

Cumulants via diagonal measures

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Abstract. In the framework of the combinatorial approach to stochastic integration initiated by Rota and Wallstrom, we focus on the representation of cumulants as the expectation of the diagonal measures of the associated product random measure. This setting turns out to be particularly suitable to manage cumulants of the process of variations of a càdlàg Lévy process, as well as to describe κ -statistics and polykays for multiplicative random measures.

1. INTRODUCTION

The goal of the present work is to highlight the advantages that can result in managing cumulants via a random measure approach, thanks to the combinatorial properties of the so-called *diagonal measures* [8].

Using the language introduced by Rota and Wallstrom [8], the present work relies on the representation of cumulants as the expectation of random variables that are invariant under translation: diagonal measures (see Theorem 3.1).

The original setting in [8] gave the birth to the first systematic theory of stochastic integration in combinatorial terms. The basic idea is the representation of the product random measure as a sum of partition-depending measures, one of this being the exact random analogue of the classical one: the *stochastic measure*. One speaks about exact random analogue because the product random measure does not vanish identically when integrating on the so called *diagonal sets*, producing therefore an “anomaly” (see [3, 5, 8]). As a consequence, several known identities concerning stochastic integrals were recovered in a more compact way as identities over the lattice of partitions, giving a unifying treatment of the subject.

In the sequel, the accent will be put in particular on the diagonal measures associated with Lévy processes: the process of variations. As pointed out in [10], the variations process of a semi-martingale is homogeneous, that is, for any real number a and for all $n \geq 1$, $(aX)^{(n)} = a^n X^{(n)}$. Moreover, for two semi-martingales X and Y , with zero quadratic covariation $[X, Y]_t = 0$ for all t , the additivity property turns into: $(X + Y)^{(n)} = X^{(n)} + Y^{(n)}$. When Y is a constant process, say $Y_t = c$, for all $t \geq 0$, the additivity turns to be a semi-invariance property,

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namely, $(X + c)^{(1)} = X^{(1)} + c$, while for all $n \geq 2$, $(X + c)^{(n)} = X^{(n)}$. Since these three properties characterize cumulants [7], and $\mathbb{E}[X_t^{(n)}] = \chi_n(X_t)$, the process of variations (and, more generally, diagonal measures) can be referred to as *functions of cumulant type*.

The contents are organized as follows: Section 2 presents a brief summary on the combinatorial theory of stochastic integration, which is not meant to be exhaustive: any unspecified result can be traced to [5] and [8]. In Section 3, Theorem 3.1 provides cumulants as (deterministic) measures, namely, the expectation of the diagonal measures. As a consequence, an alternative simpler proof of the correspondence between cumulants of a Lévy processes and cumulants of its variation processes (see identity (13)) is achieved. The advantage of this approach is that the expression provided for the cumulants does not depend on the orders of the involved variations processes, and that it can be applied to compute the cumulants of the variation processes of the associated Teugel martingales. In Section 4, Theorem 3.1 is given a statistical interpretation: more specifically, it is shown that diagonal measures naturally correspond to κ -statistics for positive random measures.

2. PRELIMINARIES: RANDOM MEASURES

Let (Z, \mathcal{Z}) denote a Polish space (that is, a complete metrizable and separable topological space), where \mathcal{Z} denotes the Borel σ -algebra of Z . For every $n \geq 1$, let $(Z^{\otimes n}, \mathcal{Z}^{\otimes n})$ denote the n -fold product space of (Z, \mathcal{Z}) . Following Rota and Wallstrom [8], a measurable set of the type $\mathcal{A} = A_1 \times \cdots \times A_n$ in $\mathcal{Z}^{\otimes n}$ will be called a rectangle, and a rectangle with equal sides will be called a cube.

The combinatorial approach to the theory of stochastic integration is based on the concept of partitions of a set: for any partition $\pi \in \mathcal{P}([n])$, and any measurable set $\mathcal{A} \in \mathcal{Z}^{\otimes n}$, consider

(i) the *diagonal* set associated with π :

$$\mathcal{A}_\pi = \{(z_1, \dots, z_n) \in \mathcal{A} : z_i = z_j \text{ if and only if } i \sim_\pi j\};$$

(ii) the *superdiagonal* set associated with π :

$$\mathcal{A}_{\geq \pi} = \{(z_1, \dots, z_n) : i \sim_\pi j \Rightarrow z_i = z_j\},$$

where $i \sim_\pi j$ denotes the equivalence relation on $[n]$: $i \sim_\pi j$ if and only if i and j belong to the same block of π .

