-}(a,b|z) = 1$ for $a < z \le b$.

Proposition 2.2. *For $x,y \in (a,b]$, with a absorbing boundary and b reflecting boundary ($a < b$), one has:*

$$\int_0^{+\infty} \beta^{(a,b)}(x,t|y)\,dt = \begin{cases} s(x)\,H[a,y], & a < y \le x \le b \\ s(x)\,H[a,x], & a < x \le y \le b. \end{cases} \tag{29}$$

Proof. Let $k(x)$ a cost rate incurred whenever the process $X(t)$ is in the state x. The function

$$b(a,b|y) = \int_0^{+\infty} dt \int_a^b \beta^{(a,b)}(x,t|y)\,k(x)\,dx \qquad (a < y \le b) \tag{30}$$

satisfies the differential equation (21) to solve with an absorbing boundary condition in a and a reflecting boundary condition in b:

$$\lim_{y \downarrow a} u(y) = 0 \quad , \quad \lim_{y \uparrow b} \left\{ h^{-1}(y)\, \frac{du(y)}{dy} \right\} = 0\,.$$

Recalling (22) and by imposing the reflecting condition in b one has
$\beta = \int_a^b k(\xi) s(\xi)\,d\xi$, so that

$$\frac{1}{h(y)}\,\frac{du(y)}{dy} = \int_y^b k(\xi)\,s(\xi)\,d\xi\,.$$

Integrating again in the interval (a, y) and making use of the absorbing condition in a, for $a < y \leq b$ it follows:

$$
\begin{aligned}
b(a,b|y) &= \int_a^y dz\, h(z) \int_z^b k(\xi) s(\xi)\, d\xi = \\
&= H[a,y] \int_y^b k(z)\, s(z)\, dz + \int_a^y k(z)\, s(z)\, H[a,z]\, dz\,.
\end{aligned}
\tag{31}
$$

By choosing $k(z)$ as in (24), from (30) and (31) one obtains:

$$
\int_x^{x+\Delta} dz \int_0^{+\infty} \beta^{(a,b)}(z,t|y)\, dt = \begin{cases} H[a,y] \displaystyle\int_x^{x+\Delta} s(z)\, dz & , \quad a < y \leq x \leq b \\ \displaystyle\int_x^{x+\Delta} s(z)\, H[a,z]\, dz & , \quad a < x \leq y \leq b\,, \end{cases}
$$

from which, by dividing both sides by Δ and taking the limit as $\Delta \to 0$, (29) follows.

□

Note that (29) describes the asymptotic conditional average $L_1^{\infty}(x|y)$ of the local time when $X(t)$ is restricted to $(a, b]$, with an absorbing boundary in a and a reflecting boundary in b. Furthermore, taking into account (27) for $n = 1$, we integrate both sides of (29) with respect to x in the interval (a, b). Proceeding in this way, we re-obtain $t_1^-(a, b|y)$ as in (28).

Case (ii). Let $\eta^{(a,b)}(x, t|y)$ be the transition pdf of $X(t)$ in the presence of a reflecting boundary in a and an absorbing boundary in b. The function $\eta^{(a,b)}(x, t|y)$ is the unique solution of (2) with initial condition (4) and boundary conditions

$$
\lim_{x \downarrow a} \left\{ A_1(x)\, \eta^{(a,b)}(x,t|y) - \frac{1}{2} \frac{\partial}{\partial x} \Big[A_2(x)\, \eta^{(a,b)}(x,t|y) \Big] \right\} = 0\,,
$$

$$
\lim_{x \uparrow b} [A_2(x)\, h(x)\, \eta^{(a,b)}(x,t|y)] = 0\,.
$$

It is also the unique solution of (3) with initial condition (4) and boundary conditions

$$
\lim_{y \downarrow a} \left[h^{-1}(y)\, \frac{\partial}{\partial y} \eta^{(a,b)}(x,t|y) \right] = 0 \quad , \quad \lim_{y \uparrow b} \eta^{(a,b)}(x,t|y) = 0\,.
$$

Clearly,

$$
\begin{aligned}
\eta^{(a,b)}(x,t|y) &= \alpha^{(a,b)}(x,t|y) + \\
&+ \int_0^t g^-(a,b,\tau|y)\, \eta^{(a,b)}(x,t-\tau|a)\, d\tau \quad , \quad x, y \in [a,b)\,,
\end{aligned}
\tag{32}
$$

where $\alpha^{(a,b)}(x, t|y)$ denotes the transition pdf in the presence of two absorbing boundaries a, b and $g^-(a, b, t|yy)$ the FPT pdf to a before crossing b.

Furthermore, let $\gamma^+(a, b, t|y)$ be the FPT pdf to b in the presence of a reflecting boundary in a starting from $y \in [a, b)$. It is immediate to prove that $Q^+(a, b|y) = \int_0^{+\infty} \gamma^+(a, b, t|y)\, dt = 1$, so that the first passage to b occurs with certainty. The functions $\eta^{(a,b)}(x, t|y)$ and the FPT pdf $\gamma^+(a, b, t|y)$ are related as follows:

$$
\int_a^b \eta^{(a,b)}(x,t|y)\, dx = 1 - \int_0^t \gamma^+(a,b,\tau|y)\, d\tau \quad , \quad a \leq y < b\,.
\tag{33}
$$

Let $t_n^+(a,b|y)$ be the n-th order moment of FPT to a in the presence of a reflecting boundary in a. By virtue of (33) one has:

$$
\begin{aligned}
t_n^+(a,b|y) &= \int_0^{+\infty} t^n \gamma^+(a,b,t|y)\,dt = \\
&= n\int_0^{+\infty} dt\; t^{n-1}\int_a^b \eta^{(a,b)}(x,t|y)\,dx \quad (n=1,2,\cdots)\;.
\end{aligned} \tag{34}
$$

Alternatively, the moments $t_n^+(a,b|y)$ can be iteratively computed as:

$$
\begin{aligned}
t_n^+(a,b|y) = n\int_y^b dz\; h(z)\int_a^z s(u)\, t_{n-1}^+(a,b|u)\,du \\
(n=1,2,\cdots;\; a\le y<b)\;,
\end{aligned} \tag{35}
$$

with $t_0^+(a,b|z) = Q^+(a,b|z) = 1$ for $a \le z < b$.

