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The role of Hill equation in the study of torsional instability in suspension bridge models

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Abstract. The present survey aims to highlight how the theory of Hill equation turns out to be a crucial tool of investigation when dealing with torsional stability in suspension bridges models. More precisely, we first recall some basic facts about Hill stability and instability domains and then we explain how these results allow to give some theoretical answers concerning the origin of torsional instability, the shape of torsional oscillations and the rule governing the energy transfer between different oscillating modes. Most of the results stated in the paper have been obtained in [7] and [8].

1. Introduction

If there exists a boundary between linear and nonlinear systems, J.J. Stoker in the introduction of his nice book *Nonlinear Vibrations in Mechanical and Electrical Systems* [28] locates on this boundary the Hill equation:

$$y'' + p(t)y = 0 \quad , \quad p(t) = p(t+T)\ . \tag{1.1}$$

Namely a linear, second order differential equations with real, periodic coefficients. The motivations of this choice come, on one hand, from the fact that the treatment of important questions of stability of any periodic *nonlinear* oscillation leads inevitably to such equations. On the other hand, the vibration phenomena encountered in equations of this type have features which place them, in a sense, in a position between those of nonlinear and of linear equations.

The name Hill equation comes from the fundamental contribution given to its theory by George W. Hill [15] who introduced the equation in his study of celestial mechanics: he gave a new definition of the equation of the first approximation of the motions of the moon. Even today, Hill definition is of basic importance in this field of research. M. H. Poincaré in his memoire of Hill [26] predicts: *"... je crois que ceux qui s'occupent des petites planètes seront étonnés des facilitités qu'ils rencontreront le jour où en ayant pénétré l'esprit ils les appliqueront à ce nouvel object. Mais jusqu'ici c'est pour la Lune qu'elles ont fait leurs preuves; quand elles*

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s'étendront à un domaine plus vaste, on ne devra pas oublier que c'est à M. Hill que nous devons un instrument si précieux". In fact, this model equation appears in many areas of applied mathematics aiming to study the stability of periodic motions, such us problems in mechanics, astronomy, theory of elettric circuit. One of the simplest example is that of the so-called inverted pendulum. Consider a force varying periodically in time and assume that it acts on the pendulum and tends to move it back to an equilibrium position, contrary to what one may expect, the mass may not remain in a neighborhood of the equilibrium position and an increasing in the force may cause wider and wider oscillations. The theory of stability and instability regions of Hill equation provides the precise theoretical explanation of this and many other phenomena, see [11, 18] and Section 2.

Quoting Poincaré, a "nouvel object" to which Hill theory has been recently applied is the study of structural instability in long span suspension bridges. From the engineering point of view, these kind of structures pose great technical difficulty due to their slenderness and they tend to gave problems, some of which have witnessed in our history. We cannot avoid to recall the celebrated Tacoma Narrow Bridge collapse occurred in 1940. It is well-know, since the federal report [1], that the crucial event in the collapse was a sudden change from a vertical to a torsional mode of oscillation. A similar kind of destructive torsional oscillations was observed in several other suspension bridges, among the others, we recall the Brighton Chair Pier collapsed in 1823 and the Matukituki Suspension Footbridge collapsed in 1977, we refer to the books [14] and [17] for an up to now survey on the subject. Despite what so far remarked, this research topic is currently of great interest due to the utility of these constructions to overcome geographical barriers [3, 9, 12, 20]. From the purely theoretical point of view, the interest for the mathematics behind these problems has been highlighted in several papers by J. Mckenna [21, 22, 23] and in the recent book [14]. Besides the fundamental question of the origin of torsional instability, further interesting questions may be posed, concerning the shape of torsional oscillations and the rule governing the energy transfer between different oscillating modes, see [7, 8] for more details. The aim of the present survey is to highlight how, once more, the theory of Hill equation turns out to be a powerful tool of investigation and allows to give some theoretical answers to the above mentioned questions.

The paper is organized as follows: in Section 2 we recall some theoretical facts about Hill equation and its stability regions. In particular, to make the paper self contained, we shortly explain how the so-called Zhukovskii's stability test may be derived from the general theory. In Section 3.1 we briefly describe the suspension bridges model considered and we relate the nonlinear problems arising to the study of the stability of some families of Hill equations. In the subsequent Section 3.2 we discuss how the theory of Hill equation may be exploited to give a sufficient condition for torsional stability and a sound answer to the origin of torsional instability. Finally, in Section 3.3, a precise criterion governing the energy transfer between different oscillating modes is outlined. Here, the specific choice of potential considered yields particular kind of Hill equations, the so-called Mathieu equations. The detailed proofs of all the results collected in the paper may be found in [30], for Hill theory, and in [7] and [8], for the application to suspension bridges models. Nevertheless, for the sake of completeness, we put some sketch of them in Section 4.

2. Hill equation: solution set and stability domains

The problem of deciding stability questions to (1.1) is, in general, quite difficult and it requires the studying of solutions in considerable detail. For this reason, we start by briefly describing the general form of solutions to (1.1). Clearly, (1.1) may be reduced to the canonical (vector) form

$$J\,\frac{d\mathbf{x}}{dt} = H(t)\mathbf{x} \tag{2.1}$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y' \\ y \end{bmatrix}$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $H(t) = \begin{bmatrix} 1 & 0 \\ 0 & p(t) \end{bmatrix}$.

Next we introduce some notations that will be useful in the sequel. Let Ω denote the set of absolutely continuous matrix functions $F(t)$ such that $F(t)$ is $T-$periodic ($F(t+T) = F(t)$) or $T-$antiperiodic ($F(t+T) = -F(t)$), $\det F(t) = 1$ and $F(0) = Id$. Ω becomes a metric space when endowed of a suitable matrix norm, see [30, Chapter VIII, Section 1] for the details. Furthermore, we denote with $\overline{\mathbb{R}}^3$ the set of all real matrices K with Tr $K = 0$. $\overline{\mathbb{R}}^3$ turns out to be isomorphic to $\mathbb{R}^3$ since any matrix $K \in \overline{\mathbb{R}}^3$ writes $K = \begin{bmatrix} -x & y-z \\ y+z & x \end{bmatrix}$ with $(x, y, z) \in \mathbb{R}^3$. Hence, the eigenvalues λ_i of K satisfy $\lambda_i^2 = x^2 + y^2 - z^2 = \det K$. We denote with $\widetilde{\Pi}$ the cone $x^2 + y^2 = z^2$ and $\widetilde{\mathcal{H}}$ the exterior of the cone $x^2 + y^2 > z^2$. The cone $\widetilde{\Pi}$ is the locus of singular matrices $K \in \overline{\mathbb{R}}^3$, indeed $\det K = 0$ and $\lambda_i = 0$ for some $i = 1, 2$, while in $\widetilde{\mathcal{H}}$ $\det K > 0$ and the eigenvalues λ_i are real. The set $\widetilde{\Pi}$ may be decomposed as $\widetilde{\Pi} = \widetilde{\Pi}_- \cup \mathbf{0} \cup \widetilde{\Pi}_+$, where $\widetilde{\Pi}_\pm$ are the lower and upper sheet of the cone Π without vertex and $\mathbf{0}$ corresponds to the null matrix. Finally, we denote with $\widetilde{\mathcal{O}}_+$ the domain between the upper sheet of the cone $x^2 + y^2 = z^2$ and the upper sheet of the hyperboloid $x^2 + y^2 - z^2 = -\pi^2/T^2$, namely $\widetilde{\mathcal{O}}_+ := \{(x, y, z) \in \mathbb{R}^3 : -\pi^2/T^2 < x^2 + y^2 - z^2 < 0 \text{ and } z > 0\}$. Clearly, in $\widetilde{\mathcal{O}}_+$, $\det K < 0$ and the eigenvalues are pure imaginary. We are so ready to state the following theorem which gives a detailed characterization of solutions to (2.1):

