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Some remarks on stochastic diffusion processes with jumps

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Abstract. We consider stochastic diffusion processes subject to jumps that occur at random times. We assume that after each jump the process is reset to a random state from which it can evolve with a different dynamics. For this kind of processes the transition probability density function and its moments are analyzed. Moreover, the first passage time problem is studied. The results are applied to the processes with jumps constructed on the Wiener diffusion process.

1. Introduction and background

In the last decades, great attention has been paid to the description of biological, physical and engineering systems subject to various types of jumps. A jump, or catastrophe, is a random event that shifts the state of an evolutionary process in a certain level from which the process can re-start. The notion of catastrophe was introduced by Brockwell in the 80's to evaluate the dangerous of extinction of some wild species subject to phenomenons such as pollution, epidemics, fires or any other external agent. Specifically, Brockwell studied birth-death stochastic processes with catastrophes causing the reduction of the population size n to $n-j$, with an assigned probability; he introduced models with geometric, binomial and uniform distribution (cf. [1]) and he obtained interesting results regarding the extinction time and the mean size of the population (cf. [2], [3]). In this direction, the studies have been focused on the birth and death processes subject to total catastrophes, whose effect corresponds to the total extinction of the population (cf. [6], [7], [11], [17], [22], [23], [24], [25], [28]). Later, taking into account that many real phenomena are either reasonably modeled, or well approximated, by diffusion processes, the idea of catastrophe has been extended to diffusion processes. These include examples from molecular motions of enumerable particles subject to interactions, security price fluctuations in a perfect market, some communication systems with noise, neurophysiological activity with disturbances, variations of population growth, changes in species numbers subject to competition and other community

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relationships, gene substitutions in evolutionary development, etc.
In the present paper, we study a stochastic diffusion process subject to catastrophes analyzing the transition probability density function (pdf) and its moments as well as the first passage time (FPT) problem. Since the study of a diffusion process with jumps involves diffusion processes without jumps, in the following we remark some characteristics of a diffusion process without jumps. In Section 2, we construct a stochastic diffusion process with jumps. Specifically, we consider jumps down and we alternatively use the term *catastrophe* as synonymous of jump. In particular, we suppose that catastrophes occur at time intervals following a general probability distribution and we suppose that return points are randomly chosen. Moreover, we consider the possibility that, after each jump, the process can evolve with a different dynamics respect to the previous process; we also include the circumstance that the inter-jump intervals and the return points are not identically distributed. For this type of processes, we analyze the transition pdf, its moments and the FPT problem. In Section 3, we analyze two particular distributions of the inter-jump intervals: degenerate and exponential distribution. Finally, in Section 4, some obtained results are applied to the Wiener process with jumps.

Let $\{Z(t),\ t \geq t_0\}$ be a diffusion process characterized by drift $A_1(x)$ and infinitesimal variance $A_2(x)$, defined in the interval $\mathcal{D}_Z = (r_1, r_2)$. We denote by

$$h(x) = \exp\left\{-2\int^x \frac{A_1(\xi)}{A_2(\xi)}\, d\xi\right\} \quad , \quad s(x) = \frac{2}{A_2(x)\, h(x)}$$

the scale function and the speed density of $Z(t)$, respectively. A characterization of $Z(t)$ is given in terms of differential stochastic equations. Specifically, the sample paths of $Z(t)$ are time dependent functions described by the following stochastic differential equation

$$dZ(t) = A_1[Z(t)]\, dt + \sqrt{A_2[Z(t)]}\, dB(t)\ ,$$
$$Z(t_0) = z_0 \quad \text{a.s.}\ ,$$

where $B(t)$ is a standard Brownian motion and z_0 denotes the initial state of $Z(t)$.

As is well-known a probabilistic characterization of $Z(t)$ is specified by the transition pdf

$$f_Z(x,t|y,\tau) = \frac{\partial}{\partial t} P[Z(t) < x | Z(\tau) = y]$$

that satisfies the Fokker-Planck equation

$$\text{(1)} \qquad \frac{\partial f_Z(x,t|y,\tau)}{\partial t} = -\frac{\partial}{\partial x}[A_1(x) f_Z(x,t|y,\tau)] + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}[A_2(x) f_Z(x,t|y,\tau)]\ ,$$

and the Kolmogorov equation

$$\text{(2)} \qquad \frac{\partial f_Z(x,t|y,\tau)}{\partial \tau} + A_1(y)\,\frac{\partial f_Z(x,t|y,\tau)}{\partial y} + \frac{1}{2}\,A_2(y)\,\frac{\partial^2 f_Z(x,t|y,\tau)}{\partial y^2} = 0\ ,$$

that must be solved by using the delta initial condition:

$$\text{(3)} \qquad \lim_{t\to\tau} f_Z(x,t|y,\tau) = \lim_{\tau\to t} f_Z(x,t|y,\tau) = \delta(x-y)\ .$$

The initial condition is not always sufficient to determine uniquely the transition pdf; but, as proved by Feller (cf. [12], [13]), due to the nature of the end points r_1, r_2, suitable boundary conditions have to be considered (cf. also [21]). When r_1 and r_2 are natural boundaries, condition (3) allows to determine $f(x,t|y,\tau)$ univocally by solving (1) or (2).

The n-th conditional moment of the process $Z(t)$ is given by

$$m_Z^{(n)}(t|y,\tau) = \int_{x\in\mathcal{D}} x^n f_Z(x,t|y,\tau)\, dx \ ;$$

in particular,

$$E[Z(t)|Z(\tau)=y] = m_Z^{(1)}(t|y,\tau) \ ,$$

$$Var[Z(t)|Z(\tau)=y] = m_Z^{(2)}(t|y,\tau) - \left[m_Z^{(1)}(t|y,\tau)\right]^2$$

denote the conditional mean and the variance of $Z(t)$, respectively.

1.1. **The first passage time problem.** In many concrete problems, it is interesting to analyze the time at which a process reaches firstly a particular state depending on time. This instant is the random variable "first passage time" (FPT).

Formally, let $S(t)$ be a continuous function of t, called *threshold*. The random variable FPT is defined as:

$$T_Z = \begin{cases} \inf_{t\geq\tau}\{t: \ Z(t) > S(t)\} & , \quad Z(\tau) = y < S(\tau) \\ \inf_{t\geq\tau}\{t: \ Z(t) < S(t)\} & , \quad Z(\tau) = y > S(\tau) \ . \end{cases}$$