Proposition 2.3. *For $x,y \in [a,b)$, with a reflecting boundary and b absorbing boundary, one has:*

$$
\int_0^{+\infty} \eta^{(a,b)}(x,t|y)\,dt = \begin{cases} s(x)\,H[x,b] \quad , & a\le y\le x<b \\ s(x)\,H[y,b] \quad , & a\le x\le y<b\;. \end{cases} \tag{36}
$$

Proof. The proof proceeds similarly to that of Proposition 2.2.

□

Hence, (36) gives the asymptotic conditional average $L_1^\infty(x|y)$ of the local time when $X(t)$ is restricted to $[a,b)$ with a reflecting boundary in a and an absorbing boundary in b. Moreover, recalling (34) for $n=1$ and integrating both sides of (36) with respect to x in the interval (a,b), we obtain $t_1^+(a,b|y)$ as given in (35).

Equations (29) and (36) show that the reflecting boundary acts on the asymptotic average of the local time only by limiting the state-space of the process.

3. Diffusions in the presence of a single boundary

In this section we consider diffusion processes restricted by a single boundary. In Section 3.1 we consider a single absorbing boundary and in Section 3.2 we examine the case of a single reflecting boundary.

3.1. **Single absorbing boundary.**

We define the following first passage time random variable

$$
\mathcal{T} = \begin{cases} \inf\limits_{t\ge 0}\{t : X(t)>a\} \quad , & X(0)=y<a \\ \inf\limits_{t\ge 0}\{t : X(t)<a\} \quad , & X(0)=y>a\;. \end{cases} \tag{37}
$$

and denote by $g(a,t|y)$ its pdf. As is well known,

$$
f(x,t|y) = \int_0^t g(a,\tau|y)\,f(x,t-\tau|a)\,d\tau \quad , \quad y<a\le x \quad \text{or} \quad x\le a<y\;. \tag{38}
$$

Furthermore, let $\alpha(x,t|y)$ be the transition pdf of $X(t)$ in the presence of an absorbing boundary $a \in I$:

$$\alpha(x,t|y)= \tag{39}$$

$$=\begin{cases} \dfrac{\partial}{\partial x}P\{X(t)<x; X(\tau)<a,\ \forall\tau<t \mid X(0)=y\} & , \quad x<a,\ y<a \\[2ex] \dfrac{\partial}{\partial x}P\{X(t)<x; X(\tau)>a,\ \forall\tau<t \mid X(0)=y\} & , \quad x>a,\ y>a\ . \end{cases}$$

This is the unique solution of (2) with initial condition (4) and boundary condition $\lim_{x\to a}[A_2(x)\ \ h(x)\,\alpha(x,t|y)]=0$; furthermore, it is the unique solution of (3) with initial condition (4) and boundary condition $\lim_{y\to a}\alpha(x,t|y)=0$. By a direct enumeration of the paths going from y to x, one is led to:

$$\alpha(x,t|y)=f(x,t|y)- \tag{40}$$

$$-\int_0^t g(a,\tau|y)\,f(x,t-\tau|a)\,d\tau \quad ,x<a,\ y<a \quad \text{or} \quad x>a,\ y>a\ .$$

The pdf $g(a,t|y)$ and $\alpha(x,t|y)$ are intimately related as:

$$\begin{aligned} &\int_{r_1}^{a}\alpha(x,t|y)\,dx=1-\int_0^t g(a,\tau|y)\,d\tau \quad , \quad y<a \\ &\int_{a}^{r_2}\alpha(x,t|y)\,dx=1-\int_0^t g(a,\tau|y)\,d\tau \quad , \quad y>a\ . \end{aligned} \tag{41}$$

We denote by $P(a|y)=\int_0^{+\infty}g(a,t|y)\ dt$ the first-passage time probability to a starting from y and by $t_n(a|y)$ the n-th order moment of $\mathcal{T}$.

Proposition 3.1. *For $x,y\in(a,r_2)$, with a absorbing boundary, one has:*
(i) *Let r_2 be a nonattracting end point (natural or entrance). Then, $P(a|y)=1$ for $y>a$ and*

$$\int_0^{+\infty}\alpha(x,t|y)\,dt=\begin{cases} s(x)H[a,y] & , \quad a<y\le x \\ s(x)H[a,x] & , \quad a<x\le y\ . \end{cases} \tag{42}$$

Furthermore, if $S[x,r_2)$ converges, then the moments $t_n(a|y)$ are finite and can be iteratively computed as:

$$t_n(a|y)=n\int_a^y dz\ h(z)\int_z^{r_2}s(u)\,t_{n-1}(a|u)\,du \qquad (n=1,2,\cdots;y>a)\ , \tag{43}$$

with $t_0(a|z)=P(a|z)=1$ for $z>a$.
(ii) *Let r_2 be a natural attracting end point. Then, $P(a|y)=H[y,r_2)/H[a,r_2)$ for $y>a$ and*

$$\int_0^{+\infty}\alpha(x,t|y)\,dt=\begin{cases} s(x)P(a|x)H[a,y] & , \quad a<y\le x \\ s(x)P(a|y)H[a,x] & , \quad a<x\le y\ . \end{cases} \tag{44}$$