Theorem 2.1 (Floquet-Lyapunov theorem, [30, Chapter VIII, Section 1]**).** *The fundamental matrix $X(t)$ of (2.1) may be expressed as*

$$X(t) = F(t)e^{tK}\ , \tag{2.2}$$

where $F \in \Omega$ and $K \in \overline{\mathbb{R}}^3$. Furthermore, the matrix K may always be chosen in the domain $\overline{\mathbb{R}}^1 := \widetilde{\mathcal{H}} \cup \widetilde{\Pi} \cup \widetilde{\mathcal{O}}_+$ and when this is done the matrices K and $F(t)$ are uniquely defined.

Let $\mathbf{a} \neq \mathbf{0}$ be an arbitrary vector and $\mathbf{x}(t) = F(t)\mathbf{a}$, where $F \in \Omega$. Let $\varphi_{\mathbf{x}}$ denote the rotation of the vector $\mathbf{x}$ in time T, that is the angle through which it turns in that time. Since $F(T) = \pm Id$, it follows that $\varphi_{\mathbf{x}} = n\pi$, with n integer. It can be proved that n only depends on the matrix $F(t)$ and we denote it n_F. Theorem 2.1 may be stated for general linear differential equations with periodic coefficients, but when H is as in (2.1), namely for Hill equation, n_F turns out to be nonnegative, see [30, Chapter VIII, Section 3.1]. Let Ω_n be the set of matrices $F(t)$ such that

$n_F = n$, then

$$\Omega = \bigcup_{n=0}^{n=+\infty} \Omega_n \quad \text{and} \quad \Omega_n \cap \Omega_m = 0 \text{ if } n \neq m \, . \tag{2.3}$$

Summarizing, by Theorem 2.1 the motion defined by system (2.1) may be decomposed as follows:

(i) a motion on the plane $\{y\}$ obeying the equation $\mathbf{y'} = e^{tK}\mathbf{y}$. The corresponding trajectories are either hyperbolas ($K \in \widetilde{\mathcal{H}}$), ellipses ($K \in \widetilde{\mathcal{O}}$), straight lines ($K \in \widetilde{\Pi}_{\pm}$) or all points of the plane are fixed ($K = \mathbf{0}$);

(ii) the whole plane $\{y\}$ is subjected to an area-preserving deformation by $F(t)$ and rotated through an angle $n\pi$ in time T. Since the transformation need not be orthogonal, the plane will generally not only rotate but will also be deformed.

We recall that a solution to (2.1) is said to be *stable* if all the solutions are bounded and *unstable* if an unbounded solution exists. Hence, for what so far stated, the stability to (2.1) (and in turns to (1.1)) turns out to be related to which of the sets $\widetilde{\mathcal{H}}, \widetilde{\Pi}, \widetilde{\mathcal{O}}_+$ the matrix K belongs. Namely, to the eigenvalues λ_i of the matrix K, usually called *Floquet characteristic exponents.* We note that the matrix K is related to $X(T)$ (monodromy matrix) by the relation:

$$\rho_i = exp(\lambda_i T) \quad \text{for } i = 1, 2 \, ,$$

where the ρ_i are the eigenvalues of $X(T)$ and are named *Floquet multipliers.* Since, in general, these multipliers are not explicitly known, the stability analysis to (2.1) is not an easy topic to handle. In this perspective, several stability tests may been found in literature each one based on a different strategy, such as estimating the eigenvalues of $H(t)$ and, in turn, the characteristic exponents, or studying the rotation of solutions. To our future scopes, here below we recall the Zhukovskii's stability test. First we need to specify what we mean exactly for stability and instability domains of (2.1).

Denote with $\widetilde{\mathcal{L}}^3$ the set of all Hamiltonians $H(t)$. Each matrix $H(t) \in \widetilde{\mathcal{L}}^3$ determines a unique fundamental matrix $X(t)$ ($X(0) = Id$) of system (2.1). Via formula (2.2), $X(t)$ determines a unique pair $F(t)$ and K where $F \in \Omega$ and $K \in \overline{\mathbb{R}}^1$. We write this correspondence as $\widetilde{\mathcal{L}}^3 = \Omega \times (\widetilde{\mathcal{H}} \cup \widetilde{\Pi} \cup \widetilde{\mathcal{O}}_+)$. Now we define $\mathcal{H}_n = \Omega_n \times \widetilde{\mathcal{H}}$, $\mathcal{O}_n = \Omega_n \times \widetilde{\mathcal{O}}_+$ and $\Pi_n = \Omega_n \times \widetilde{\Pi}$, with Ω_n as in (2.3), and we set

$$\mathcal{H} = \bigcup_{n=0}^{n=+\infty} \mathcal{H}_n \quad , \quad \mathcal{O} = \bigcup_{n=0}^{n=+\infty} \mathcal{O}_n \quad , \quad \Pi = \bigcup_{n=0}^{n=+\infty} \Pi_n \, .$$

Furthermore, recalling that $\widetilde{\Pi} = \widetilde{\Pi}_- \cup \mathbf{0} \cup \widetilde{\Pi}_+$, we set $\Pi^{**} = \bigcup_{n=0}^{n=+\infty} (\Omega_n \times \mathbf{0}) = \bigcup_{n=0}^{n=+\infty} \Pi_n^{**}$ and $\Pi^* = \Pi \setminus \Pi^{**} = \bigcup_{n=0}^{n=+\infty} \Pi_n^*$. Hence,

$$\widetilde{\mathcal{L}}^3 = \mathcal{H} \cup \Pi \cup \mathcal{O} \quad \text{and} \quad \Pi = \Pi^* \cup \Pi^{**} \, .$$