We denote by

$$g_Z\left[S(t),t|y,\tau\right] = \frac{d}{dt} P(T_Z < t)$$

the FPT pdf. In literature various approaches have been proposed to obtain informations on the FPT for diffusion processes (cf. [26]). However, due to the continuity of the sample-paths of a diffusion process the following integral equation holds:

$$f_Z(x,t|y,\tau) = \int_0^t g_Z\left[S(\theta),\theta|y,\tau\right] f_Z\left[x,t|S(\theta),\theta\right] d\theta \ , \tag{4}$$

for $x \leq S(t)$ and $y > S(\tau)$ or $x \geq S(t)$ and $y < S(\tau)$. Eq. (4) is a first-kind Volterra integral equation in the unknown function $g_Z[S(t),t|y,\tau]$. The kernel of (4), $f_Z[x,t|S(\theta),\theta]$, exhibits a singularity of the type $1/\sqrt{t-\theta}$ as $\theta \uparrow t$. As proved in [4], [14], [26], if $S(t) \in C^1[t_0,\infty)$, the FPT pdf satisfies the following second-kind Volterra integral equation

$$\begin{aligned} g_Z[S(t),t|y,\tau] = \rho\,\{&-2\Psi\left[S(t),t|y,\tau\right] + \\ &+2\int_\tau^t g_Z[S(\theta),\theta|y,\tau]\Psi[S(t),t|S(\theta),\theta]\, d\theta\Big\} \quad , \quad y \neq S(\tau) \end{aligned} \tag{5}$$

with

$$\rho = \mathrm{sgn}[S(\tau)-y] = \begin{cases} 1 & , \quad y < S(\tau) \\ -1 & , \quad y > S(\tau) \end{cases}$$

and

$$\begin{aligned} \Psi[S(t),t|z,\theta] = &\left\{S'(t) - A_1[S(t)] + \frac{1}{2}\, A_2'[S(t)] + k(t)\right\} f_Z[S(t),t|z,\theta] + \\ &+\frac{1}{2}\, A_2[S(t)]\, \frac{\partial}{\partial x} f_Z(x,t|z,\theta)\Big|_{x=S(t)} \ , \end{aligned}$$

where $S'(t) = dS(t)/dt$, $A'_2[S(t)] = dA_2(x)/dx|_{x=S(t)}$. Here $k(t)$ is an arbitrary continuous function that can be chosen so that the singularity of the kernel is removed. Specifically, if $S(t) \in C^2[t_0, \infty)$, one has

$$\lim_{\theta \to t} \Psi[S(t), t|S(\theta), \theta] = 0 \tag{6}$$

if, and only if,

$$k(t) = \frac{1}{2}\left\{ A_1[S(t)] - \frac{A'_2[S(t)]}{4} - S'(t) \right\} . \tag{7}$$

Hence, choosing $k(t)$ as in (7), the kernel of the equation (4) becomes non singular so that a simple numerical procedure can be used (cf. [4]). Specifically, denoting by $h > 0$ the integration step and setting $t = \tau + kh$, $k = 1, 2, \cdots$, equation (5) becomes:

$$\begin{aligned} g_Z[S(\tau+kh), \tau+kh|y,\tau] &= \rho\{-2\Psi[S(\tau+kh), \tau+kh|y,\tau]+ \\ &+2\int_\tau^{\tau+kh} g_Z[S(\theta),\theta|y,\tau]\Psi[S(\tau+kh),\tau+kh|S(\theta),\theta]\,d\theta\} \quad , \quad y \neq S(\tau) . \end{aligned} \tag{8}$$

Note that by taking $k(t)$ as in (7), condition (6) is satisfied. Hence, from equation (8), using a composite trapezium rule, one obtains the following approximate solution $\breve{g}_Z$ to g_Z:

$$\breve{g}_Z[S(\tau+h), \tau+h|y,\tau] = -2\Psi[S(\tau+h), \tau+h|y,\tau] ,$$

and for $k = 2, 3, \cdots$

$$\begin{aligned} \breve{g}_Z[S(\tau+kh), \tau+kh|y,\tau] = \rho\Bigg\{ &- 2\,\Psi[S(\tau+kh), \tau+kh|y,\tau]+ \\ +2h\sum_{j=1}^{k-1} \breve{g}_Z[S(\tau+jh), \tau+jh|y,\tau]&\Psi[S(\tau+kh), \tau+kh|S(\tau+jh), \tau+jh]\Bigg\} . \end{aligned}$$

We note that this approach has also been used to determine some closed form expressions for the FPT pdf of specific diffusion processes in the presence of particular time dependent thresholds (cf. [4], [14]). Moreover, in [10] this methodology has been extended to Gauss-Markov processes.

Some particular considerations can be made if $Z(t)$ is time homogeneous and the threshold S is time independent. Indeed, in this case Eq. (4) can be re-written as follows:

$$f_Z(x,t|y) = \int_0^t g_Z(S,\theta|y) f_Z(x, t-\theta|S)\, d\theta ,$$

$$(x \leq S \text{ and } y > S \text{ or } x \geq S \text{ and } y < S) ,$$

from which, considering the Laplace transform (LT), one has:

$$g_\lambda(S|y) = \frac{f_\lambda(x|y)}{f_\lambda(x|S)} \quad , \quad (x \leq S \text{ and } y > S \text{ or } x \geq S \text{ and } y < S) ,$$

where

$$g_\lambda(S|y) = \int_0^{+\infty} e^{-\lambda t} g_Z(S,t|y)\, dt \quad , \quad f_\lambda(S|y) = \int_0^{+\infty} e^{-\lambda t} f_Z(S,t|y)\, dt$$

are the LT with respect to t of the functions g_Z and f_Z, respectively. We can determine the FPT probability $P_Z(S|y) = \int_0^\infty g_Z(S,t|y)\,dt = g_\lambda(S|y)|_{\lambda=0}$ and, if $P_Z(S|y)=1$ the moments of T_Z can be evaluated by virtue of

$$t_n(S|y) = \int_0^\infty t^n g_Z(S,t|y)\,dt = (-1)^n \frac{d^n g_\lambda(S|y)}{d\lambda^n}|_{\lambda=0} \quad , \quad n=1,2,\cdots .$$

We note that, a recursive formula to evaluate $t_n(S|y)$ also exists (cf. [26], [27]):

$$t_n(S|y) = \begin{cases} n\displaystyle\int_y^S dz h(z) \int_{r_1}^z s(u) t_{n-1}(S|u)\,du \quad , \quad y<S \\ \\ n\displaystyle\int_S^y dz h(z) \int_z^{r_2} s(u) t_{n-1}(S|u)\,du \quad , \quad y>S \end{cases} \qquad n=1,2,\cdots ,$$

with $t_0(S|y)=1$ for $S \neq y$.

2. Stochastic diffusion processes with random jumps

Now we consider stochastic diffusion processes subject to jumps. In this context, an *inter-jump interval* is the time interval elapsing between two consecutive jumps. In [5], [6], [8], [9] some general results for the transient pdf and steady-state density of diffusion processes in the presence of catastrophes have been obtained. In these works the catastrophes occur according to a Poisson process. The effect of each catastrophe is to reset the process to a fixed and particular point of the diffusion interval, so that the process restarts following the same previous behaviour; moreover, the inter-jump intervals are identically distributed following an exponential law.

In the present paper we suppose that catastrophes occur at times that follow a general distribution and the return points can be random variables. Moreover, we consider the possibility that, after each jump, the process can evolve with a different dynamics respect to the previous processes; we also suppose that the inter-jump intervals and the return points are not identically distributed.