Proof. If r_2 is a nonattracting end point, then $H[x,r_2)$ diverges, so that from the first of (13) one has $P(a|y)=\lim_{b\uparrow r_2}P^-(a,b|y)=1$. Moreover, $\alpha(x,t|y)=$

$\lim_{b\uparrow r_2} \alpha^{(a,b)}(x,t|y)$, so that by virtue of (13) and (19), relation (42) follows. If $S[x,r_2)$ converges, making use of (13), from (18) one has:

$$t_n(a|y) = \lim_{b\uparrow r_2} t_n(a,b|y) = nH[a,y] \int_y^{r_2} s(z) t_{n-1}(a|z) +$$

$$+n \int_a^y du\, h(u) \int_u^y s(z) t_{n-1}(a|z)\, dz\ ,$$

from which (43) immediately follows.

Instead, if r_2 is a natural attracting end point, then $H[x,r_2)$ converges, so that from (13) one has $P(a|y) = \lim_{b\uparrow r_2} P^-(a,b|y) = H[y,r_2)/H[a,r_2)$. Furthermore, by virtue of (13) and (19), one obtains (44).

□

Hence, (42) and (44) describe the asymptotic conditional average $L_1^\infty(x|y)$ of the local time when $X(t)$ is restricted to (a,r_2), with an absorbing boundary in a. We note that if r_2 is a nonattracting end point and $S[x,r_2)$ converges, then by integrating both sides of (42) with respect to x in the interval (a,r_2) we have (43) for $n=1$.

Proposition 3.2. *For $x,y \in (r_1,a)$, with a absorbing boundary, one has:*
(i) *Let r_1 be a nonattracting end point (natural or entrance). Then, $P(a|y)=1$ for $y<a$ and*

$$\int_0^{+\infty} \alpha(x,t|y)\, dt = \begin{cases} s(x)H[x,a] & , \quad y \le x < a \\ s(x)H[y,a] & , \quad x \le y < a\ . \end{cases} \tag{45}$$

Furthermore, if $S(r_1,x]$ converges, then the moments $t_n(a|y)$ are finite and can be iteratively computed as:

$$t_n(a|y) = n \int_y^a dz\, h(z) \int_{r_1}^z s(u)\, t_{n-1}(a|u)\, du \qquad (n=1,2,\cdots; y<a)\ , \tag{46}$$

with $t_0(a|z) = P(a|z) = 1$ for $z<a$.
(ii) *Let r_1 be a natural attracting end point. Then, $P(a|y) = H(r_1,y]/H(r_1,a]$ for $y<a$ and*

$$\int_0^{+\infty} \alpha(x,t|y)\, dt = \begin{cases} s(x)P(a|y)H[x,a] & , \quad y \le x < a \\ s(x)P(a|x)H[y,a] & , \quad x \le y < a\ . \end{cases} \tag{47}$$

Proof. The proof proceeds similarly to that of Proposition 3.1.

□

Hence, (45) and (47) describe the asymptotic conditional average $L_1^\infty(x|y)$ of the local time when $X(t)$ is restricted to (r_1,a), with a absorbing boundary. We note that if r_1 is a nonattracting end point and $S(r_1,rx]$ converges, then by integrating both sides of (45) with respect to x in the interval (r_1,a) we obtain (46) for $n=1$.

3.2. **Diffusion process in the presence of a reflecting boundary.** Let $r(x,t|y)$ be the transition pdf of $X(t)$ in the presence of a reflecting boundary $a \in I$. Such density is the unique solution of (2) with initial condition (4) and boundary condition

$$\lim_{x\to a} \left\{ A_1(x) r(x,t|y) - \frac{1}{2}\frac{\partial}{\partial x}\left[A_2(x)\, r(x,t|y)\right] \right\} = 0\ ; \tag{48}$$

furthermore, it is the unique solution of (3) with initial condition (4) and boundary condition

$$\lim_{y\to a}\left[h^{-1}(y)\,\frac{\partial}{\partial y}r(x,t|y)\right]=0\,. \tag{49}$$

The following relation among $r(x,t|y)$, $\alpha(x,t|y)$ and $g(a,t|y)$ holds:

$$r(x,t|y)=\alpha(x,t|y)+\int_0^t g(a,\tau|y)\,r(x,t-\tau|a)\,d\tau\,, \tag{50}$$
$$x\ge a,\ y\ge a\quad\text{or}\quad x\le a,\ y\le a\,.$$

Proposition 3.3. *For $x,y\in(r_1,a]$, with a reflecting boundary, one has:*
(i) *If r_1 is a natural attracting end point, one has:*

$$\int_0^{+\infty} r(x,t|y)\,dt=\begin{cases} s(x)H(r_1,y] & ,\quad y\le x\le a\\ s(x)H(r_1,x] & ,\quad x\le y\le a\,.\end{cases} \tag{51}$$

(ii) *If r_1 is a nonattracting end point (natural or entrance), then*

$$\int_0^{+\infty} r(x,t|y)\,dt=+\infty\qquad(x\le a,\ y\le a)\,. \tag{52}$$

Proof. In the case (i), (51) follows from (29) by taking the limit as $a\downarrow r_1$ and recalling that $H(r_1,z)$ is finite. Instead, in the case (ii) $H(r_1,z)$ diverges, implying that (52) holds.