Since, from general theorems of linear algebra, there always exists a constant nonsingular real matrix S such that

- if $H(t) \in \mathcal{H}$ one has $e^{tK} = S \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{-\lambda t} \end{bmatrix} S^{-1}$, for some $\lambda \in \mathbb{R}$;

- if $H(t) \in \mathcal{O}$ one has $e^{tK} = S \begin{bmatrix} \cos(\beta t) & -\sin(\beta t) \\ \sin(\beta t) & \cos(\beta t) \end{bmatrix} S^{-1}$, for some $\beta \in (0, \pi/T)$;
- if $H(t) \in \Pi^*$ one has $e^{tK} = S \begin{bmatrix} 1 & \pm t \\ 0 & 1 \end{bmatrix} S^{-1}$ while if $H(t) \in \Pi^{**}$ one has $e^{tK} = Id$,

it follows that the $\mathcal{O}_n$ are *stability* domains for H, namely all corresponding solutions are bounded, the $\mathcal{H}_n$ are *instability* domains and the Π_n are *boundary* sets. In particular, if $H(t) \in \Pi_n^*$ unstable solutions exist while if $H(t) \in \Pi_n^{**}$ all solutions are stable. In Section 4.1 we show how the following statement can be obtained by relating the rotation of solutions $\varphi_{\mathbf{x}}$ to their position with respect the stability domains.

Theorem 2.2 ([30, Chapter VIII, Section 3]). *Let r_1 and r_2 be arbitrary functions satisfying: r_i'' exists, it is Lebesgue summable and*

$$r_i(t+T) = r_i(t) > 0 \quad , \quad \int_0^T r_i^{-2}(t)\, dt = 1 \, .$$

If

$$\frac{n^2\pi^2}{r_1^4} - \frac{r_1''}{r_1} \le p(t) \le \frac{(n+1)^2\pi^2}{r_2^4} - \frac{r_2''}{r_2} \, ,$$

for some integer $n = 0, 1, 2, \cdots$, then $H(t) \in \mathcal{O}_n$ and the trivial solution to the Hill equation (1.1) *is stable.*

Taking $r_1 \equiv r_2 = const$, Theorem 2.2 yields the following corollary

Corollary 2.1 (**Zhukovskii's test** [30, Chapter VIII,Section 3]). *If*

$$\frac{n^2\pi^2}{T^2} \le p(t) \le \frac{(n+1)^2\pi^2}{T^2} \, ,$$

for some integer $n = 0, 1, 2, \cdots$, then $H(t) \in \mathcal{O}_n$ and the trivial solution to the Hill equation (1.1) *is stable.*

We have already remarked that, in general, the determination of the stability regions of an Hill equation is a very difficult task and one aims, at least, to estimate their location. On the contrary, in the particular case $p(t) = a + 2q\cos(2t)$, with a, q real constants, very precise informations are known. In this case (1.1) is named *Mathieu equation* and writes

$$y'' + (a + 2q\cos(2t))\, y = 0 \, ,$$

where, for our future purposes, we assume $q > 0$. Here, $T = \pi$ and the sets $\mathcal{O}_n, \mathcal{H}_n, \Pi_n$ in $\widetilde{\mathcal{L}}^3$ define projections on the (q, a)-plane which we shall continue to denote with the same letters. In other words, we say that a point (q, a) belongs to one of the sets in $\widetilde{\mathcal{L}}^3$ if the corresponding Hamiltonian (2.1) belongs to that set.

It is known that the Mathieu equation admits solutions which are either π or 2π-periodic only if a belongs to the countably infinite sets of the so-called Mathieu characteristic values $\{a_n(q)\}_{n\ge 0}$ and $\{b_n(q)\}_{n\ge 1}$, see [24, 29]. The characteristic curves do not intersect, that is, we have

$$a_0(q) < b_1(q) < a_1(q) < \cdots < b_n(q) < a_n(q) < b_{n+1}(q) < \cdots \qquad \forall\, n \ge 2 \, .$$

It is also known that periodic solutions cannot coexist, hence the characteristic curves correspond to the boundary sets Π_n^* and they divide the (q,a)-plane into stable and unstable regions, see Figure 1 where the lines correspond to the characteristic curves, the instability regions $\mathcal{H}_n$ are shaded and the stability regions $\mathcal{O}_n$ are white. More precisely, for $n \geq 0$ we have

$$\mathcal{O}_n := \{(q,a)\,:\, q>0,\ a_n(q)<a<b_{n+1}(q)\}\ ,$$

while for $n \geq 1$ we have

$$\mathcal{H}_n := \{(q,a)\,:\, q>0,\ b_n(q)<a<a_n(q)\}\ .$$

Furthermore, in the last section we will exploit the expansions:

$$\begin{cases} a_0(q)=o(q),\quad b_1(q)=1-q+o(q),\quad a_1(q)=1+q+o(q)\ , \\ b_n(q)=n^2+o(q)\ \text{ and }\ a_n(q)=n^2+o(q)\quad \forall\, n\geq 2\ , \end{cases} \qquad \text{as } q\to 0 \tag{2.4}$$

and

$$a_n(q)\sim -2q \quad , \quad b_n(q)\sim -2q \qquad \text{as } q\to\infty\ , \tag{2.5}$$

see [24, Sections 2.151 and 12.30].

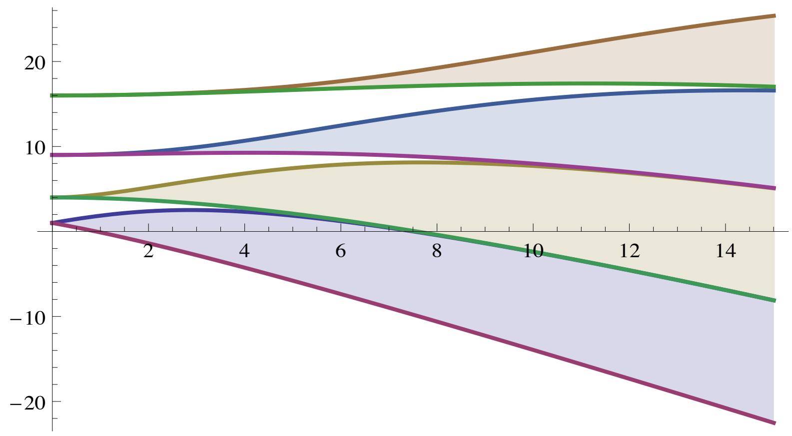

Figure 1. The Mathieu stability diagram in the (q,a)-plane. The instability regions $\mathcal{H}_n$ are shaded, the stability regions $\mathcal{O}_n$ are white.