Let $\left\{\widetilde{X}_k(t),\, t \geq t_0 \geq 0\right\}$ be a diffusion stochastic process defined on the diffusion interval $\mathcal{D}_k$, for $k=0,1,\cdots$, characterized by drift $A_1^{(k)}(x)$ and infinitesimal variance $A_2^{(k)}(x)$. Now, we construct the stochastic process with random catastrophes $X(t)$ as follows. Starting from the initial state $\rho_0 = x_0$ at time t_0, the process $X(t)$ evolves according to the process $\widetilde{X}_0(t)$ until a random catastrophe occurs that shifts the process to a random state ρ_1. From here, $X(t)$ restarts according to $\widetilde{X}_1(t)$ until another catastrophe occurs resetting the process to ρ_2 and so on. In general, the effect of the k-th catastrophe ($k=1,2,\cdots$) is to shift the state of $X(t)$ in a certain level ρ_k, randomly chosen according to a pdf $\phi_k(\cdot)$. Then, the process evolves like $\widetilde{X}_k(t)$, until a new catastrophe occurs. The process $X(t)$ consists of independent cycles $\mathcal{I}_1,\, \mathcal{I}_2\, \cdots$, whose durations are described by the independent random variables $I_1,\, I_2, \cdots$, that represent the time intervals between two consecutive catastrophes. For $k=1,2,\cdots$, the random variable I_k is distributed with pdf $\psi_k(\cdot)$.

We denote by $\Theta_1, \Theta_2, \cdots$ the times in which the jumps occur and we set $\Theta_0 = t_0$

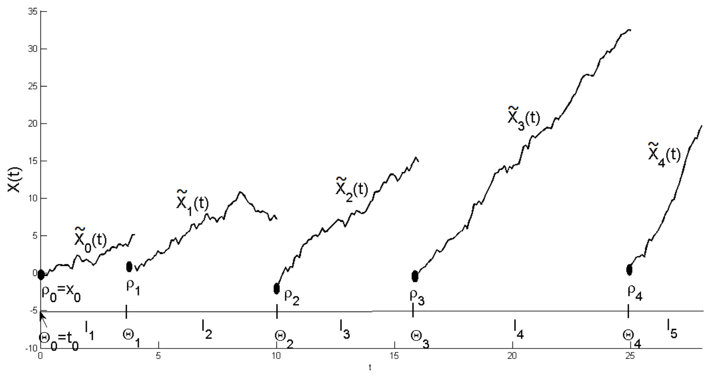

FIGURE 1. A sample path of the diffusion process with jumps $X(t)$.

the initial time. For $k = 1, 2, \cdots$, let $\gamma_k(\tau)$ be the pdf of the random variable Θ_k. Of course, the variables I_k and Θ_k are related. Indeed we have:

$$\Theta_1 = I_1 \quad , \quad \Theta_k = I_1 + I_2 + \cdots I_k, \; k > 1 \; .$$

Hence, the pdf $\gamma_k(\cdot)$ of Θ_k and the pdf $\psi_k(\cdot)$ of I_k are related, indeed $\gamma_1(t) = \psi_1(t)$ and $\gamma_k(t) = \psi_1(t) * \psi_2(t) * \cdots * \psi_k(t)$, where $*$ denotes the convolution operator.

Therefore, with reference to Figure 1, one has that $\mathcal{I}_0$ is the time interval starting at t_0 and finishing when the first jump occurs; for $k \geq 1$, $\mathcal{I}_k$ is the time elapsing between the k-th and the $(k+1)$-th jump; I_k is the duration of the k-th inter-jump interval, for $k \geq 1$. We note explicitly that $\Theta_0 = t_0$ and Θ_k for $k \geq 1$ are time instants, whereas I_k are interval widths. Furthermore, $\rho_0 = x_0$ and ρ_k is the return state in correspondence of the k-th jump, for $k \geq 1$. Finally, $\widetilde{X}_k(t)$ is a stochastic diffusion process which starts from ρ_k at the random time Θ_k; in the considered model, we are interested in the evolution of $\widetilde{X}_k(t)$ until the random time Θ_{k+1}.

In the following, given two random variables X and Y, we write $X \stackrel{d}{=} Y$ if they are identically distributed; whereas, given two diffusion processes $X(t)$ and $Y(t)$, we write $X(t) \stackrel{d}{=} Y(t)$ if they are characterized by the same drift and infinitesimal variance so that their transition pdf's are equal.
In the remaining part of this section, we assume that $\widetilde{X}_k(t)$ are diffusion processes and we analyze the pdf of the process with jumps, its moments and the FPT problem. Note that, these characteristics will be expressed in terms of the same characteristics of the involved processes without jumps.

2.1. **The probability density function and its moments.** The transition pdf f of the diffusion process with jumps can be expressed in terms of the transition densities $\widetilde{f}_k$ of the processes $\widetilde{X}_k(t)$. Indeed, considering the age of the process with jumps, we have the following expression of the transition pdf of the process $X(t)$:

$$f(x,t|\rho_0,t_0) = \left(1 - \int_0^{t-t_0} \psi_1(s)\, ds\right) \widetilde{f}_0(x,t|\rho_0,t_0) +$$

$$(9) \qquad +\sum_{k=1}^{\infty}\int_{t_0}^{t}\left(1-\int_0^{t-\tau}\psi_k(s)\,ds\right)\left(\int_{z\in\mathcal{D}_k}\phi_k(z)\,\widetilde{f}_k(x,t|z,\tau)\,dz\right)\gamma_k(\tau)\,d\tau\ .$$

Note that $\gamma_k(\tau)d\tau \approx P(\tau < \Theta_k < \tau + d\tau)$. We analyze the right hand side of (9). The first term represents the case in which there are not jumps in the interval (t_0,t) of width $t-t_0$, so that $X(t)$ evolves as $\widetilde{X}_0(t)$. The factor $1-\int_0^{t-t_0}\psi_1(s)\,ds = P(I_1 > t-t_0)$ represents the probability that the first jump occurs after the time t. With the second term in (9), we consider the circumstance that one or more jumps occur in (t_0,t). In this case, the last jump, the k-th one, occurs at the time $\tau \in (t_0,t)$; then the process $X(t)$ evolves according to $\widetilde{X}_k(t)$ to reach x at time t, starting from an initial point randomly chosen in $\mathcal{D}_k$. The integral in z takes into account that the return point, that is the initial point of $\widetilde{X}_k(t)$, is a random variable with pdf $\phi_k(z)$. The factor $1-\int_0^{t-t_0}\psi_k(s)\,ds$ represents the probability that the k-th jump is in the last one.
From (9) an expression for the n-th moments of $X(t)$ follows:

$$m^{(n)}(t|\rho_0,t_0) = \left(1-\int_0^{t-t_0}\psi_1(s)\,ds\right)\widetilde{m}_0^{(n)}(t|\rho_0,t_0)+$$

$$(10) \qquad +\sum_{k=1}^{\infty}\int_{t_0}^{t}\left(1-\int_0^{t-\tau}\psi_k(s)\,ds\right)\left(\int_{z\in\mathcal{D}_k}\phi_k(z)\,\widetilde{m}_k^{(n)}(t|z,\tau)\,dz\right)\gamma_k(\tau)\,d\tau\ ,$$

where $\widetilde{m}_k^{(n)}$ represents the conditional n-th moment of $\widetilde{X}_k(t)$.

2.2. **The first passage time problem.** We focus on the FPT problem for the process $X(t)$ through a constant threshold S.