Since the product space is itself a Polish space (and therefore, it is second countable), every diagonal set can be written as the union of at most countably many rectangles whose kernel is π (where a rectangle $A_1 \times \cdots \times A_n$ is said to have kernel π if $A_i \cap A_j = \emptyset$ for $i \not\sim_\pi j$). Thanks to the σ -additivity property of the random measures defined later on, one can and will focus the attention only on such rectangles.

It is easy to check that the following properties hold:

1. $\mathcal{A}_{\geq \pi} = \bigcup_{\sigma \geq \pi} \mathcal{A}_\sigma$ (in particular, $\mathcal{A}_{\geq \hat{0}} = \bigcup_{\sigma \geq \hat{0}} \mathcal{A}_\sigma = \mathcal{A}$);
2. $\mathcal{A}_\pi \cap \mathcal{A}_\sigma = \emptyset$ for $\sigma \neq \pi$.

Definition 2.1. A *random measure* Φ on (Z, \mathcal{Z}) is a finitely additive set function admitting a σ -additive extension, that maps each $A \in \mathcal{Z}$ to a random variable $\Phi(A)$ in some Banach space of random variables on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

and such that $\Phi(\emptyset) = 0$ a.s.. Φ is said to be a *completely random measure* (for short, CR-measure) if it has “independent increments”, that is if it maps pairwise disjoint sets to a system of independent random variables.

Remark 2.1. Usually, random measures are defined over $L^2(\Omega)$, or $\bigcap_{p \geq 1} L^p(\Omega)$ (see [5]). In the literature, completely random measures are also called *independently scattered measures*. Unlike the deterministic product measure, the σ -additive extension of the product $\Phi^{\otimes n}$ needs not be uniquely determined (see [3]). When this is not the case, and accordingly with the definitions given in [5] and [8], Φ will be referred to as a *good random measure*.

On the space (Z, \mathcal{Z}) , a σ -finite *non-atomic measure* ν is given to control the random measure Φ as follows:

$$\mathbb{E}[\Phi(B)^2] = \nu(B) \quad , \quad \mathbb{E}[\Phi(B)\Phi(C)] = \nu(B \cap C) \quad ,$$

where, as usual, \mathbb{E} denotes the expectation on the fixed probability space. The most important feature of a non-atomic measure ν is that, for every integer N , any measurable set A , with $\nu(A) > 0$ and $\nu(A) < \infty$, can be partitioned into N measurable subsets, pairwise disjoint and with the same measure:

$$(1) \quad A = \bigcup_{i=1}^N A_{iN} \quad , \quad \text{with } \nu(A_{iN}) = \frac{\nu(A)}{N} \quad .$$

Remark 2.2. By definition, a measure ν is non-atomic if for every measurable set \mathcal{A} with $0 < \nu(\mathcal{A}) < \infty$, for every $r \in (0, \nu(\mathcal{A}))$ there exists a measurable set $B \subset \mathcal{A}$ with $\nu(B) = r$. The non-atomicity of the measure ν is needed to ensure that the random field Φ is composed of infinitely divisible distributions (see, for instance, [5, Proposition 5.3.2]), and to ensure that the class of elementary (simple) functions of n variables is dense in the space $L^2(\nu^{\otimes n})$ of functions that are square-integrable with respect to $\nu^{\otimes n}$, for every n (see, for instance, [5, Lemma 5.5.2]). The most important example of non-atomic measure is the Lebesgue measure on the real line.

For good CR-measures, the main idea in [8] was to consider the restrictions of the product measure to diagonal and superdiagonal sets as measures themselves. In the sequel, as in [10], the setting will be that of the product of n jointly good random measures, according to the following definition.

Definition 2.2. Let Φ_1, \dots, Φ_n be completely random measures given on the same space (Z, \mathcal{Z}) . Φ_1, \dots, Φ_n are *jointly good* if the (finitely additive) product vector measure $\Phi_1 \otimes \dots \otimes \Phi_n$ can be extended to a unique σ -additive random measure on the product space $(Z^{\otimes n}, \mathcal{Z}^{\otimes n})$.

If Φ_1, \dots, Φ_n are assumed to be jointly good on a Polish space (Z, \mathcal{Z}) , the following definitions are well-posed.

Definition 2.3. For every $\pi \in \mathcal{P}([n])$ and $\mathcal{A} \in \mathcal{Z}^{\otimes n}$, define:

$$(2) \quad St_{\pi}^{(\Phi_1, \dots, \Phi_n)}(\mathcal{A}) := \Phi_1 \otimes \dots \otimes \Phi_n(\mathcal{A}_{\pi})$$

and

$$(3) \quad St_{\geq \pi}^{(\Phi_1, \dots, \Phi_n)}(\mathcal{A}) := \Phi_1 \otimes \dots \otimes \Phi_n(\mathcal{A}_{\geq \pi}) \quad .$$

It is easy to verify, by using the additivity property of $\Phi_1 \otimes \dots \otimes \Phi_n$, and the Möbius inversion theory on the lattice $\mathcal{P}([n])$ [6, 11], that the measures $St_\pi^{(\Phi_1, \dots, \Phi_n)}$ and $St_{\geq \pi}^{(\Phi_1, \dots, \Phi_n)}$ satisfy the combinatorial identities summarized in the next proposition (see [10]).