□

Proposition 3.4. *For $x,y\in[a,r_2)$, with a reflecting boundary, one has:*
(i) *If r_2 is a natural attracting end point, one has:*

$$\int_0^{+\infty} r(x,t|y)\,dt=\begin{cases} s(x)H[x,r_2) & ,\quad a\le y\le x\\ s(x)H[y,r_2) & ,\quad a\le x\le y\,.\end{cases} \tag{53}$$

(ii) *If r_2 is a nonattracting end point (natural or entrance), then*

$$\int_0^{+\infty} r(x,t|y)\,dt=+\infty\qquad(x\ge a,\ y\ge a)\,. \tag{54}$$

Proof. It is analogue to the proof of Proposition 3.3 by taking the limit as $b\uparrow r_2$ in (36).

□

Hence, (51) and (53) describe the asymptotic conditional average $L_1^\infty(x|y)$ of the local time when $X(t)$ is restricted by a reflecting boundary in a. Moreover, the reflecting boundary acts on $L_1^\infty(x|y)$ only by limiting the state-space of the process.

4. Unrestricted diffusions

In this section we consider the case of unrestricted diffusion process $X(t)$, with space-state $I=(r_1,r_2)$, where r_1 and r_2 are unattainable end points. We denote by $f(x,t|y)$ the transition pdf of the unrestricted diffusion process $X(t)$.

Proposition 4.1. *For $x,y\in I$ one has:*

(i) *If* r_1 *is a natural attracting end point and* r_2 *is a nonattracting end point* (*natural or entrance*), *one has:*

$$\int_0^{+\infty} f(x,t|y)\,dt = \begin{cases} s(x)H(r_1,y] & , \quad y \le x \\ s(x)H(r_1,x] & , \quad x \le y\,. \end{cases} \tag{55}$$

(ii) *If* r_1 *is a nonattracting end point* (*natural or entrance*) *and* r_2 *is a natural attracting end point, then:*

$$\int_0^{+\infty} f(x,t|y)\,dt = \begin{cases} s(x)H[x,r_2) & , \quad y \le x \\ s(x)H[y,r_2) & , \quad x \le y\,. \end{cases} \tag{56}$$

(iii) *If* r_1 *and* r_2 *are both natural attracting end points, one has:*

$$\int_0^{+\infty} f(x,t|y)\,dt = \begin{cases} s(x)\,\dfrac{H(r_1,y]H[x,r_2)}{H[r_1,r_2)} & , \quad y \le x \\[2ex] s(x)\,\dfrac{H(r_1,x]H[y,r_2)}{H[r_1,r_2)} & , \quad x \le y\,. \end{cases} \tag{57}$$

(iv) *If* r_1 *and* r_2 *are both nonattracting end points* (*natural or entrance*), *then*

$$\int_0^{+\infty} f(x,t|y)\,dt = +\infty\,. \tag{58}$$

Proof. In the case (i) $H(r_1,x] < +\infty$ and $H[x,r_2] = +\infty$, so that taking the limit as $a \downarrow r_1$ in (42) one obtains (55). Instead, in the case (ii) $H(r_1,x] = +\infty$ and $H[x,r_2] < +\infty$, so that taking the limit as $a \uparrow r_2$ in (45) one obtains (56). Furthermore, if r_1 and r_2 are both natural attracting end points one has $H(r_1,x] < +\infty$ and $H[x,r_2] < +\infty$, so that taking the limit as $a \downarrow r_1$ in (44) one obtains (56). This result can be also derived from (47) taking the limit as $a \uparrow r_2$. Finally, if r_1 and r_2 are both nonattracting end points $H(r_1,x] = +\infty$ and $H[x,r_2] = +\infty$, so that from (42), or equivalently from (45), one has (58). $\square$

Hence, (55), (56) and (57) describe the asymptotic conditional average $L_1^\infty(x|y)$ of the local time for the unrestricted process $X(t)$. We note that if r_1 and r_2 are nonattracting end points $L_1^\infty(x|y)$ diverges.

5. Asymptotic average of local time for the Wiener process

In this section we assume that $\{X(t), t \ge 0\}$ is the Wiener process with drift and infinitesimal variance

$$A_1 = \mu \quad , \quad A_2 = \sigma^2 \qquad (\mu \in \mathbb{R}, \sigma > 0)\,,$$

respectively, with space-state $\mathbb{R}$. The scale function is:

$$h(x) = B\,\exp\left\{-\frac{2\mu\,x}{\sigma^2}\right\} \qquad (B > 0,\ x \in \mathbb{R})\,. \tag{59}$$

The end points $r_1 = -\infty$ and $r_2 = +\infty$ are both natural boundaries. In particular, the boundary $r_1 = -\infty$ is nonattracting for $\mu \ge 0$ and it is attracting otherwise. Furthermore, $r_2 = +\infty$ is nonattracting for $\mu \le 0$ and it is attracting for $\mu > 0$. Moreover, the Wiener process is transient for $\mu \ne 0$ and it is recurrent for $\mu = 0$. The transition pdf $f(x,t|y)$ is normal with mean $y + \mu\,t$ and variance $\sigma^2\,t$.

For $x, y \in \mathbb{R}$, from Proposition 4.1 one has:

$$\int_0^{+\infty} f(x,t|y)\,dt = \frac{1}{|\mu|}\,\exp\left\{\frac{\mu(x-y)}{\sigma^2} - \frac{|\mu(x-y)|}{\sigma^2}\right\} \qquad (\mu \neq 0) \tag{60}$$

whereas the integral diverges when $\mu = 0$. For the transient Wiener process, (60) gives the asymptotic average of the local time for $x, y \in \mathbb{R}$.