Let

$$T_{\rho_0}(t_0) = \inf\{t \ge t_0 \,:\, X(t) > S\} \quad , \quad X(t_0) = \rho_0 < S$$

be the FPT random variable of $X(t)$ through S and let $g(S,t|\rho_0,t_0)$ be its pdf.
For $k = 0,1,\cdots$, let

$$\widetilde{T}_k(\theta) = \inf\{t \ge \theta \,:\, \widetilde{X}_k(t) > S\} \quad , \quad \widetilde{X}_k(\theta) = \rho_k < S$$

be the random variable FPT through S of the process $\widetilde{X}_k(t)$ without jumps, which starts from ρ_k at the time θ, and let $\widetilde{g}_k(S,t|\rho_k,\theta)$ be its pdf.
We obtain an expression for g as follows. Starting from ρ_0 at time t_0, the process reaches the threshold S for the first time at t if one, and only one, of the following cases occurs:

i) there are no jumps between t_0 and t, so that $X(t) \overset{d}{=} \widetilde{X}_0(t)$ and hence
$$g(S,t|\rho_0,t_0) = \widetilde{g}_0(S,t|\rho_0,t_0)\ ;$$

ii) for $k \ge 1$, k jumps happen in (t_0,t), the k-th jump occurs at time $\tau \in [t_0,t]$, and S is not crossed before τ. Recalling that $\widetilde{X}_k(t)$ evolves in the time interval $\mathcal{I}_{k+1} = [\Theta_k,\Theta_{k+1}]$ and making use of the independence of cycles $\mathcal{I}_1,\mathcal{I}_2\ \cdots$, the probability that none of the processes $\widetilde{X}_0(t),\widetilde{X}_1(t),\cdots,$ $\widetilde{X}_{k-1}(t)$ crosses S before τ is given by
$$\prod_{j=0}^{k-1}\left[1-P(\widetilde{T_j}(\Theta_j) < \Theta_{j+1})\right]\ .$$

Therefore we can conclude that the FPT pdf of $X(t)$ through the threshold S is given by

$$g(S,t|\rho_0,t_0)=\left(1-\int_0^{t-t_0}\psi(s)\,ds\right)\widetilde{g}_0(S,t|\rho_0,t_0)+$$

$$+\sum_{k=1}^{\infty}\int_{t_0}^{t}\left(1-\int_0^{t-\tau}\psi(s)\,ds\right)\left(\int_{z\in\mathcal{D}_k}\phi_k(z)\widetilde{g}_k(S,t|z,\tau)\,dz\right)\gamma_k(\tau)\,d\tau\times$$

$$\times\left\{\prod_{j=0}^{k-1}\Big[1-P(\widetilde{T_j}(\Theta_j)<\Theta_{j+1})\Big]\right\}. \tag{11}$$

3. Some particular cases

In this Section we focus on two distributions of inter-jump intervals: deterministic and exponential pdf. To simplify the discussion, assume that the return states $\rho_1,\rho_2,\cdots$ are fixed and different from the threshold S. In other words, we choose the pdf of the return point ρ_k as $\phi_k(z)=\delta(z-\ \rho_k)$, where $\delta(\cdot)$ denotes the Dirac delta function. Under this assumption, from (9), one has

$$f(x,t|\rho_0,t_0)=\left(1-\int_0^{t-t_0}\psi_1(s)\,ds\right)\widetilde{f}_0(x,t|\rho_0,t_0)+$$

$$+\sum_{k=1}^{\infty}\int_{t_0}^{t}\left(1-\int_0^{t-\tau}\psi_k(s)\,ds\right)\widetilde{f}_k(x,t|\rho_k,\tau)\,\gamma_k(\tau)\,d\tau\,, \tag{12}$$

and, from (10), it follows

$$m^{(n)}(t|\rho_0,t_0)=\left(1-\int_0^{t-t_0}\psi_1(s)\,ds\right)\widetilde{m}_0^{(n)}(t|\rho_0,t_0)+$$

$$+\sum_{k=1}^{\infty}\int_{t_0}^{t}\left(1-\int_0^{t-\tau}\psi_k(s)\,ds\right)\widetilde{m}_k^{(n)}(t|\rho_0,\tau)\,\gamma_k(\tau)\,d\tau\,. \tag{13}$$

Moreover, from (11) in this case one has that the FPT pdf of $X(t)$ through the threshold S is given by

$$g(S,t|\rho_0,t_0)=\left(1-\int_0^{t-t_0}\psi(s)\,ds\right)\widetilde{g}_0(S,t|\rho_0,t_0)+$$

$$+\sum_{k=1}^{\infty}\int_{t_0}^{t}\left(1-\int_0^{t-\tau}\psi(s)\,ds\right)\widetilde{g}_k(S,t|\rho_k,\tau)\gamma_k(\tau)\,d\tau\times$$

$$\times\left\{\prod_{j=0}^{k-1}\Big[1-P(\widetilde{T_j}(\Theta_j)<\Theta_{j+1})\Big]\right\}.$$

3.1. **Deterministic inter-jumps.** Let $\tau_0 = t_0$ and we assume that the jumps occur in the time instants $\tau_1, \tau_2, \cdots, \tau_N$. Then the process $X(t)$ consists of a combination of processes $\widetilde{X}_k(t)$ with $\widetilde{X}_k(\tau_k) = \rho_k$. Hence,

$$X(t) = \sum_{k=0}^{N} \widetilde{X}_k(t)\mathbf{1}_{(\tau_k,\tau_{k+1})}(t) \quad , \quad X(\tau_k) = \rho_k \ ,$$

where $\tau_{N+1} = \infty$ and

$$\mathbf{1}_{(\tau_k,\tau_{k+1})}(t) = \begin{cases} 1 & , \quad t \in (\tau_k, \tau_{k+1}) \\ 0 & , \quad t \notin (\tau_k, \tau_{k+1}) \ . \end{cases}$$

In this case we assume that a finite number of jumps occurs. After the time τ_N, the process $X(t) = \widetilde{X}_N(t)$. For $k = 0, 1, \cdots, N$, $\Theta_k = \tau_k$ a.s. and I_k are degenerate random variables; in particular the pdf of Θ_k is $\phi_k(t) = \delta(t - \tau_k)$ and the pdf of I_k is $\psi_k(t) = \delta\left[t - (\tau_k - \tau_{k-1})\right]$.
Denoting by

$$H(x) = \int_{-\infty}^{x} \delta(u)\, du = \begin{cases} 0 & , \quad x < 0 \\ 1 & , \quad x > 0 \ , \end{cases}$$

the Heaviside unit step function, we note that

$$\int_a^b \delta(s - \tau_k)\, ds = H(b - a - \tau_k) \ .$$

Hence, from (12) one has:

$$\begin{aligned} f(x,t|\rho_0,t_0) &= \left[1 - H\left(t - \tau_1\right)\right] \widetilde{f}_0(x,t|\rho_0,t_0) + \\ &+ \sum_{k=1}^{\infty} \int_{t_0}^{t} \delta\left(\tau - \tau_k\right) \left[1 - H\left(t - \tau - (\tau_k - \tau_{k-1}\right)\right] \widetilde{f}_k(x,t|\rho_k,\tau)\, d\tau = \\ &= \left[1 - H\left(t - \tau_1\right)\right] \widetilde{f}_0(x,t|\rho_0,t_0) + \\ &+ \sum_{k=1}^{\infty} H\left(t - \tau_k\right) \left[1 - H\left(t - \tau_k - (\tau_k - \tau_{k-1})\right)\right] \widetilde{f}_k\left(x,t|\rho_k,\tau_k\right) \ . \end{aligned}$$