Proposition 2.1. *Let Φ_1, \dots, Φ_n be jointly good CR-measures. Then:*

- (i) $\Phi_1 \otimes \dots \otimes \Phi_n = \sum_{\sigma \in \mathcal{P}([n])} St_\sigma^{(\Phi_1, \dots, \Phi_n)}$;
- (ii) $St_{\geq \pi}^{(\Phi_1, \dots, \Phi_n)} = \sum_{\substack{\sigma \in \mathcal{P}([n]) \\ \sigma \geq \pi}} St_\sigma^{(\Phi_1, \dots, \Phi_n)}$;
- (iii) $St_\pi^{(\Phi_1, \dots, \Phi_n)} = \sum_{\substack{\sigma \in \mathcal{P}([n]) \\ \sigma \geq \pi}} \mu(\pi, \sigma) St_{\geq \sigma}^{(\Phi_1, \dots, \Phi_n)}$;
- (iv) $St_{\hat{0}}^{(\Phi_1, \dots, \Phi_n)} = \sum_{\sigma \in \mathcal{P}([n])} \mu(\hat{0}, \sigma) St_{\geq \sigma}^{(\Phi_1, \dots, \Phi_n)}$,

where $\mu(\sigma, \pi)$ is the Möbius function on the interval $[\sigma, \pi] = \{\tau \in \mathcal{P}([n]) : \sigma \leq \tau \leq \pi\}$.

The measure $St_{\hat{0}}^{(\Phi_1, \dots, \Phi_n)}$ is called the *stochastic measure* of order n : it is concentrated on the so-called completely non-diagonal subset $\mathcal{A}_{\hat{0}}$ of \mathcal{A} , that is, on the subset of \mathcal{A} whose elements are the n -tuples with coordinates all distinct among themselves. In this direction, rectangles with kernel $\hat{0}$ are called triangles, since they have only trivial diagonal subsets. Moreover, note that the product measure $\Phi_1 \otimes \dots \otimes \Phi_n$ can be recovered as $St_{\geq \hat{0}}^{(\Phi_1, \dots, \Phi_n)}$. More generally, when $\Phi_j = \Phi$ for every $j = 1, \dots, n$, $St_{\geq \pi}^{[n]} := St_{\geq \pi}^{(\Phi, \dots, \Phi)}$ is always a product measure, in the sense specified in the following proposition [8].

Proposition 2.2. *The measure $St_{\geq \pi}^{[n]}$ is the product of the completely non-diagonal measures related to the blocks of $\pi \in \mathcal{P}([n])$:*

$$(4) \quad St_{\geq \pi}^{[n]} = \bigotimes_{b \in \pi} St_1^b,$$

where St_1^b is a short for $St_1^{(\Phi^{\otimes |b|})}$.

Remark 2.3. Thanks to Proposition 2.2 and identities (ii)–(iii) of Proposition 2.1, in dealing with random measures, it is sufficient to focus on the measures $St_{\hat{0}}^{[n]}$ and $St_1^{[n]}$.

Definition 2.4. If Φ_1, \dots, Φ_n are jointly good CR-measure, the n -th *diagonal measure* of $A \in \mathcal{Z}$ is defined by $\Delta_n(A) := St_1^{(\Phi_1, \dots, \Phi_n)}(A^{\otimes n})$.

It is easy to check that $\Delta_n(A \cup B) = \Delta_n(A) + \Delta_n(B)$ whenever A and B are disjoint, and that diagonal measures satisfy the following intersection property [8]:

$$(5) \quad St_1^{(\Phi_1, \dots, \Phi_n)}(A_1 \times \dots \times A_n) =$$

$$= St_1^{(\Phi_1, \dots, \Phi_n)} \left(\left(\bigcap_{i \in [n]} A_i \right)^{\otimes n} \right) = \Delta_n \left(\bigcap_{i \in [n]} A_i \right).$$

3. CUMULANTS AND DIAGONAL MEASURES

Jointly multiplicative random measures are defined accordingly with the definition of multiplicative measure given in [5, 8].