3. Application of Hill theory to suspension bridges

3.1. **A nonlinear model for suspension bridge.** We briefly describe the mathematical model suggested in [7] and [13]. The roadway of a suspension bridge is seen as a long narrow rectangular thin plate hinged at the two opposite short edges and free on the remaining two edges. Let L denote its length and 2ℓ denote its width; a realistic assumption is that $2\ell \cong L/100$. The rectangular plate $\Omega \subset \mathbb{R}^2$ is then

$$\Omega = (0,L)\times(-\ell,\ell)\ .$$

Then, by considering the bending energy of the plate and the potential energy due to the cable-hanger system, one ends up with the following nonlinear problem where u denotes the downwards displacement of the deck:

$$\begin{cases} mu_{tt}+\delta u_t+\dfrac{E\,d^3}{12(1-\sigma^2)}\,\Delta^2 u+h(y,u<)=f & (x,y,t)\in\Omega\times(0,T)\\ u(0,y,t)=u_{xx}(0,y,t)=u(L,y,t)=u_{xx}(L,y,t)=0 & (y,t)\in(-\ell,\ell)\times(0,T)\\ u_{yy}(x,\pm\ell,t)+\sigma u_{xx}(x,\pm\ell,t)=0 & (x,t)\in(0,L)\times(0,T)\\ u_{yyy}(x,\pm\ell,t)+(2-\sigma)u_{xxy}(x,\pm\ell,t)=0 & (x,t)\in(0,L)\times(0,T)\\ u(x,y,0)=\phi_0(x,y)\ ,\ \ u_t(x,y,0)=\phi_1(x,y) & (x,y)\in\Omega\ . \end{cases}$$

Here, m is the mass density, $\delta>0$ is the damping coefficient, h is the restoring force due to the cables, f is an external force, d denotes the thickness of the plate, E is the Young modulus and σ is the Poisson ratio which depends by the material of the plate. As concerns the boundary conditions, we have already mentioned that they are hinged on short edges, namely Navier boundary conditions arise, and free on the remaining edges. We refer to [7] for a detailed description and derivation of the model. Since the goal of the stability analysis we are going to outline is to show that the principal cause of torsional instability are internal resonances which mainly depend on the structure of the bridge itself, following a suggestion of Irvine [16, p.176], the isolated version of the above problem is considered. Furthermore, following Bartoli-Spinelli [4, p.180] and Plaut-Davis [25, § 3.5] and assuming that the action of the hangers on the roadway is confined in the union of two thin strips parallel and adjacent to the two long edges of the plate Ω, the restoring force due to the cables-hangers system will be

$$h(y,u)=\Upsilon(y)\left(k_1\,u+k_2\,u^3\right)\ ,$$

where Υ is the characteristic function of the set $(-\ell,-\ell+\delta)\cup(\ell-\delta,\ell)$ for some $\delta>0$ and $k_1,k_2>0$ depend on the elasticity of the cables and hangers. Finally, by exploiting the change of variable (without renaming)

$$u(x,y,t)=\sqrt{\frac{k_1}{k_2}}\;u\left(\frac{\pi x}{L}\,,\,\frac{\pi y}{L}\,,\,\sqrt{\frac{k_1}{m}}\,t\right)\quad,\quad \gamma=\frac{E\,d^3}{12k_1(1-\sigma^2)}\,\frac{\pi^4}{L^4}$$

one reduces to the following dimensionless and isolated problem:

$$\begin{cases} u_{tt}+\gamma\Delta^2 u+\Upsilon(y)(u+u^3)=0 & \text{in } \Omega\times(0,\infty)\\ u(0,y,t)=u_{xx}(0,y,t)=u(\pi,y,t)= & \\ =u_{xx}(\pi,y,t)=0 & \text{for } (y,t)\in(-\ell,\ell)\times(0,\infty)\\ u_{yy}(x,\pm\ell,t)+\sigma\cdot u_{xx}(x,\pm\ell,t)=0 & \text{for } (x,t)\in(0,\pi)\times(0,\infty)\\ u_{yyy}(x,\pm\ell,t)+ & \\ +(2-\sigma)\cdot u_{xxy}(x,\pm\ell,t)=0 & \text{for } (x,t)\in(0,\pi)\times(0,\infty)\\ u(x,y,0)=\phi_0(x,y)\ , & \\ u_t(x,y,0)=\phi_1(x,y) & \text{for } (x,y)\in\Omega \end{cases} \tag{3.1}$$

The conserved energy of (3.1) is given explicitly as

$$E(u)=\int_\Omega \frac{1}{2}\,u_t^2\,dxdy+ \tag{3.2}$$

$$+\int_\Omega \left(\frac{\gamma}{2}(\Delta u)^2+\gamma(1-\sigma)(u_{xy}^2-u_{xx}u_{yy})+\Upsilon(y)\left(\frac{u^2}{2}+\frac{u^4}{4}\right)\right)\,dxdy\ .$$

The above functional is well-defined in

$$H^2_*(\Omega):=\big\{w\in H^2(\Omega)\,:\,w=0\text{ on }\{0,L\}\times(-\ell,\ell)\big\}$$

which is a Hilbert space when endowed with the scalar product

$$(u,v)_{H^2_*}:=\int_\Omega[\Delta u\Delta v+(1-\sigma)(2u_{xy}v_{xy}-u_{xx}v_{yy}-u_{yy}v_{xx})]\,dxdy\ .$$

Let $H^{-2}_*(\Omega)$ denotes the dual space of $H^2_*(\Omega)$, we say that

$$u\in C^0(\mathbb{R}_+;H^2_*(\Omega))\cap C^1(\mathbb{R}_+;L^2(\Omega))\cap C^2(\mathbb{R}_+;H^{-2}_*(\Omega))$$

is a solution of (3.1) if it satisfies the initial conditions and if

$$\langle u''(t),v\rangle+\gamma(u(t),v)_{H^2_*}+(h(y,u(t)),v)_{L^2}=0\qquad\forall\,v\in H^2_*(\Omega),\ \forall\,t\in(0,T)\ .$$

Then, from [7] and [13, Theorem 3.6] we recall

Theorem 3.1. [7, Theorem 4.1]. *Let $\phi_0\in H^2_*(\Omega)$ and $\phi_1\in L^2(\Omega)$. Then there exists a unique solution $u=u(t)$ of* (3.1) *and its energy* (3.2) *satisfies*

$$E(u(t))\equiv\int_\Omega\frac{1}{2}\,\phi_1^2\,dxdy+$$

$$+\int_\Omega\left(\frac{\gamma}{2}(\Delta\phi_0)^2+\gamma(1-\sigma)((\phi_0)_{xy}^2-(\phi_0)_{xx}(\phi_0)_{yy})+\Upsilon(y)\left(\frac{\phi_0^2}{2}+\frac{\phi_0^4}{4}\right)\right)\,dxdy\ .$$