Taking into consideration the definition of the Heaviside unit step function, it follows:

$$f(x,t|\rho_0,t_0) = \sum_{k=0}^{\infty} \widetilde{f}_k\left(x,t|\rho_k,\tau_k\right) \mathbf{1}_{(\tau_k,\tau_{k+1})}(t) = \tag{14}$$

$$= \begin{cases} \widetilde{f}_0(x,t|\rho_0,t_0) & , \quad t \in \mathcal{I}_1 \\ \widetilde{f}_k(x,t|\rho_k,\tau_k) & , \quad t \in \mathcal{I}_{k+1},\ (k = 1, 2, \cdots) \ . \end{cases}$$

The conditional moments of $X(t)$ follow from (13):

$$m^{(n)}(t|\rho_0, 0) = \sum_{k=0}^{\infty} \widetilde{m}_k^{(n)}(t|\rho_k,\tau_k)\mathbf{1}_{(\tau_k,\tau_{k+1})}(t) = \tag{15}$$

$$= \begin{cases} \widetilde{m}_0^{(n)}(t|\rho_0, t_0) \quad , \quad t \in \mathcal{I}_1 \\ \widetilde{m}_k^{(n)}(t|\rho_k, \tau_k) \quad , \quad t \in \mathcal{I}_{k+1} \ (k = 1, 2, \cdots) \ . \end{cases}$$

Now we discuss the FPT problem. Since $\Theta_k = \tau_k$ *a.s.*, one has

$$1 - P[\widetilde{T}_k(\tau_k) < \tau_{k+1}] = 1 - \int_{\tau_k}^{\tau_{k+1}} \widetilde{g}_k(S, \tau|\rho_k, \tau_k)\, d\tau \ ;$$

so, following the procedure used to obtain (14), from (11), one has:

(16)
$$g(S, t|\rho_0, t_0) =$$

$$= \begin{cases} \widetilde{g}_0(S, t|\rho_0, t_0) \quad , & t \in \mathcal{I}_1 \\ \prod_{j=0}^{k-1} \left[1 - \int_{\tau_j}^{\tau_{j+1}} \widetilde{g}_j(S, \tau|\rho_i, \tau_j)\, d\tau \right] \widetilde{g}_k\left(S, t|\rho_k, \tau_k\right) \quad , & t \in \mathcal{I}_k \quad (k = 2, 3, \cdots) \ . \end{cases}$$

We note that, when the processes $\widetilde{X}_k(t)$ are time homogeneous, assuming $t_0 = 0$, the expression (16) becomes

(17)
$$g(S, t|\rho_0) =$$

$$= \begin{cases} \widetilde{g}_0(S, t|\rho_0) \quad , & t \in \mathcal{I}_1 \\ \prod_{j=0}^{k-1} \left[1 - \int_0^{\tau_{j+1}-\tau_j} \widetilde{g}_j(S, \tau|\rho_j)\, d\tau \right] \widetilde{g}_k\left(S, t - \tau_k|\rho_k\right) \quad , & t \in \mathcal{I}_k \ \ (k = 2, 3, \cdots) \ . \end{cases}$$

In particular, if the inter-jumps are characterized by the same amplitude $A > 0$, i.e. $I_k = A$ *a.s.*, $\rho_k = \rho$, $\widetilde{X}_k(t) \stackrel{d}{=} \widetilde{X}(y)$, the expression (17) becomes

$$g(S, t|\rho) = \begin{cases} \widetilde{g}(S, t|\rho) \quad , & t \in \mathcal{I}_1 \\ \left[1 - \int_0^{A} \widetilde{g}(S, \tau|\rho)\, d\tau \right]^k \widetilde{g}(S, t - \tau_k|\rho) \quad , & t \in \mathcal{I}_k \ \ (k = 2, 3, \cdots) \ , \end{cases}$$

with $\widetilde{g}(S, t|\rho_0) = \widetilde{g}_k(S, t|\rho_0)$.

3.2. **Exponentially distributed inter-jumps.** For $k \geq 1$ we assume that $\rho_k = \rho$ and I_k are identically distributed with pdf $\psi_k(s) = \psi(s) = \xi e^{-\xi s}$ for $s > 0$. In this case Θ_k is the sum of k exponentially distributed random variables so that the pdf of Θ_k is an Erlang distribution with parameters (k, ξ), and

$$\phi_k(t) = \begin{cases} \dfrac{\xi^k t^{k-1} e^{-\xi t}}{(k-1)!} \quad , & t > 0 \\ 0 \quad , & \text{otherwise} \ . \end{cases}$$

From (12) the transition pdf of $X(t)$ follows:

(18)
$$f(x, t|\rho, t_0) = e^{-\xi(t-t_0)} \widetilde{f}_0(x, t|\rho, t_0) + e^{-\xi t} \sum_{k=1}^{\infty} \int_{t_0}^{t} \frac{\xi^k \tau^{k-1}}{(k-1)!} \widetilde{f}_k(x, t|\rho_k, \tau)\, d\tau$$

and, from (10) one has the conditional moments of $X(t)$:

$$m^{(n)}(t|\rho, t_0) = e^{-\xi(t-t_0)}\widetilde{m}_0^{(n)}(t|\rho, t_0) + e^{-\xi t}\sum_{k=1}^{\infty}\int_{t_0}^{t}\frac{\xi^k\tau^{k-1}}{(k-1)!}\widetilde{m}_k^{(n)}(t|\rho,\tau)\,d\tau\ . \tag{19}$$

Moreover, if $\widetilde{X}_k(t) \stackrel{d}{=} \widetilde{X}(t)$ and $\rho_k = \rho$ for $k = 0, 1, 2, \cdots$ from (18) and (19) it follows:

$$f(x,t|\rho,t_0) = e^{-\xi(t-t_0)}\widetilde{f}(x,t|\rho,t_0) + \xi\int_{t_0}^{t} e^{-\xi(t-\tau)}\widetilde{f}(x,t|\rho,\tau)\,d\tau \tag{20}$$

and

$$m^{(n)}(t|\rho,t_0) = e^{-\xi(t-t_0)}\widetilde{m}^{(n)}(t|\rho,t_0) + \xi\int_{t_0}^{t} e^{-\xi(t-\tau)}\widetilde{m}^{(n)}(t|\rho,\tau)\,d\tau\ . \tag{21}$$

Relations (20) and (21) are in agreement with the analogue results in [5] and [15].