Definition 3.1. Let Φ_1, \dots, Φ_n be (jointly good) random measures over the same Polish space. Φ_1, \dots, Φ_n are *jointly multiplicative* if, for every partition $\pi \in \mathcal{P}([n])$, the following factorization over the blocks of π holds:

$$(6) \quad \mathbb{E}[St_\pi^{(\Phi_1, \dots, \Phi_n)}] = \bigotimes_{b \in \pi} \mathbb{E}[St_1^{(\Phi_j: j \in b)}].$$

It can be easily proved, just rearranging the corresponding proof in [8, Proposition 8], that jointly multiplicative random measures can be characterized in the following way (it is a consequence of the fact that the product of a non-atomic measure does not charge diagonals).

Proposition 3.1. *Let Φ_1, \dots, Φ_n be (jointly good) CR-measures on a fixed Polish space. Then, the following statements are equivalent:*

- (i) Φ_1, \dots, Φ_n are jointly multiplicative;
- (ii) for every $b \subseteq [n]$, the deterministic measure $\mathbb{E}[St_1^{(\Phi_j: j \in b)}(\cdot)]$ is non-atomic.

Example 3.1. The main examples of multiplicative good CR-measures are listed below (see [5, 8]):

- the *Gaussian measure*, such that $\Phi(A) \sim \mathcal{N}(0, \nu(A))$ for every A with $\nu(A) < \infty$, has diagonal measures given by $\Delta_1(A) = 0$, $\Delta_2(A) = \nu(A)$, $\Delta_n(A) = 0$, for all $n \geq 3$;
- the *Poisson measure*, such that $\Phi(A)$ has a Poisson distribution of rate $\nu(A)$ for every A with $\nu(A) < \infty$, has diagonal measures given by $\Delta_n(A) = \Phi(A)$, for all $n \geq 2$, while $\Delta_1(A) = \Phi(A) - \nu(A)$;
- the *Compound Poisson measure* given by $\Phi(A) = X_1 + \dots + X_{\Psi(A)}$, where $\Psi(\cdot)$ is a Poisson measure of control $\nu(A)$, has diagonal measure given by $\Delta_n(A) = X_1^n + \dots + X_{\Psi(A)}^n$, for all $n \geq 1$;
- let Φ denote the random measure spanned by a càdlàg² Lévy process $\mathbf{X} = \{X_t\}_{t \geq 0}$, with control given by the Lebesgue measure on the positive half-line of the real numbers, namely $\Phi([s, t]) = X_t - X_s$ for every $s < t$. Then, $\Delta_n([0, t]) = X_t^{(n)}$, where $\mathbf{X}^{(n)} = \{X_t^{(n)}\}_{t \geq 0}$ is the n -th process of variations of \mathbf{X} .

The representation of cumulants as deterministic real measures is established via the next statement.

²When working with Lévy processes, càdlàg is a short for *continue à droite, limite à gauche*, indicating that the paths X_t are right continuous and admit a left-limit.

Theorem 3.1. For $n \in \mathbb{N}$, let Φ_1, \dots, Φ_n be jointly good and multiplicative completely random measures. Then, for every $A_1, \dots, A_n \in \mathcal{Z}$ of finite measure,

$$(7) \quad \chi(\Phi_1(A_1), \dots, \Phi_n(A_n)) = \mathbb{E}[\text{St}_{\hat{1}}^{(\Phi_1, \dots, \Phi_n)}(A_1 \times \dots \times A_n)],$$

where $\chi(\Phi_1(A_1), \dots, \Phi_n(A_n))$ denotes the multidimensional cumulant of the random vector $(\Phi_1(A_1), \dots, \Phi_n(A_n))$.

Proof. By virtue of Proposition 2.1,

$$(8) \quad \Phi_1 \otimes \dots \otimes \Phi_n(A_1 \times \dots \times A_n) = \sum_{\pi \in \mathcal{P}([n])} \text{St}_{\pi}^{(\Phi_1, \dots, \Phi_n)}(A_1 \times \dots \times A_n).$$

Since the random measures are jointly multiplicative, taking the expectation on both sides of the above equation yields:

$$(9) \quad \mathbb{E}[\Phi_1 \otimes \dots \otimes \Phi_n(A_1 \times \dots \times A_n)] = \sum_{\pi \in \mathcal{P}([n])} \prod_{b \in \pi} \mathbb{E}[\text{St}_{\hat{1}}^{(\Phi_j : j \in b)}(\bigotimes_{j \in b} A_j)],$$

from which it trivially follows that:

$$(10) \quad \chi(\Phi_j(A_j) : j \in b) = \mathbb{E}[\text{St}_{\hat{1}}^{(\Phi_j : j \in b)}(\bigotimes_{j \in b} A_j)].$$

In particular, when $\pi = \hat{1}$, the desired relation follows. □

Applying (7) for $\Phi = \Phi_j$ for all $j = 1, \dots, n$, and the diagonal measures Δ_n associated with its product $\Phi^{\otimes n}$, the following statements hold.