The proof is achieved by performing a suitable Galerkin method, namely by projecting the original problem on the finite dimensional system spanned by the eigenfunctions of the following biharmonic eigenvalue problem:

$$\begin{cases}\Delta^2u=\lambda u & \text{in }\Omega\\ u(0,y)=u_{xx}(0,y)= & \\ =u(\pi,y)=u_{xx}(\pi,y)=0 & \text{for }y\in(-\ell,\ell)\\ u_{yy}(x,\pm\ell)+\sigma u_{xx}(x,\pm\ell)= & \\ =u_{yyy}(x,\pm\ell)+(2-\sigma)u_{xxy}(x,\pm\ell)=0 & \text{for }x\in(0,\pi)\ .\end{cases}\tag{3.3}$$

A nontrivial function $u\in H^2_*(\Omega)$ is an eigenfunction of (3.3) if there exists $\lambda\in\mathbb{R}$ such that

$$\int_\Omega[\Delta u\Delta v+(1-\sigma)(2u_{xy}v_{xy}-u_{xx}v_{yy}-u_{yy}v_{xx})-\lambda uv]\,dxdy=0$$

for all $v\in H^2_*(\Omega)$.

From [13, Theorem 7.6], the set of eigenvalues of (3.3) may be ordered in an increasing sequence $\{\lambda_j\}$ of strictly positive numbers diverging to $+\infty$ and any eigenfunction belongs to $C^\infty(\overline{\Omega})$. Furthermore, the set of eigenfunctions of (3.3) is a complete system in $H^2_*(\Omega)$. Much more important from our point of view is that the set of eigenfunctions (given explicitly in [13]) may be divided into two families:

(3.4) - **longitudinal** eigenfunctions:

$$w_k(x)\sim c\sin(mx)\text{ as }y\to0\quad\text{with eigenvalues }\lambda=\mu_k\ ,$$

(3.5) - **torsional** eigenfunctions :

$$v_l(x) \sim cy\sin(mx) \text{ as } y \to 0 \quad \text{with eigenvalues } \lambda = \nu_l \ ,$$

for some $m, k, l \in \mathbb{N}_+$ and some constant c. The eigenvalues μ_k, ν_l solve very complicated equations, nevertheless they that can be numerically evaluated. See [13] and [5] where the dependence of the eigenvalue from the width of the plate and from the shape of the deck has been investigated.

3.2. **Stability analysis.** As already explained, the proof of Theorem 3.1 makes use of a Galerkin method. The solution of (3.1) is the limit (in a suitable topology) of a sequence of solutions of approximated problems in finite dimensional spaces. Consider an orthogonal complete system $\{u_k\}_{k\geq 1} \subset H^2_*(\Omega)$ of eigenfunctions (either of type (3.4) or (3.5)) of (3.3) such that $\|u_k\|_{L^2} = 1$ and let

$$u^m(t) = \sum_{i=1}^{m} g_i^m(t)u_i \quad \text{and} \quad g^m(t) := (g_1^m(t), \cdots, g_m^m(t))^T$$

then the vector valued function g^m solves

$$\begin{cases} (g^m(t))'' + \gamma\,\Lambda_m g^m(t) + \Phi_m(g^m(t)) = 0 \quad \forall\, t \in (0,T) \\ g^m(0) = ((\phi_0, u_1)_{L^2}, \cdots, (\phi_0, u_m)_{L^2})^T, \\ (g^m)'(0) = ((\phi_1, u_1)_{L^2}, \cdots, (\phi_1, u_m)_{L^2})^T \end{cases} \tag{3.6}$$

where $\Lambda_m := \mathrm{diag}(\lambda_1, \cdots, \lambda_m)$ and $\Phi_m : \mathbb{R}^m \to \mathbb{R}^m$ is the map defined by

$$\Phi_m(\xi_1, \cdots, \xi_m) := \left(\left(h\left(y, \sum_{j=1}^{m} \xi_j u_j \right), u_1 \right)_{L^2}, \cdots, \left(h\left(y, \sum_{j=1}^{m} \xi_j u_j \right), u_m \right)_{L^2} \right)^T .$$

The proof of Theorem 3.1 follows by showing that $u^m \to u$ in $C^0([0,T]; H^2_*(\Omega)) \cap C^1([0,T]; L^2(\Omega))$, see [13] for the details. Hence, in this sense, the u^m approximate u and we are allowed to perform our stability analysis on the finite dimensional system (3.6). We refer to [6] for a precise estimate of the error committed when replacing u with u^m. Before starting the stability analysis, we need to define what we mean by longitudinal modes. This is a classical definition in a linear regime while it is by no means standard how to characterize modes in nonlinear regimes; contrary to the linear case, the frequency of a nonlinear mode depends on the energy or, equivalently, on the amplitude of its oscillations. Let μ_k be as in (3.4), namely an eigenvalue corresponding to a longitudinal eigenfunction. We select the corresponding longitudinal mode simply by setting equal to zero all the other components in (3.6). More precisely, after a couple of straightforward computations, we obtain problem (3.7) below.

Definition 3.2. Let μ_k be as in (3.4) and w_k be the corresponding eigenfunction. We call *k-th longitudinal mode* at energy $E(\phi_0^k, \phi_1^k) > 0$ the unique (periodic) solution $\bar{\varphi}_k$ of the Cauchy problem:

$$\begin{cases} \bar{\varphi}_k''(t) + (\gamma\mu_k + a_k)\bar{\varphi}_k(t) + b_k\bar{\varphi}_k^3(t) = 0 \quad \forall\, t > 0 \\ \bar{\varphi}_k(0) = \phi_0^k, \ \ \bar{\varphi}_k'(0) = \phi_1^k \ , \end{cases} \tag{3.7}$$

where $a_k := \int_\Omega \Upsilon(y)\, w_k^2(x,y)\, dxdy$ and $b_k := \int_\Omega \Upsilon(y)\, w_k^4(x,y)\, dxdy$. Furthermore, system (3.7) admits the conserved quantity

$$E = \frac{(\bar\varphi_k')^2}{2} + (\gamma\mu_k + a_k)\frac{\bar\varphi_k^2}{2} + b_k\, \frac{\bar\varphi_k^4}{4} \equiv \tag{3.8}$$

$$\equiv E(\phi_0^k, \phi_1^k) = \frac{(\phi_1^k)^2}{2} + (\gamma\mu_k + a_k)\frac{(\phi_0^k)^2}{2} + b_k\, \frac{(\phi_0^k)^4}{4}\ .$$