Concerning the FPT pdf, assuming that the return states are deterministic and recalling that in this case the inter-jumps interval are independent and identically distributed, from (11), one has:

$$\begin{gathered} g(S,t|\rho,t_0) = e^{-\xi(t-t_0)}\widetilde{g}(S,t|\rho,t_0 + \\ +\sum_{k=1}^{\infty}\int_{t_0}^{t}\frac{(\xi\tau)^{k-1}e^{-\xi\tau}}{(k-1)!}\xi e^{-\xi(t-\tau)}\widetilde{g}_k(S,t|\rho,\tau)\,d\tau\left\{\prod_{j=0}^{k-1}\left[1-P(\widetilde{T}_j(\Theta_j)<\Theta_{j+1})\right]\right\}\ . \end{gathered} \tag{22}$$

Now we assume that $\widetilde{X}_k(t) \stackrel{d}{=} \widetilde{X}(t)$ is a time homogeneous process. So that, we have $P(\widetilde{T}_j(\Theta_j) < \Theta_{j+1}) = P(\widetilde{T}(0) < I_{j+1}) = P(\widetilde{T}(0) < I)$, where $\widetilde{T}(0)$ is the FPT of $\widetilde{X}_0(t)$ through the threshold S and $I_k \stackrel{d}{=} I$. Hence, for time homogeneous diffusion process Eq. (22) becomes:

$$\begin{gathered} g(S,t-t_0|\rho) = e^{-\xi(t-t_0)}\widetilde{g}(S,t-t_0|\rho)+ \\ +\sum_{k=1}^{\infty}\int_0^{t-t_0}\frac{\left(\xi\tau\left[1-P(\widetilde{T}(0)<I)\right]\right)^{k-1}}{(k-1)!}\xi e^{-\xi t}\widetilde{g}(S,t-\tau|\rho)d\tau\left[1-P(\widetilde{T}(0)<I)\right] = \\ = e^{-\xi(t-t_0)}\widetilde{g}(x,t-t_0|\rho)+ \\ +\xi\left[1-P(\widetilde{T}(0)<I)\right]e^{-\xi t}\int_0^{t-t_0} e^{\xi\tau\left[1-P(\widetilde{T}(0)<I)\right]}\widetilde{g}(S,t-\tau|\rho)\,d\tau\ . \end{gathered} \tag{23}$$

4. The Wiener process

Let $\{Z(t), t \geq 0\}$ be a time homogeneous Wiener process defined in $\mathcal{D}_Z = \mathbb{R}$, with drift and infinitesimal variance $A_1(x) = \mu,\ A_2(x) = \sigma^2$, with $\mu \in \mathbb{R},\ \sigma > 0$. The sample paths of $Z(t)$ are described by the stochastic differential equation

$$dZ(t) = \mu\,dt + \sigma\,dB(t)\ ,$$

with the initial condition $Z(t_0) = z_0$ *a.s.*. From the Feller's classification, the end points $r_i = \pm\infty$ are natural. Solving (1) and (2), with the initial condition (3), the transition pdf results to be a Gaussian density:

$$f_Z(x,t|y) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\exp\left\{-\frac{(x-y-\mu t)^2}{2\sigma^2 t}\right\}, \tag{24}$$

with the mean and the variance:

$$E[Z(t)|Z(0) = y] = y + \mu\, t, \qquad Var[Z(t)|Z(0) = y] = \sigma^2\, t, \tag{25}$$

respectively. The FPT pdf through a constant threshold $S \neq y$ is (cf., for instance, [26]):

$$g_Z(S, t|y) = \frac{|S - y|}{\sqrt{2\pi\sigma^2 t^3}} \exp\left\{-\frac{(S - y - \mu t)^2}{2\sigma^2 t}\right\} . \tag{26}$$

Eq. (26) identifies an inverse Gaussian pdf (Wald distribution).

4.1. **The Wiener process with jumps.** Let $\widetilde{X}_k(t)$ be the Wiener diffusion processes with drift $A_1^k = \mu_k$ and infinitesimal variance $A_2^k = \sigma_k^2$ and let $X(t)$ be the process with jumps constructed as described in Section 2.

4.2. **Wiener process with deterministic jumps.** We suppose that $\tau_0 = t_0 = 0$, $\tau_1, \tau_2, \cdots, \tau_N$ are the instants in which jumps occur and $\mathcal{I}_k = [\tau_{k-1}, \tau_k]$, $k = 2, 3, \cdots, N$ with $\mathcal{I}_{N+1} = [\tau_N, \tau_{N+1}]$ and $\tau_{N+1} = \infty$. After the time τ_N, the process $X(t) = \widetilde{X}_N(t)$. In Figure 2 a sample path of the Wiener process $X(t)$ with deterministic catastrophes' instants $(4, 8, 13, 17, 20, 22)$ is plotted. The red line is a sample path of $\widetilde{X}_0(t)$. The coefficients are $\mu_k = 0.5$ (on the left), $\mu_k = 0.5 + k$ (on the right) and the infinitesimal variance is $\sigma_k^2 = 2$, for all k. The return points are $\rho_k = 0$ on the left and $\rho_k = -k$ on the right. From (14), by taking into account

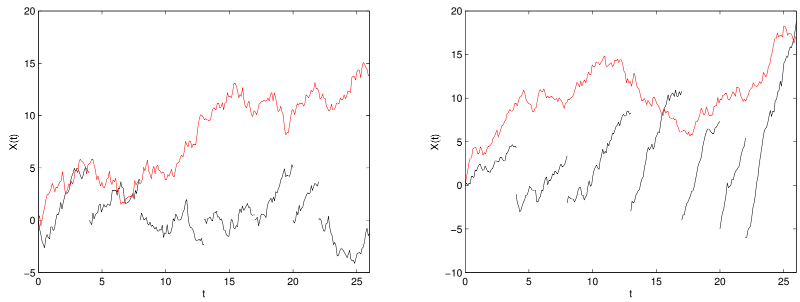

FIGURE 2. A sample path of the process $X(t)$ (black line) for $x_0 = 0$, and deterministic catastrophes' instants $(4, 8, 13, 17, 20, 22)$. The coefficients are $\mu_k = 0.5$ (on the left), $\mu_k = 0.5 + k$ (on the right) and the infinitesimal variance is $\sigma_k^2 = 2$. The return points are $\rho_k = 0$ on the left and $\rho_k = -k$, $k = 0, 1, \cdots$ on the right. The red line is a sample path of the process $\widetilde{X}_0(t)$.

(24), the pdf of $X(t)$ is

$$f(x, t|\rho_0) = \sum_{k=0}^{N} \frac{1}{\sqrt{2\pi\sigma_k^2(t - \tau_k)}} \exp\left\{-\frac{[x - \rho_k - \mu_k(t - \tau_k)]^2}{2{\sigma_k}^2(t - \tau_k)}\right\} \mathbf{1}_{(\tau_k, \tau_{k+1})}(t) .$$

From (15) and (25), the mean of $X(t)$ is

$$E[X(t)|\rho_0] = \sum_{k=0}^{N} [\rho_k + \mu_k(t - \tau_k)]\, \mathbf{1}_{(\tau_k, \tau_{k+1})}(t) .$$

In Figure 3 and Figure 4 the pdf $f(1,t|0,0)$ and the mean $E[X(t)|0,0]$ of $X(t)$ are shown, respectively, with the same choices of Figure 2. Concerning the FPT pdf,

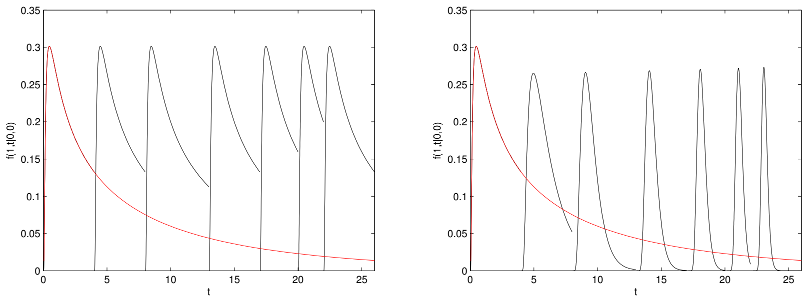

FIGURE 3. The pdf $f(1,t|0,0)$ (black line) and the pdf $\widetilde{f}_0(1,t|0,0)$ (red line) with deterministic jumps, for the same choices of Figure 2.