Theorem 3.2. Let Φ be a multiplicative good CR-measure on the non-atomic Polish space (Z, \mathcal{Z}, ν) . For every measurable set $A \in \mathcal{Z}$, with $\nu(A) < \infty$:

$$(11) \quad \chi_n(\Phi(A)) = \mathbb{E}[\Delta_n(A)].$$

The following corollary points out the relationship between multivariate and univariate cumulants for any random measure, obtained straightforwardly from the intersection property of diagonal measures.

Corollary 3.1. Let Φ be a multiplicative good CR-measure on the Polish space (Z, \mathcal{Z}, ν) . Then, for every $A_1, \dots, A_n \in \mathcal{Z}$, $\nu(A_j) < \infty$,

$$(12) \quad \chi_n(\Phi(\bigcap_{j \in [n]} A_j)) = \chi(\Phi(A_1), \dots, \Phi(A_n)).$$

Example 3.2. Let Φ denote the random measure spanning a Lévy process $\mathbf{X} = \{X_t\}_{t \geq 0}$, with $X_0 = 0$ a.s., namely $\Phi([s, t]) = X_t - X_s$ if $s < t$. Then, for every $n \in \mathbb{N}$ and every choice of real numbers $0 \leq t_1 < t_2 < \dots < t_n$, identity (12) gives:

$$\chi(X_{t_1}, \dots, X_{t_n}) = \chi_n(X_{t_1}).$$

Theorem 3.2 entails, as a consequence, that the additivity of cumulants is inherited by the additivity of measures on disjoint measurable sets. Likewise, Corollary 3.1 provides the vanishing of cumulants of independent entries $(\Phi(A_i), \Phi(A_j))$, if $A_i \cap A_j = \emptyset$ as a consequence of $\Phi(\emptyset) = 0$ a.s.

3.1. An application to the process of variation of a Lévy process. Let $\mathbf{X} = \{X_t\}_{t \geq 0}$ denote a càdlàg Lévy process on \mathbb{R}_+ , admitting moments of all orders (see [1, 9] for any unspecified definition). Recall that for every integer $n \geq 1$, the n -th process of variations $\mathbf{X}^{(n)} = \{X_t^{(n)}\}_{t \geq 0}$ associated with \mathbf{X} is the Lévy process defined by:

- (i) $X_t^{(1)} = X_t$;
- (ii) $X_t^{(2)} = \sigma^2 t + \sum_{0 < s \leq t} (\Delta X_s)^2$;
- (iii) for every integer $n \geq 3$, $X_t^{(n)} = \sum_{0 < s \leq t} (\Delta X_s)^n$,

where σ^2 denotes the variance of the Gaussian component of \mathbf{X} (as in the Khintchine formula for its characteristic function), and ΔX_s is the jump in s , namely $\Delta X_s = X_s - X_{s-}$, with $X_{s-} = \lim_{t \rightarrow s-} X_t$. Cumulants of the variations process are related to the process \mathbf{X} via:

$$(13) \quad \chi(X_t^{(c_1)}, \dots, X_t^{(c_n)}) = \chi_{c_1 + \dots + c_n}(X_t),$$

where $c_j \in \mathbb{N}$, and with a slight abuse of notation, χ denotes both a multidimensional and a unidimensional cumulant.

Consider the n -dimensional variations process $(X^{(1)}, \dots, X^{(n)})$ associated with $\mathbf{X} = \{X_t\}_{t \geq 0}$. For every $t \geq 0$, the characteristic function of the random vector $(X_t^{(1)}, \dots, X_t^{(n)})$ is given by:

$$(14) \quad \begin{aligned} \varphi_t(z_1, \dots, z_n) = \\ = \exp \left\{ -\frac{1}{2} t z_1^2 \sigma^2 + i t z_2 \sigma^2 + t \int_{\mathbb{R}} \left(\exp \left\{ i \sum_{j=1}^n z_j x^j \right\} - 1 - i z_1 x \right) \mu(dx) \right\}, \end{aligned}$$

where μ denotes the Lévy measure of \mathbf{X} , and σ^2 the variance of its Gaussian component (see, for instance, [10]). Similarly, for every $t \geq 0$ and every choice of non-negative integers c_j 's, the characteristic function of the n -dimensional variations process $(X_t^{(c_1)}, \dots, X_t^{(c_n)})$ is given by:

$$(15) \quad \begin{aligned} \varphi_t(z_1, \dots, z_n) = \\ = \exp \left\{ -\frac{1}{2} t \sigma^2 \sum_{\substack{h,j=1,\dots,n \\ c_h=c_j=1}} z_h z_j + i t \sigma^2 \sum_{\substack{h=1,\dots,n \\ c_h=2}} z_h + \right. \\ \left. + t \int_{\mathbb{R}} \left(\exp \left\{ i \sum_{c_j \geq 2} z_j x^{c_j} \right\} - 1 - i z_1 x \right) \mu(dx) \right\}. \end{aligned}$$