In order to study the torsional stability of a longitudinal mode $\bar\varphi_k$ with respect to a chosen torsional component, we linearize the equation of system (3.6) corresponding to a torsional eigenvalue ν_l as in (3.5) around $(0, \cdots, \bar\varphi_k, \cdots, 0) \in \mathbb{R}^m$. In each case we obtain a **Hill equation** of the type

$$\xi''(t) + A_{l,k}(t)\xi(t) = 0\ , \tag{3.9}$$

where

$$A_{l,k}(t) = \gamma\nu_l + \bar a_l + d_{l,k}\overline{\varphi}_k^2(t) \tag{3.10}$$

with $\bar a_l := \int_\Omega \Upsilon(y)\, v_l^2(x,y)\, dxdy$, $d_{l,k} := \int_\Omega \Upsilon(y)\, w_k^2(x,y) v_l^2(x,y)\, dxdy$, w_k and v_l are the eigenfunctions corresponding to μ_k and ν_l. As already explained in Section 2, the notion of stability to (3.9) is standard, hence the above procedure enables us to define the torsional stability of a longitudinal mode.

Definition 3.3. We say that the k-th longitudinal mode $\bar\varphi_k$ at energy $E(\phi_0^k, \phi_1^k)$, namely the unique periodic solution of (3.7), is *stable with respect to the l-th torsional mode* if the trivial solution of the Hill equation (3.9) is stable.

We are so ready to exploit the stability theory for the Hill equation outlined in the previous section. More precisely, by applying Corollary 2.1 we prove

Theorem 3.4 ([7, Theorem 4.4])**.** *Let μ_k and ν_l be as in* (3.4) *and* (3.5)*. Assume furthermore that*

$$\sqrt{\frac{\gamma\nu_l + \bar a_l}{\gamma\mu_k + a_k}} \notin \mathbb{N} \tag{3.11}$$

with a_k and $\bar a_l$ as above. Then there exists $E_k^l > 0$ and a strictly increasing function Λ such that $\Lambda(0) = 0$ and such that the k-th longitudinal mode $\overline{\varphi}_k$ at energy $E(\phi_0^k, \phi_1^k)$ (that is, the solution of (3.7)*) is stable with respect to the l-th torsional mode provided that*

$$E \le E_k^l\ .$$

The sketch of the proof of Theorem 3.4 is postponed to Section 4. We note that this is not a perturbation result and the critical thresholds E_k^l are explicitly given. As highlighted in the previous section, the stability analysis of Hill equation has to deal with with the difficulty to determine precisely the boundaries of the stability domains, see also [10]. For (3.9), the usual difficulties are further increased by the fact that instead of a single equation, it represents a family of Hill equations having coefficients with periods depending on the energy of the solution of (3.7). In this respect, condition (3.11) is essentially a technical condition related to the particular stability test applied in the proof furthermore it has probability 1 to occur among all random choices of the positive real numbers involved. We refer to [13] for an alternative sufficient condition for torsional stability when (3.11) fails.

Mechanical interpretation of the results

Theorem 3.4 provides only a sufficient condition for stability. Numerical experiments confirm the existence of an **energy threshold** where stability is lost, see Figure 2. Notice that all the numerical experiments performed in [13] yield qualitatively similar results.

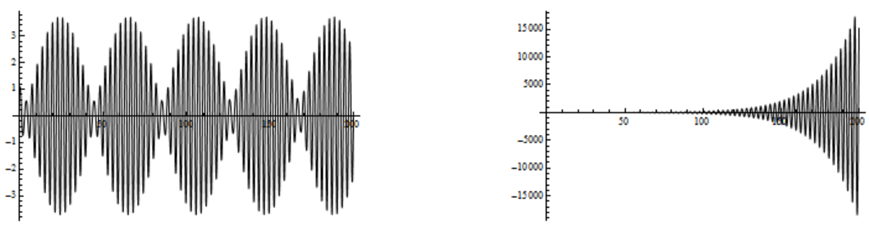

FIGURE 2. Plot of the solution to (3.9) when $k = 14$, $l = 1$ and all the data are fixed according to the Tacoma parameters. The energy is concentrated on the 14−th longitudinal mode and the following initial data are taken: $\bar{\varphi}_{14}(0) = 0.79$ (left) and 0.8 (right), $\bar{\varphi}'_{14}(0) = 0$ and $\xi_1(0) = \xi'_1(0) = 1$. Passing from the left to the right the amplitude of the oscillations (suddenly) increases of a factor 10000 and this testifies the occurred energy transfer from the 14−th longitudinal mode to the 1−st torsional mode.

Summarizing, theoretical and numerical results suggest the following explanations of the origin of torsional instability: *it is due to an* **internal resonance** *which generates an energy transfer between different oscillation modes. When the bridge is oscillating* **longitudinally** *with a suitable amplitude, part of the energy is suddenly transferred to a torsional mode giving rise to wide* **torsional** *oscillations.*

Also we note that from the critical energies computed in [13], taking into account the Tacoma parameters, it follows that the **tenth** longitudinal mode (i.e. when the motion is concentrated on the eigenfunctions behaving like $c\sin(10x)$ as $y \to 0$) seems to be the **most prone** to generate the **second torsional** mode (i.e when the motion is concentrated on the eigenfunctions behaving like $cy\sin(2x)$ as $y \to 0$). This result is in complete accordance to what observed in the federal report [1, V-10]: Farquharson witnessed the collapse and wrote that *the motions, which a moment before had involved a number of waves* (*nine or* **ten**) *had shifted almost instantly to* **two**.

3.3. **A criterion governing the energy transfer.** The analysis of the previous section is performed for small energies. One may wonder what happens for large energies and why the energy initially concentrated on a selected longitudinal modes decides to move to a suitable torsional mode instead to another. Exploiting the theory of Hill equation, in [8] we give a detailed explanation of how the stability is lost for the selected longitudinal mode and which torsional mode first captures its energy. Here we limit ourselves to briefly recall the idea behind the proofs and to explain where the above mentioned theory turns out to be of crucial utility. More precisely, we consider the following simple prototype problem like (3.6) but with