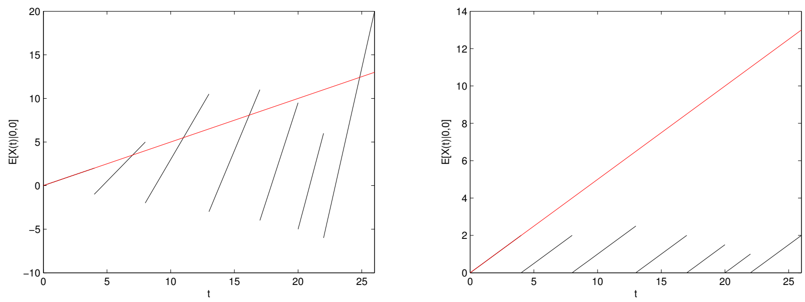

FIGURE 4. The mean $E[X(t)|0,0]$ (black line) and $E[\widetilde{X}_0(t)|0,0]$ (red line) with deterministic jumps, for the same choices of Figure 2.

since the Wiener processes $\widetilde{X}_k(t)$ are time homogeneous, the expression of $g(S,t|\rho_0)$ is given by (17) where $\widetilde{g}_k(S,\tau|\rho_k)$ is defined in (26); so, for $\rho_k < S$ and $k \geq 1$, one has:

$$g(S,t|\rho_0) =$$

$$= \begin{cases} \dfrac{(S-\rho_0)}{\sqrt{2\pi\sigma_0^2 t^3}} \exp\left\{-\dfrac{(S-\rho_0-\mu_0 t)^2}{2\sigma_0^2 t}\right\} \quad , & t \in \mathcal{I}_1 \\[2ex] \displaystyle\prod_{j=0}^{k-1}\left[1-\int_0^{\tau_{j+1}-\tau_j} \frac{(S-\rho_j)}{\sqrt{2\pi\sigma_j^2\tau^3}} \exp\left\{-\frac{(S-\rho_j-\mu_j\tau)^2}{2\sigma_j^2\tau}\right\} d\tau\right] \times & \\ \times \dfrac{(S-\rho_k)}{\sqrt{2\pi\sigma_k^2(t-\tau_k)^3}} \exp\left\{-\dfrac{[S-\rho_k-\mu_k(t-\tau_k)]^2}{2\sigma_k^2(t-\tau_k)}\right\} \quad , & t \in \mathcal{I}_k \ , \end{cases}$$

where

$$\int_0^{\tau_{j+1}-\tau_j} \frac{(S-\rho_j)}{\sqrt{2\pi\sigma_j^2\tau^3}} \exp\left\{-\frac{(S-\rho_j-\mu_j\tau)^2}{2\sigma_j^2\tau}\right\} d\tau =$$

$$= \frac{1}{2}\,\text{Erfc}\left[\frac{S-\rho_j+\mu_j(\tau_{j+1}-\tau_j)}{\sqrt{2(\tau_{j+1}-\tau_j)\sigma_j^2}}\right] +$$

$$+\frac{1}{2}\exp\left\{-\frac{2\mu_j(S-\rho_j)}{\sigma_j^2}\right\}\text{Erfc}\left[\frac{S-\rho_j-\mu_j(\tau_{j+1}-\tau_j)}{\sqrt{2(\tau_{j+1}-\tau_j)\sigma_j^2}}\right] ,$$

with

$$\text{Erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}\,dt$$

the complementary error function.
In Figure 5 the FPT pdfs' $g(5,t|0)$ (black line) and $\widetilde{g}_0(5,t|0)$ (red line) are plotted for the same choices of Figure 2.

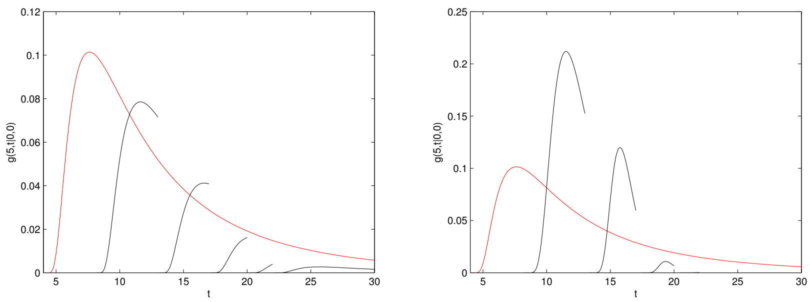

FIGURE 5. FPT pdfs' $g(5,t|0)$ (black line) and $\widetilde{g}_0(5,t|0)$ (red line), with deterministic jumps, are plotted for the same choices of Figure 2.

4.3. **Wiener process with exponentially distributed jumps.** In this case, assuming that I_k are identically distributed by $\psi_k(s) \equiv \psi(s) = \xi e^{-\xi s}$, the expression (18) holds, with $\widetilde{f}_k(x,t|\rho)$ defined in (24). Moreover, making use of the moments of the single process $\widetilde{X}_k(t)$, also the moments of $X(t)$ can be evaluated via (19). Similarly, recalling (26), from (22) the FPT pdf can be written.

Now we consider a special case; specifically, we suppose that the processes $\widetilde{X}_k(t)$ are equals, hence $\widetilde{X}_k(t) \stackrel{d}{=} \widetilde{X}(t)$ are Wiener processes with $A_1 = \mu$ and $A_2(t) = \sigma^2$, and we also consider $\rho_k = \rho$. In this case, setting $t_0 = 0$ and making use of (24), from (20) one has:

$$f(x,t|\rho) = \frac{e^{-\xi t}}{\sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{(x-\rho-\mu t)^2}{2\sigma^2 t}\right\}+$$

$$+\xi\int_0^t \frac{e^{-\xi(t-\tau)}}{\sqrt{2\pi\sigma^2(t-\tau)}} \exp\left\{-\frac{[x-\rho-\mu(t-\tau)]^2}{2\sigma^2(t-\tau)}\right\} d\tau ,$$

where

$$\int_0^t \frac{e^{-\xi(t-\tau)}}{\sqrt{2\pi\sigma^2(t-\tau)}} \exp\left\{-\frac{[x-\rho-\mu(t-\tau)]^2}{2\sigma^2(t-\tau)}\right\} d\tau = \frac{e^{(x-\rho)\left(\mu-\sqrt{\mu^2+2\sigma^2\xi}\right)}}{2\sqrt{\mu^2+2\sigma^2\xi}}\times$$

$$\times\left[\text{Erfc}\left(\frac{x-\rho-t\sqrt{\mu^2+2\sigma^2\xi}}{\sqrt{2t\sigma^2}}\right)-\right.$$

$$\left.-e^{(2(x-\rho)\sqrt{\mu^2+2\sigma^2\xi})/\sigma^2}\text{Erfc}\left(\frac{x-\rho+t\sqrt{\mu^2+2\sigma^2\xi}}{\sqrt{2\theta\sigma^2}}\right)\right] .$$