Thanks to (15), it follows that, if $c_1 + \dots + c_n \geq 3$:

$$(16) \quad \chi(X_t^{(c_1)}, \dots, X_t^{(c_n)}) = t \int_{\mathbb{R}} x^{c_1 + \dots + c_n} \mu(dx) = t L_{c_1 + \dots + c_n},$$

while, if $c_1 + \dots + c_n = 2$ (in the case $n = 2$), then one has:

$$(17) \quad \chi(X_t^{(c_1)}, \dots, X_t^{(c_n)}) = t\sigma^2 + tL_2,$$

where L_m denotes the m -th Lévy moment of \mathbf{X} (that is, the m -th moment of the Lévy measure μ). From these identities, (13) follows trivially remarking that if $c_1 + \dots + c_n \geq 3$, then:

$$(18) \quad \chi_{c_1+\dots+c_n}(X_t) = tL_{c_1+\dots+c_n},$$

while if $c_1 + \dots + c_n = 2$ (in the case $n = 2$):

$$(19) \quad \chi_2(X_t) = t\sigma^2 + tL_2.$$

In general, identity (13) is proved by differentiating the respective cumulant generating functions (as for the computation of any cumulant), and by checking that the two quantities are equal. One of the advantages of the random measure approach is that (13) follows simply by taking an expectation. For any integer $c \geq 1$, consider the random measure spanning the variation process of the Lévy process \mathbf{X} , namely $\Phi_c([s, t]) = X_t^{(c)} - X_s^{(c)}$, as introduced in [4]. In particular, $\Phi_c([0, t]) = X_t^{(c)}$ if $X_0 = 0$ a.s.. For every choice of non-negative integers c_1, \dots, c_n , the authors in [4] showed that the diagonal measure associated with the product random measure $\Phi_{c_1} \otimes \dots \otimes \Phi_{c_n}$ corresponds to the variation of order $c_1 + \dots + c_n$. More precisely, if $St_1^{(c_1, \dots, c_n)}$ stands for $St_1^{(\Phi_{c_1}, \dots, \Phi_{c_n})}$, and following the notation introduced in [4], one has:

$$(20) \quad \begin{aligned} St_1^{(c_1, \dots, c_n)}([0, t]^{\otimes n}) &= \Phi_{c_1} \otimes \dots \otimes \Phi_{c_n} ([0, t]^{\otimes n})_1 = \\ &= \Phi_{c_1+\dots+c_n}([0, t]) = X_t^{(c_1+\dots+c_n)}, \end{aligned}$$

and therefore, if $m = c_1 + \dots + c_n$,

$$(21) \quad St_1^{(c_1, \dots, c_n)}([0, t]^{\otimes n}) = St_1^{(\Phi_1, \dots, \Phi_1)}([0, t]^{\otimes m}),$$

namely, the diagonal measure associated with $\Phi_1 = \Phi$.

Remark 3.1. For the random measures $\Phi_{c_1}, \dots, \Phi_{c_n}$ spanning the processes of variations of orders c_1, \dots, c_n , Proposition 2.2 has been generalized to yield:

$$\Phi_{c_1} \otimes \dots \otimes \Phi_{c_n} ((A^{\otimes n})_{\geq \sigma}) = \prod_{j=1}^m \Phi_{\sum_{i \in B_j} c_j}(A),$$

for $\sigma \in \mathcal{P}([n])$, where B_1, \dots, B_m denote the blocks of σ (see [10, Lemma 4.4]).

Most importantly, the random measures spanning the processes of variation of a Lévy process are jointly-multiplicative.

Proposition 3.2. *The random measures $\Phi_{c_1}, \dots, \Phi_{c_n}$ generating the variations processes of orders c_1, \dots, c_n of a (càdlàg) Lévy process \mathbf{X} , with moments of all orders, are jointly multiplicative.*

Proof. For every $b \subseteq [n]$, it follows from (20) that:

$$\mathbb{E}[St_1^{(c_j: j \in b)}([0, t]^{\otimes |b|})] = \mathbb{E}[X_t^{(a_b)}] = L_{a_b} t,$$

where $a_b := \sum_{j \in b} c_j$. □

For the sake of convenience, Theorem 3.1 will be reformulated explicitly for random measures spanning the processes of variation of a Lévy process.