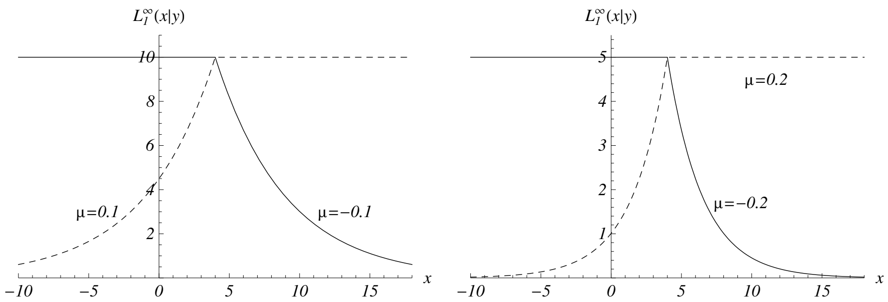


FIGURE 1. The asymptotic average of the local time for the Wiener process with $x \in \mathbb{R}$, $y = 4$ and $\sigma = 1$. The solid curves refer to $\mu < 0$ and the dashed curves are for $\mu > 0$.

As shown in Figure 1, due to the nature of end points, $L_1^\infty(x|y)$ is a non-decreasing function of x for $\mu > 0$ and it is non-increasing for $\mu < 0$.

One reflecting boundary. Let $a \in \mathbb{R}$ be a reflecting boundary. For $x, y \in (-\infty, a]$, from Proposition 3.3 one has:

$$\int_0^{+\infty} r(x,t|y)\,dt = \frac{1}{|\mu|}\,\exp\left\{\frac{\mu(x-y)}{\sigma^2} + \frac{\mu\,|x-y|}{\sigma^2}\right\} \tag{61}$$
$$(x \le a,\ y \le a,\ \mu < 0)\ ,$$

whereas the integral diverges when $\mu \ge 0$. Similarly, for $x, y \in [a, +\infty)$, from Proposition 3.4 one obtains:

$$\int_0^{+\infty} r(x,t|y)\,dt = \frac{1}{\mu}\,\exp\left\{\frac{\mu(x-y)}{\sigma^2} - \frac{\mu\,|x-y|}{\sigma^2}\right\} \tag{62}$$
$$(x \ge a,\ y \ge a,\ \mu > 0)\ ,$$

whereas the integral diverges when $\mu \le 0$. Hence, (61) and (62) give the average of the local time for the Wiener process in the presence of a reflecting boundary in a.

One absorbing boundary. Let $a \in \mathbb{R}$ be an absorbing boundary. Then, the FPT pdf $g(a,t|y)$ of such a process is given by

$$g(a,t|y) = \frac{|a-y|}{\sigma\sqrt{2\,\pi\,t^3}}\,\exp\left[-\frac{(a-y-\mu\,t)^2}{2\,\sigma^2\,t}\right] \qquad (a \neq y)\ , \tag{63}$$

while the FPT probability is

$$P(a|y) = \begin{cases} 1 \quad , & [y > a,\ \mu \le 0] \text{ or } [y < a,\ \mu \ge 0] \\ e^{-2\mu(y-a)/\sigma^2} \quad , & [y > a,\ \mu > 0] \text{ or } [y < a,\ \mu < 0]\,. \end{cases}$$

For $x, y \in (a, +\infty)$ or $x, y \in (-\infty, a)$, with a absorbing boundary, from Proposition 3.1 and Proposition 3.2 one obtains:

$$\int_0^{+\infty} \alpha(x,t|y)\,dt = \frac{1}{|\mu|}\left[\exp\left\{\frac{\mu(x-y)}{\sigma^2} - \frac{|\mu\,(x-y)|}{\sigma^2}\right\} - \right.$$

$$\left. - \exp\left\{\frac{\mu(x-y)}{\sigma^2} - \frac{|\mu\,(x-2a+y|}{\sigma^2}\right\}\right] \qquad (\mu \neq 0) \tag{64}$$

$$\int_0^{+\infty} \alpha(x,t|y)\,dt = \frac{|x-2a+y|}{\sigma^2} - \frac{|x-y|}{\sigma^2} \qquad (\mu = 0)$$

Relations (64) give the average of the local time for the Wiener process in the presence of an absorbing boundary in a. We note that the average of the local time is always finite.

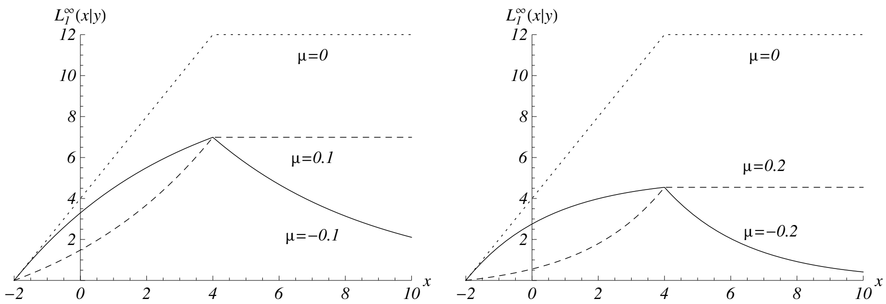


FIGURE 2. The asymptotic average of the local time for the Wiener process in the presence of an absorbing boundary in $a = -2$, with $x \in (-2, +\infty)$, $y = 4$ and $\sigma = 1$. The solid curves refer to $\mu < 0$, the dotted lines represent the case $\mu = 0$ and the dashed curves are for $\mu > 0$.