Moreover, the mean of $X(t)$ can be evaluated from (21) with $n = 1$ and $\widetilde{\mu}^{(1)}(t|\rho)$ given in (25); so, for $t_0 = 0$ it follows:

$$E[X(t)|X(0) = \rho] = e^{-\xi t}(\rho + \mu t) + \xi \int_0^t e^{-\xi(t-\tau)}[\rho + \mu(t-\tau)]\, d\tau$$

with

$$\int_0^t e^{-\xi(t-\tau)}[\rho + \mu(t-\tau)]\, d\tau = \frac{\rho}{\xi} + \frac{\mu}{\xi^2} - e^{-\xi t}\left[-\frac{\rho}{\xi} + \frac{\mu}{\xi^2} + \frac{\mu t}{\xi}\right] .$$

On the left of Figure 6 the pdf's $f(1, t|0, 0)$ (black line) and $\widetilde{f}_0(1, t|0, 0)$ (red line) for $\rho_k = 0$ ($k = 0, 1, \cdots$) and exponentially distributed inter-jumps with $1/\xi = 4$ are plotted. Each $X_k(t)$ is a Wiener diffusion process with drift $A_1 = \mu$ and infinitesimal variance $A_2 = \sigma^2$, where $\mu = 0.5$ and $\sigma^2 = 2$. On the right of Figure 6 the mean $E[X(t)|0, 0]$ (black line) and $E[\widetilde{X}_0(t)|0, 0]$ (red line) are plotted for the same choices of the left side.
Concerning the FPT pdf, recalling that $\widetilde{g}_j(S, \tau|\rho) = \widetilde{g}(S, \tau|\rho)$ is defined in (26), from (23) one has:

$$g(S, t|\rho) = e^{-\xi t}\, \frac{S-\rho}{\sqrt{2\pi\sigma^2 t^3}}\, \exp\left\{-\frac{(S-\rho-\mu t)^2}{2\sigma^2 t}\right\} + \xi\left[1 - P(\widetilde{T}(0) < I)\right] e^{-\xi t}\times$$

$$\times \int_0^t e^{\xi\tau\left[1-\widetilde{T}(0)<I\right)]}\frac{S-\rho}{\sqrt{2\pi\sigma^2(t-\tau)^3}}\, \exp\left\{-\frac{(S-\rho-\mu(t-\tau))^2}{2\sigma^2(t-\tau)}\right\} d\tau ,$$

with

$$P(\widetilde{T}(0) < I) = \int_0^\infty d\theta\, \xi e^{-\xi\theta} \int_0^\theta \frac{S-\rho}{\sqrt{2\pi\sigma^2 v^3}}\, \exp\left\{-\frac{(S-\rho-\mu v)^2}{2\sigma^2 v}\right\} dv =$$

$$= \int_0^\infty d\theta\, \xi e^{-\xi\theta}\left\{-\frac{1}{2}\, \text{Erfc}\left[\frac{\rho-S+\mu\theta}{\sqrt{2\sigma^2\theta}}\right] - \frac{1}{2}\, \text{Erfc}\left[\frac{\rho-S-\mu\theta}{\sqrt{2\sigma^2\theta}}\right]\right\} =$$

$$= -\frac{1}{2}\,\xi\left\{L\left[\text{Erfc}\left(\frac{\rho-S+\mu\theta}{\sqrt{2\sigma^2\theta}}\right)\right] + L\left[\text{Erfc}\left(\frac{\rho-S-\mu\theta}{\sqrt{2\sigma^2\theta}}\right)\right]\right\} ,$$

where L is the Laplace Transform.

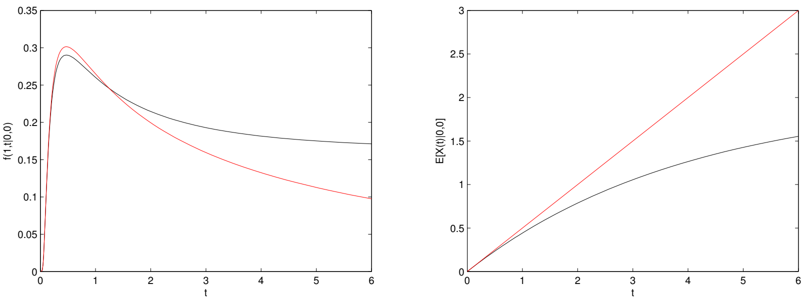

FIGURE 6. On the left the pdf's $f(1,t|0,0)$ (black line) and $\widetilde{f}_0(1,t|0,0)$ (red line) for $\rho_k = 0$ $(k = 0, 1, \cdots)$ and exponentially distributed inter-jumps with mean 4. Each $X_k(t)$ is a Wiener diffusion process with drift $A_1 = \mu$ and infinitesimal variance $A_2 = \sigma^2$, where $\mu = 0.5$ and $\sigma^2 = 2$. On the right the mean $E[X(t)|0,0]$ (black line) and $E[\widetilde{X}_0(t)|0,0]$ (red line) are plotted for the same choices of the left side.

5. Conclusions and future developments

In this paper we have studied stochastic diffusion processes subject to jumps.

A jump, or catastrophe, is a random event that shifts the state of the process in a certain level from which the process can re-start. We have constructed diffusion processes with jumps by supposing that catastrophes occur at time interval following a general distribution and the return points are randomly chosen. Moreover, we have considered the possibility that, after each jump, the process can evolve with a different dynamics respect to the previous processes; we have also supposed that the inter-jump intervals and the return points are not identically distributed. For this type of process, we have analyzed the transition pdf, its moments and the FPT problem. Two particular cases have been considered: deterministic inter-jumps intervals and catastrophes occurring following a Poisson process. Then, the obtained results have been applied to the Wiener process and some closed form expressions have been obtained for the transition pdf and for the FPT pdf.

Future studies on this topic can be made by considering other inter-jumps distributions, or by assuming that the single processes $\widetilde{X}_k(t)$ are of different nature (as Ornstein Uhlenbeck process, Lognormal process, ...). Some considerations in this direction can be found in [16] and [20] where a Gompertz diffusion process has been considered to describe the growth of a tumor mass subject to an intermittent treatment involving the reduction of tumor size and a rise of growth rate.

We note that the considerations made in the present paper also concern time non homogeneous diffusion processes. Therefore, one can consider more general models based on time non homogeneous process. Moreover, alternating processes can be analyzed in which after each jump a random death time is considered. In this direction some studies related to a neuronal model in the presence of refractoriness have already been conducted in [18] and [19].

We pinpoint that since the stochastic processes with jumps considered in the present paper are very general, future studies will involve the application of them in several different contexts in order to model various dynamic systems.

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