Proposition 3.3. *Let \mathbf{X} be a càdlàg Lévy process, with finite moments of all order. For every $n \in \mathbb{N}$, and every choice of non-negative integers c_1, \dots, c_n :*

$$(22) \quad \chi(X_t^{(c_1)}, \dots, X_t^{(c_n)}) = \mathbb{E} \left[S_t^{(c_1, \dots, c_n)}([0, t]^{\otimes n}) \right].$$

As a consequence, (13) can be proved in few lines.

Proof. As already recalled, for a random measure spanning a Lévy process, the diagonal measures correspond to the processes of variations: $\Delta_n([0, t]) = X_t^{(n)}$. Then, from (11) and by virtue of Theorem 3.1, the identity (13) follows simply by taking the expectation in (20). □

Teugel martingales. Let $\mathbf{X} = \{X_t\}_{t \geq 0}$ denote a càdlàg Lévy process, with finite moments of every order. Then, for every integer $n \geq 1$, the n -th *Teugel martingale* $\mathbf{Y}^{(n)} = \{Y_t^{(n)}\}_{t \geq 0}$ is the square integrable martingale, with independent increments, defined as:

$$Y_t^{(n)} := X_t^{(n)} - \mathbb{E}[X_t^{(n)}] = X_t^{(n)} - \chi_n(X_t).$$

In particular, the additivity property of cumulants and the stationarity of the increments of the Lévy process \mathbf{X} , yield that $\mathbf{Y}^{(n)}$ is itself a Lévy process. In particular, indeed, for every $0 \leq s < t$,

$$\begin{aligned} Y_t^{(n)} - Y_s^{(n)} &= \left(X_t^{(n)} - \chi_n(X_t) \right) - \left(X_s^{(n)} - \chi_n(X_s) \right) = \\ &= \left(X_t^{(n)} - X_s^{(n)} \right) - \left(\chi_n(X_t) - \chi_n(X_s) \right) = \\ &= X_{t-s}^{(n)} - \chi_n(X_{t-s}) = Y_{t-s}^{(n)}, \end{aligned}$$

since $X_t \stackrel{\text{Law}}{=} X_{t-s} + X_s$, and X_{t-s} is independent of X_s . Hence, for every integer $r \geq 1$, and integers $h_1, \dots, h_r \geq 1$, we can consider the multidimensional variation process of $Y_t^{(n)}$, that is,

$$(Y^{(n)})_t^{(h_1)}, \dots, (Y^{(n)})_t^{(h_r)}.$$

Then, one can easily derive that:

$$\begin{aligned} \chi((Y^{(n)})_t^{(h_1)}, \dots, (Y^{(n)})_t^{(h_r)}) &= \chi_{h_1 + \dots + h_r}(Y_t^{(n)}) = \\ &= \begin{cases} \chi_1(X_t^{(n)}) - \chi_n(X_t) = 0 & \text{if } h_1 + \dots + h_r = 1 \\ \chi_{h_1 + \dots + h_r}(X_t^{(n)}) = \chi_{n(h_1 + \dots + h_r)}(X_t) & \text{if } h_1 + \dots + h_r \geq 2. \end{cases} \end{aligned}$$

In particular, then, if $r, n \geq 2$,

$$(23) \quad \chi_r(Y_t^{(n)}) = \chi_r(X_t^{(n)}) = \chi_{rn}(X_t) = \chi_n(X_t^{(r)}) = \chi_n(Y_t^{(r)}).$$

4. DIAGONAL MEASURES AND κ -STATISTICS

Throughout this section, assume that Φ is a positive good CR-measure on the fixed non-atomic Polish space (Z, \mathcal{Z}, ν) , that is $\Phi(A) \geq 0$ a.s. for every measurable set A in \mathcal{Z} , with $\nu(A)$ finite.

Example 4.1. This is the case for the random measure on \mathbb{R}_+ spanning a Lévy process that is a *subordinator* (namely, a Lévy processes that is a.s. increasing in time), as for the Poisson and the Gamma processes (see [1]).

Few statistical definitions, in the setting of simple random sampling, will be needed.

Definition 4.1.

- An estimator V of a population characteristic θ is called *unbiased* if $\mathbb{E}[V] = \theta$.
- Given a finite population $\mathbf{x} = \{x_1, \dots, x_N\}$ of size N , consider a sequence of statistics $T = \{T_1, \dots, T_N\}$, with T_n function of n variables for all $n \geq 1$ (usually, $\theta := T_N(\mathbf{x})$ is a parameter to be estimated). T is said to be *inherited on the average* if, for every $n \leq N$, the average over all possible samples $\mathbf{y} = (y_1, \dots, y_n)$ of size n , drawn from the population \mathbf{x} , equals $T_N(x_1, \dots, x_N)$, in symbol:

$$\mathbb{E}[T_n(\mathbf{y})|\mathbf{x}] = T_N(\mathbf{x}) .$$

- For an infinite population, the inheritance on the average is satisfied in the limit, as:

$$\lim_{n \rightarrow \infty} \mathbb{E}[T_n(\mathbf{y})] = \theta ,$$

where $\mathbf{y} = (y_1, \dots, y_n)$ denotes a sample of size n drawn from the population.