In Figure 2, $L_1^\infty(x|y)$ is plotted as function of $x \in (a, +\infty)$, with $a = -2$ absorbing boundary.

Two absorbing boundaries. Making use of (59), for $a < y < b$ from (13) one has:

$$P^-(a,b|y) = \begin{cases} \dfrac{e^{-2\mu\, b/\sigma^2} - e^{-2\mu\, y/\sigma^2}}{e^{-2\mu\, b/\sigma^2} - e^{-2\mu\, a/\sigma^2}} \;, & \mu \neq 0 \\[2ex] \dfrac{b-y}{b-a}\,, & \mu = 0\,, \end{cases}$$

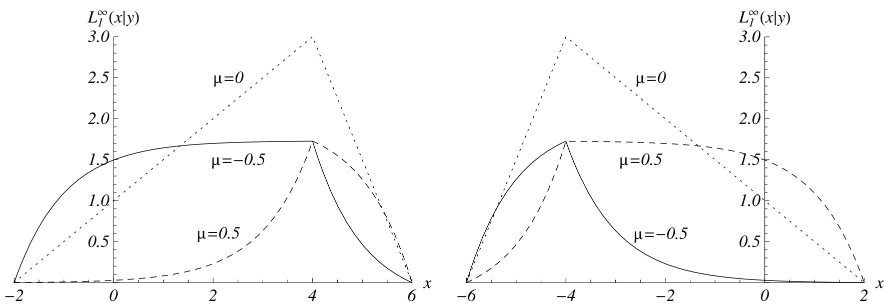
$$P^+(a,b|y) = \begin{cases} \dfrac{e^{-2\mu\, y/\sigma^2} - e^{-2\mu\, a/\sigma^2}}{e^{-2\mu\, b/\sigma^2} - e^{-2\mu\, a\sigma^2}} \;, & \mu \neq 0 \\[2ex] \dfrac{y-a}{b-a} \;, & \mu = 0\,, \end{cases}$$

so that from Proposition 2.1 one obtains the asymptotic conditional average of the local time for the Wiener process in the presence of two absorbing boundaries:

$$\int_0^{+\infty} \alpha^{(a,b)}(x,t|y)\,dt = \\ = \begin{cases} \dfrac{1-e^{-2\mu\,(y-a)/\sigma^2}}{1-e^{-2\mu\,(b-a)/\sigma^2}}\,\dfrac{1-e^{-2\mu\,(b-x)/\sigma^2}}{\mu}\,, & a<y\le x<b \\[2ex] \dfrac{e^{2\mu\,(b-y)/\sigma^2}-1}{e^{2\mu\,(b-a)/\sigma^2}-1}\,\dfrac{e^{2\mu\,(x-a)/\sigma^2}-1}{\mu}\,, & a<x\le y<b \end{cases} \qquad (\mu\neq 0)\,,$$

(65)

$$\int_0^{+\infty} \alpha^{(a,b)}(x,t|y)\,dt = \\ = \begin{cases} \dfrac{2(b-x)(y-a)}{\sigma^2(b-a)}\,, & a<y\le x<b \\[2ex] \dfrac{2(x-a)(b-y)}{\sigma^2(b-a)}\,, & a<x\le y<b \end{cases} \qquad (\mu=0)\,.$$



(a) $a=-2, b=6, y=4$

(b) $a=-6, b=2, y=-4$

FIGURE 3. The asymptotic average of the local time for the Wiener process in the presence of two absorbing boundaries in a and b, with $x\in(a,b)$ and $\sigma=1$. The solid curves refer to $\mu<0$, the dotted lines represent the case $\mu=0$ and the dashed curves are for $\mu>0$.

As shown in Figure 3, for fixed value of σ^2, by changing a,b,y,μ into $-b$, $-a$, $-y$, $-\mu$, respectively, in the asymptotic average of the local time for the Wiener process in the presence of two absorbing boundaries one obtains the corresponding specular function.

One absorbing boundary and one reflecting boundary. *Case* (i). We assume that a and b are an absorbing boundary and a reflecting boundary ($a<b$),

respectively. Then, from Proposition 2.2 one has:

$$
\int_0^{+\infty} \beta^{(a,b)}(x,t|y)\,dt =
$$
$$
= \begin{cases} \dfrac{e^{2\mu\,(x-a)/\sigma^2} - e^{2\mu\,(x-y)/\sigma^2}}{\mu} \quad , & a < y \le x \le b \\ \dfrac{e^{2\mu\,(x-a)/\sigma^2} - 1}{\mu} \quad , & a < x \le y \le b \end{cases} \qquad (\mu \neq 0) \, ,
$$

(66)

$$
\int_0^{+\infty} \beta^{(a,b)}(x,t|y)\,dt =
$$
$$
= \begin{cases} \dfrac{2(y-a)}{\sigma^2} \quad , & a < y \le x \le b \\ \dfrac{2(x-a)}{\sigma^2} \quad , & a < x \le y \le b \end{cases} \qquad (\mu = 0) \, .
$$

Case (*ii*). We assume that a and b are a reflecting boundary and an absorbing boundary ($a < b$), respectively. Then, from Proposition 2.3 one obtains:

$$
\int_0^{+\infty} \eta^{(a,b)}(x,t|y)\,dt =
$$
$$
= \begin{cases} \dfrac{1 - e^{-2\mu\,(b-x)/\sigma^2}}{\mu} \quad , & a \le y \le x < b \\ \dfrac{e^{-2\mu\,(y-x)/\sigma^2} - e^{-2\mu\,(b-x)/\sigma^2}}{\mu} \quad , & a \le x \le y < b \end{cases} \qquad (\mu \neq 0) \, ,
$$

(67)

$$
\int_0^{+\infty} \eta^{(a,b)}(x,t|y)\,dt =
$$
$$
= \begin{cases} \dfrac{2(b-x)}{\sigma^2} \quad , & a \le y \le x < b \\ \dfrac{2(b-y)}{\sigma^2} \quad , & a \le x \le y < b \end{cases} \qquad (\mu = 0) \, .
$$

Relations (66) and (67) give the asymptotic conditional average of the local time for the Wiener process in the presence of an absorbing and a reflecting boundary.