$m = 3$:

$$\begin{cases} \varphi'' + \mu^2\varphi + U_\varphi(\varphi, z_1, z_2) = 0 & \varphi(0) = x_0,\ \dot\varphi(0) = 0 \\ z_1'' + \nu_1^2 z_1 + U_{z_1}(\varphi, z_1, z_2) = 0 & z_1(0) = \varepsilon x_0,\ \dot z_1(0) = 0 \\ z_2'' + \nu_2^2 z_2 + U_{z_2}(\varphi, z_1, z_2) = 0 & z_2(0) = \varepsilon x_0,\ \dot z_2(0) = 0\ , \end{cases} \tag{3.12}$$

where μ, ν_1, ν_2 are positive real numbers, $x_0 \in \mathbb{R} \setminus \{0\}$ and $\varepsilon > 0$. As mentioned in [27, Section 10.7.1], the analysis with more than 2 degrees of freedom is much more complicated. Hence, it seems reasonable to start with $m = 3$. Then, we choose a potential U in such a way that the linearized problem becomes a system of Mathieu equations [19]. Namely, we take

$$U(\varphi, z_1, z_2) = \frac{\varphi^2 z_1^2 + \varphi^2 z_2^2 + z_1^2 z_2^2}{2}\ . \tag{3.13}$$

The advantage of this choice is that much more precise information is known on the behavior of the stability regions. Furthermore, due to the symmetry of the potential chosen, no z_l components is initially favored by the nonlinearity and the relevant parameters become the frequencies. Notice that by [2] we know that several different choices of U yield a similar response in the bridge.

Basically, in (3.12), φ is the selected longitudinal mode and z_l the torsional modes of the previous section and we are interested to detect which of the z_l takes the energy initially concentrated on φ (recall Figure 2). For $\varepsilon = 0$ (and $x_0 \neq 0$), system (3.12) admits the unique solution $(\bar\varphi, 0, 0) = (x_0 \cos(\mu t), 0, 0)$ and the conserved energy

$$E := \frac{(\bar\varphi')^2}{2} + \frac{\mu^2}{2}\bar\varphi^2 = \frac{\mu^2}{2}x_0^2\ . \tag{3.14}$$

As did for system (3.6), we linearize the z_l equations of (3.12) around this solution and we obtain the following system of Mathieu equations

$$\begin{cases} \xi_1'' + \left(\nu_1^2 + \frac{x_0^2}{2} + \frac{x_0^2}{2}\cos(2\mu t)\right)\xi_1 = 0 \\ \xi_2'' + \left(\nu_2^2 + \frac{x_0^2}{2} + \frac{x_0^2}{2}\cos(2\mu t)\right)\xi_2 = 0\ . \end{cases} \tag{3.15}$$

By a change of variables (without renaming the ξ_l's), we may rewrite the equations in (3.15) in the canonical form:

$$\xi_l'' + (\alpha_l + 2q_l \cos(2t))\,\xi_l = 0 \quad , \quad \text{for } l = 1, 2\ , \tag{3.16}$$

with

$$\alpha_l(x_0) = \frac{2\nu_l^2 + x_0^2}{2\mu^2} \quad \text{and} \quad q_l(x_0) = q(x_0) = \frac{x_0^2}{4\mu^2} \quad \text{for } l = 1, 2\ , \tag{3.17}$$

so that

$$\alpha_l(q) = \frac{\nu_l^2}{\mu^2} + 2q\ . \tag{3.18}$$

Definition 3.5. We say that the energy E in (3.13) is *activating for the torsional mode* z_l ($l = 1$ or $l = 2$) of system (3.12) if the trivial solution $\xi_l \equiv 0$ of the Mathieu equation (3.16) is unstable. Otherwise, we say that it is non-activating.

We have already recalled in Section 2 that, given $q > 0$, to each Mathieu equation we may associate a countably infinite set of the so-called Mathieu characteristic values $\{a_n(q)\}_{n\geq 0}$ and $\{b_n(q)\}_{n\geq 1}$. The characteristic curves do not intersect and divide the (q, a)-plane into stable and unstable regions $\mathcal{O}_n$ and $\mathcal{H}_n$, see Figure 1. Then, to each couple (μ, ν_l) in system (3.12) we may associate a sequence of energies $\{E^l_m\}_{m=0}^{\infty}$ as follows. The energy associated to $(\bar{\varphi}, 0, 0)$ satisfies (3.14), that is

$$E = \frac{\mu^2 x_0^2}{2} = 2\mu^4 q \ ,$$

where the second equality is due to (3.17). Hence, as E increases from $E = 0$ to $E = \infty$ the parameters $(q(x_0), \alpha_l(x_0))$ in (3.16) move along the line (3.18) in the (q, a)-plane, according to the law (3.17), see the stright lines in Figure 3. By the asymptotics (2.4) and (2.5), we infer that the straight line (3.18) intersects at least once each characteristic curve a_n and b_n provided that $n > \nu_l/\mu$; moreover, at each crossing, the line moves from some $\mathcal{H}_n$ to $\mathcal{O}_n$ or from some $\mathcal{O}_n$ to $\mathcal{H}_{n+1}$, thereby alternating its intersection with shaded and white regions in Figure 3. From this construction a sequence $\{E^l_m\}_{m=0}^{\infty}$ is naturally associated to any couple (μ, ν_l) by intersecting the straight lines (3.18) with the characteristic curves. Since the stability of the trivial solution $\xi_l \equiv 0$ of (3.16) depends on the position of (q, α_l) in the Mathieu diagram, the following statement readily follows. See [8] for the details.

Theorem 3.6 ([8])**.** *Let $\mu, \nu_1, \nu_2 > 0$ and $x_0 \in \mathbb{R} \setminus \{0\}$. Let $E > 0$ be the energy (3.13) associated to the solution $(\bar{\varphi}, 0, 0) = (x_0 \cos(\mu t), 0, 0)$ to system (3.12) for $\varepsilon = 0$. For each $l = 1, 2$ there exists an increasing divergent sequence $\{E^l_m\}_{m=0}^{\infty}$ such that $E^l_0 = 0$ and*

(i) E is non-activating whenever $E \in (E^l_{2k}, E^l_{2k+1})$ for some $k \geq 0$;
(ii) E is activating whenever $E \in (E^l_{2k+1}, E^l_{2k+2})$ for some $k \geq 0$.

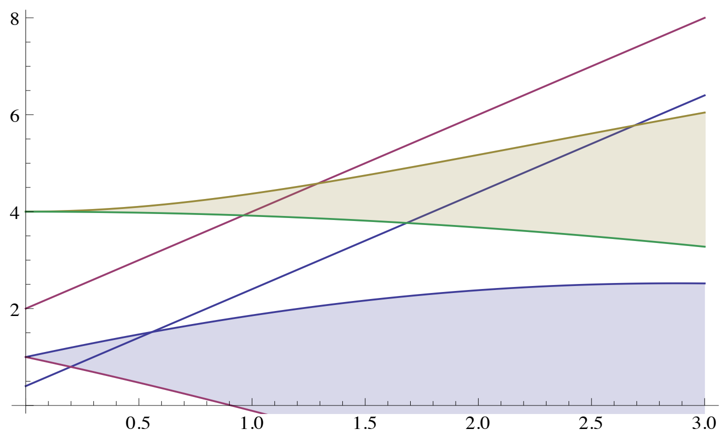

FIGURE 3. How to compute the energy sequences $\{E^l_m\}_{m=0}^{\infty}$: we intersect the boundaries of the instability regions (shaded) in the (q, a)-plane with the parametric lines (3.18) associated to each torsional mode.