Remark that unbiasedness and inheritance on the average are two structurally different concepts: indeed, unbiasedness refers to a single estimator, while inheritance is concerned with sequences of statistics. Sometimes, statistics being inherited on the average for finite populations are called *natural statistics* (see [2] for a new approach to natural statistics for spectral samples, via symbolic methods).

Definition 4.2. Given any statistical distribution, or equivalently any random variable X , the κ -statistics c_n is the unique symmetric unbiased estimator of its cumulant $\chi_n(X)$, that is, $\mathbb{E}[c_n] = \chi_n(X)$. A *polykay* $c_{r, \dots, s}$ is an unbiased estimator for a product of cumulants: $\mathbb{E}[c_{r, \dots, s}] = \chi_r(X) \cdots \chi_s(X)$, for $r, \dots, s \in \mathbb{N}$.

Moreover, sequences of κ -statistics and polykays satisfy the inheritance on the average property.

By virtue of Theorem 3.2, the n -th diagonal measure $\Delta_n(A)$ is an unbiased estimator for the n -th cumulant of $\Phi(A)$ whenever Φ is a good CR-measure. Similarly, if Φ is multiplicative, the sequence $St_\pi^{[n]}(A^{\otimes n})$ is a sequence of polykays for the parent distribution $\Phi(A)$.

More generally, the decomposition (1) yields an explicit description of the κ -statistics for $\Phi(A)$, where Φ is the random measure spanning a Lévy process that is a subordinator, in the sense of the following proposition.

Proposition 4.1. Consider the random variable $\Phi(A)$, and for every N , the i.i.d. random variables $\Phi(A_{1N}), \dots, \Phi(A_{NN})$ associated with the decomposition in (1). Then, for every integer $n \geq 1$, the sequence with general term $\sum_{i=1}^N \Phi(A_{iN})^n$ satisfies the inheritance on the average property as to estimation of $\chi_n(\Phi(A))$, namely:

$$(24) \quad \lim_{N \rightarrow \infty} \mathbb{E} \left[\sum_{i=1}^N \Phi(A_{iN})^n \right] = \chi_n(\Phi(A)) .$$

Proof. Last statement follows straightforwardly starting from the L^2 -limit relation for diagonal measures provided in [8, Proposition 12]: for a fixed integer $n \geq 1$, and a multiplicative good CR-measure Φ , if for every $m < n$ there exist positive constants c_m and d_m such that:

1. $\mathbb{E}[(\Delta_m - \langle \Delta_m \rangle)^2] \leq c_m \nu$,
2. $|\mathbb{E}[\Delta_m]| \leq d_m \nu$;

(where ν denotes the non-atomic control of Φ and Δ_n the n -th diagonal measure associated with $\Phi^{\otimes n}$), then it holds true that:

$$(25) \quad \Delta_n(A) = \lim_{N \rightarrow \infty} \sum_{i=1}^N \Phi(A_{iN})^n ,$$

where the limit is to be intended in $L^2(\Omega)$.

Since, for every Lévy process, the non-atomic control is given by the Lebesgue measure times its second Lévy moment, from the identity:

$$\mathbb{E}[X_t^{(n)} X_s^{(n)}] = L_{2n} \min\{s, t\} + L_n^2 t^2$$

(see [4]), it is easy to check that the diagonal measures associated with \mathbf{X} satisfy these requirements. Indeed, for $c_m := L_{2m}/L_2$, $d_m = |L_m|/L_2$, one has:

1. $\mathbb{E}[(X_t^{(m)} - L_m t)^2] = \mathbb{E}[(X_t^{(m)})^2] - L_m^2 t^2 = L_{2m} t = c_m L_2 t$;
2. $|\mathbb{E}[X_t^{(m)}]| = |t L_m| = |L_m| t = d_m L_2 t$,

where L_n 's are the Lévy moments of \mathbf{X} (note that, by virtue of Wolfe's Theorem, the Lévy process \mathbf{X} has finite moments of every order if and only if its Lévy measure has, see for instance [5, Lemma 5.3.4]). Since L^2 -convergence implies the convergence in L^1 -norm for finite measure spaces, the desired conclusion follows by applying Theorem 3.2:

$$(26) \quad \lim_{N \rightarrow \infty} \left\| \sum_{i=1}^N \Phi(A_{iN})^n \right\|_{L^1} = \|\Delta_n(A)\|_{L^1} = \mathbb{E}[\Delta_n(A)] .$$

□

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