6. Asymptotic average of local time for the OU process

In this section we assume that $\{X(t), t \ge 0\}$ is the Ornstein-Uhlenbeck process with drift and infinitesimal variance

$$
A_1(x) = -\nu\,x \qquad A_2 = \sigma^2 \qquad (\nu > 0, \sigma > 0) \, ,
$$

respectively, with space-state $\mathbb{R}$. The scale function is:

$$h(x) = B \exp\left\{\frac{\nu x^2}{\sigma^2}\right\} \qquad (B > 0,\ x \in \mathbb{R})\ . \tag{68}$$

The end points $r_1 = -\infty$ and $r_2 = +\infty$ are both nonattracting natural boundaries. Furthermore, the OU process is always recurrent. The transition pdf $f(x,t|y)$ is normal with mean $y\,e^{-\nu t}$ and variance $\sigma^2(1-e^{-2\nu t})/\nu$. In this section we set:

$$\mathrm{Erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{z^2}\, dz \quad , \quad \psi(x) = \mathrm{Erfi}\left(\frac{x\sqrt{\nu}}{\sigma}\right)\ .$$

For $x, y \in \mathbb{R}$, from Proposition 4.1 one has than the asymptotic average of the local time diverges.

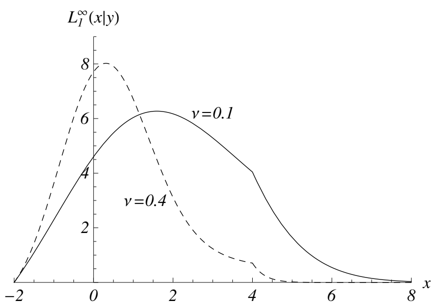
One absorbing boundary. Let $a \in \mathbb{R}$ be an absorbing boundary. Then, from Proposition 3.1 and Proposition 3.2 one has $P(a|y) = 1$ for $a \neq y$.
For $x, y \in (a, +\infty)$ from (42) it follows:

$$\int_0^{+\infty} \alpha(x,t|y)\, dt = \begin{cases} \dfrac{1}{\sigma}\sqrt{\dfrac{\pi}{\nu}}\ \exp\left\{-\dfrac{\nu x^2}{\sigma^2}\right\}[\psi(y)-\psi(a)] \quad , \quad a < y \leq x \\[2ex] \dfrac{1}{\sigma}\sqrt{\dfrac{\pi}{\nu}}\ \exp\left\{-\dfrac{\nu x^2}{\sigma^2}\right\}[\psi(x)-\psi(a)] \quad , \quad a < x \leq y\ . \end{cases} \tag{69}$$

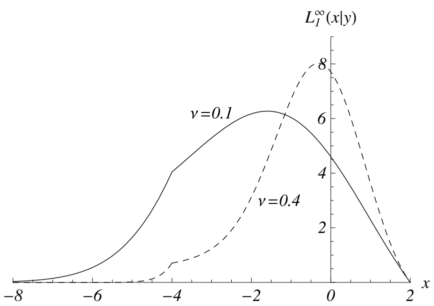
For $x, y \in (-\infty, a)$, from (45) one obtains:

$$\int_0^{+\infty} \alpha(x,t|y)\, dt = \begin{cases} \dfrac{1}{\sigma}\sqrt{\dfrac{\pi}{\nu}}\ \exp\left\{-\dfrac{\nu x^2}{\sigma^2}\right\}[\psi(a)-\psi(x)] \quad , \quad y \leq x < a \\[2ex] \dfrac{1}{\sigma}\sqrt{\dfrac{\pi}{\nu}}\ \exp\left\{-\dfrac{\nu x^2}{\sigma^2}\right\}[\psi(a)-\psi(y)] \quad , \quad x \leq y < a\ . \end{cases} \tag{70}$$

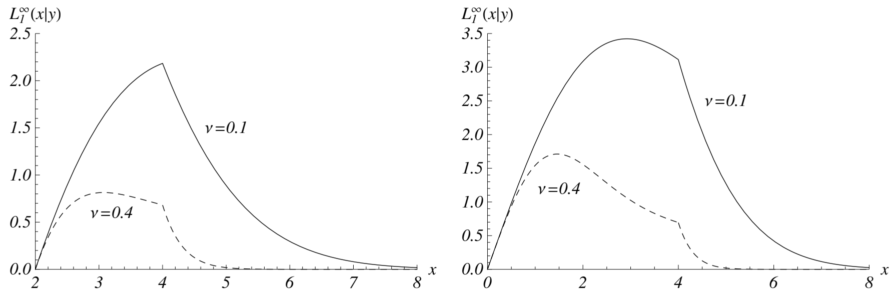
Relations (69) and (70) give the average of the local time for the OU process in the presence of an absorbing boundary in a.



(a) $a = -2, y = 4, x > -2$



(b) $a = 2, y = -4, x < 2$



(c) $a = 2, y = 4, x > 2$ (d) $a = 0, y = 4, x > 0$

FIGURE 4. The asymptotic average of the local time for the OU process in the presence of an absorbing boundary in a with $\sigma = 1$.

As shown in Figures 4(a) and 4(b), $L_1^\infty(x|y)$ for $x \in (a, +\infty)$, with a absorbing boundary, and $L_1^\infty(x|-y)$ for $x \in (-\infty, -a)$, with $-a$ absorbing boundary, are specular functions for fixed value of σ^2. Moreover, in Figure 4(a) one has $a < 0 < y$, in Figure 4(c) $0 < a < y$ and in Figure 4(d) we set $a = 0$, that corresponds to the equilibrium state of the OU process.

Two absorbing boundaries. Making use of (68), from (13) for $a < y < b$ one has:

$$P^-(a, b|y) = \frac{\psi(b) - \psi(y)}{\psi(b) - \psi(a)} \quad , \quad P^+(a, b|y) = \frac{\psi(y) - \psi(a)}{\psi(b) - \psi(a)} \, ,$$

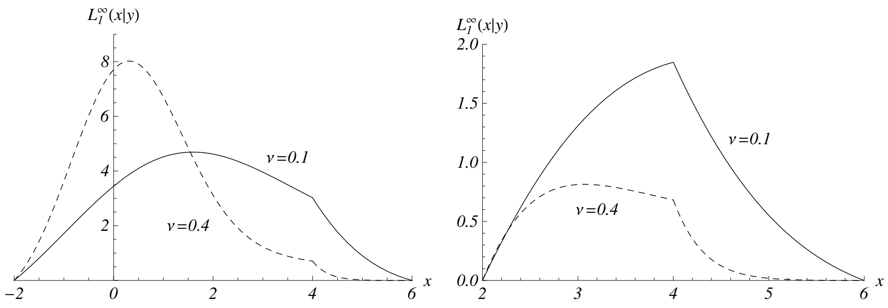
so that from Proposition 2.1 one obtains the asymptotic conditional average of the local time in the presence of two absorbing boundaries:

$$\int_0^{+\infty} \alpha^{(a,b)}(x, t|y)\, dt = \tag{71}$$

$$= \begin{cases} \dfrac{1}{\sigma} \sqrt{\dfrac{\pi}{\nu}} \, \exp\left\{-\dfrac{\nu\, x^2}{\sigma^2}\right\} \dfrac{[\psi(y) - \psi(a)]\,[\psi(b) - \psi(x)]}{\psi(b) - \psi(a)} & , \quad a < y \leq x < b \\[2ex] \dfrac{1}{\sigma} \sqrt{\dfrac{\pi}{\nu}} \, \exp\left\{-\dfrac{\nu\, x^2}{\sigma^2}\right\} \dfrac{[\psi(b) - \psi(y)]\,[\psi(x) - \psi(a)]}{\psi(b) - \psi(a)} & , \quad a < x \leq y < b \, . \end{cases}$$

In Figure 5 is plotted the asymptotic average of the local time for the OU process in the presence of two absorbing boundaries.

One absorbing boundary and one reflecting boundary. *Case* (i). We assume that a and b are an absorbing boundary and a reflecting boundary ($a < b$), respectively. Then, from Proposition 2.2 one obtains:



(a) $a = -2, b = 6, y = 4, -2 < x < 6$ (b) $a = 2, b = 6, y = 4, 2 < x < 6$

FIGURE 5. The asymptotic average of the local time for the OU process in the presence of two absorbing boundaries in a and b with $\sigma = 1$.

$$\int_0^{+\infty} \beta^{(a,b)}(x,t|y)\, dt = \tag{72}$$

$$= \begin{cases} \dfrac{1}{\sigma}\sqrt{\dfrac{\pi}{\nu}}\;\exp\left\{-\dfrac{\nu\, x^2}{\sigma^2}\right\}[\psi(y)-\psi(a)] \quad , \quad a < y \le x \le b \\[2ex] \dfrac{1}{\sigma}\sqrt{\dfrac{\pi}{\nu}}\;\exp\left\{-\dfrac{\nu\, x^2}{\sigma^2}\right\}[\psi(x)-\psi(a)] \quad , \quad a < x \le y \le b\,, \end{cases}$$

Case (*ii*). We assume that a and b are a reflecting boundary and an absorbing boundary ($a < b$), respectively. Then, from Proposition 2.3 it follows:

$$\int_0^{+\infty} \eta^{(a,b)}(x,t|y)\, dt = \tag{73}$$

$$= \begin{cases} \dfrac{1}{\sigma}\sqrt{\dfrac{\pi}{\nu}}\;\exp\left\{-\dfrac{\nu\, x^2}{\sigma^2}\right\}[\psi(b)-\psi(x)] \quad , \quad a \le y \le x < b \\[2ex] \dfrac{1}{\sigma}\sqrt{\dfrac{\pi}{\nu}}\;\exp\left\{-\dfrac{\nu\, x^2}{\sigma^2}\right\}[\psi(b)-\psi(y)] \quad , \quad a \le x \le y < b\,. \end{cases}$$

Relations (72) and (73) give the asymptotic conditional average of the local time for the OU process in the presence of an absorbing and a reflecting boundary.

7. CONCLUSION

In the present paper, a detailed study of the asymptotic behavior of the expected local time is carried out for one-dimensional time homogeneous diffusion processes in the presence of constant absorbing and/or reflecting boundaries (both not reflecting). The obtained results are applied to Wiener and Ornstein-Uhlenbeck (OU) diffusion processes.

The obtained results provide the basis to future researches on the average of residence time density and on the first-passage-time moments, aimed to extending the analysis at non-homogeneous Markov processes in the presence of time-dependent boundaries.

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