By combining Theorem 3.6 with Definition 3.5, we obtain the following theoretical criterion to determine which torsional mode captures the energy of the longitudinal mode $\bar{\varphi}$:

Corollary 3.7. *Let $E > 0$ be the energy* (3.14) *of system* (3.12) *associated to the solution $(\bar{\varphi}, 0, 0)$ for $\varepsilon = 0$. If $E \in (E^l_{2k+1}, E^l_{2k+2})$ for some $k \geq 0$ and for $l = 1$ or $l = 2$, then the torsional mode z_l captures the energy of the longitudinal mode φ.*

We complement the statement of Corollary 3.7 with some conclusions drawn from the numerical experiments collected in [8]:

(i) As long as the two couples of parameters $(q(x_0), \alpha_l(x_0))$ lie in the stability region of the Mathieu diagram, the solution $(\bar{\varphi}, 0, 0)$ is stable.
(ii) When a couple $(q(x_0), \alpha_l(x_0))$ lies in an instability region and is sufficiently far from the stability region, the corresponding residual modes become fairly large.
(iii) When the parametric lines cross a thin instability region, only small torsional oscillations appear and the bridge basically remains stable.

Mechanical interpretation of the results

The results so far stated yield the following conclusion: the **torsional stability** of a suspension bridge depends on a combination between the given energy of the system and the **ratios between torsional and longitudinal frequencies** ν_l^2/μ^2. Furthermore, since the instability regions in the Mathieu diagram become more narrow as a increases and the parametric lines take their origin when $a = \nu_l^2/\mu^2$, it would be desirable that $\nu_l >> \mu$. Namely, the larger these ratios are, more stable is the bridge.

4. Sketch of some proofs

4.1. **Proof of Theorem 2.2.** Let $\mathbf{x}(t) = F(t)\mathbf{y}(t)$ and $\mathbf{y}(t) = e^{tK}\mathbf{x}(0)$. Let $\varphi_{\mathbf{x}}$ denote the rotation of the vector solution $\mathbf{x} = \mathbf{x}(t) \neq 0$ of equation (2.1) in time T, in general $\varphi_{\mathbf{x}} = n\pi + \varphi_{\mathbf{y}}$ for some $n = 0, 1, 2, \cdots$. Assume that $H(t) \in \mathcal{O}_n$, from the definition of $\widetilde{\mathcal{O}}_+$ it is readily deduced that the eigenvalues of the corresponding matrix K are $\lambda_i = \pm i\beta$ with $0 < \beta < \pi/T$. Hence, $0 < \varphi_{\mathbf{y}} < \pi$ and $n\pi < \varphi_{\mathbf{x}} < (n+1)\pi$. On the other hand, if $H(t) \in \Pi_n^{**}$, then $e^{tK} = Id$ and $\varphi_{\mathbf{x}} = n\pi$. More precisely, there holds

Proposition 4.1 ([30, Chapter VIII, Section 1])**.** *Let $\varphi_{\mathbf{x}}$ denote the rotation of the vector solution $\mathbf{x} = \mathbf{x}(t) \neq 0$ of equation* (2.1) *in time T, then*

(1) *$H(t) \in \mathcal{O}_n$ if and only if $n\pi < \varphi_{\mathbf{x}} < (n+1)\pi$;*
(2) *$H(t) \in \Pi_n^{**}$ if and only if $\varphi_{\mathbf{x}} = n\pi$.*

Besides Proposition 4.1, the key ingredient in the proof of Theorem 2.2 is the following proposition

Proposition 4.2 ([30, Chapter VIII, Section 3.3])**.** *Let r be an arbitrary function satisfying: r'' exists, it is Lebesgue summable and*

$$r(t+T) = r(t) > 0 \quad , \quad \int_0^T r^{-2}(t)\, dt = 1 \, .$$

Then, the general form of a function $p(t)$ such that $H \in \Pi_n^{**}$ for some $n = 0, 1, 2, \cdots$ is

$$p(t) = \frac{n^2\pi^2}{r^4} - \frac{r''}{r} \,. \tag{4.1}$$

Proof. Let $X(t) = \begin{bmatrix} y_1' & y_2' \\ y_1 & y_2 \end{bmatrix}$ be the fundamental matrix of (2.1). Recall that $X(0) = Id$ and $\det X(t)$=1 (namely, y_1 and y_2 are two linearly independent solutions of the associated Hill equation). If $H \in \Pi_n^{**}$, from Theorem 2.2 and (2.1) the vector $\mathbf{y}$ rotates of an angle $-n\pi$ in time T and $\mathbf{y}(t+T) = \mathbf{y}(T)$, for some $n = 0, 1, 2, \cdots$. Let $\theta_{\mathbf{y}}$ be a continuous branch of the argument of $\mathbf{y}$, we have $\theta_{\mathbf{y}} = \arctan(y_2/y_1)$ and $(d/dt)\theta_{\mathbf{y}} = -1/\left(y_1^2 + y_2^2\right)$. Hence, $\int_0^T 1/\left(y_1^2 + y_2^2\right) = n\pi$. Set $r(t) = \sqrt{n\pi(y_1^2 + y_2^2)}$, clearly $r(t)$ satisfies the assumptions of Proposition 4.2. Furthermore, by writing $y_1(t) = (r/\sqrt{n\pi}\,)\cos\left(n\pi \int_0^t 1/r^2(s)\,ds\right)$, it is readily deduced that $-y_1''/y_1 = n^2\pi^2/r_1^4 - r''/r$ by which (4.1) follows.

Conversely, if $p(t)$ is as defined in (4.1) arguing as above one has that $y_1(t) = (r/\sqrt{n\pi})\cos(n\pi \int_0^t 1/r^2(s)\,ds)$ and $y_2(t) = (r/\sqrt{n\pi})\,\sin(n\pi \int_0^t 1/r^2(s)\,ds)$ are solutions of the corresponding Hill equation. By the assumptions on r, arguing as above, it follows that $\varphi_{\mathbf{x}} = n\pi$ for some $n = 0, 1, 2, \cdots$, hence the corresponding fundamental matrix is such that $H \in \Pi_n^{**}$.

□

The proof of Theorem 2.2 follows from what so far stated once recalled the Comparison Theorem below the proof of which follows by studying the dependence of system (2.1) from a parameter and we omit it.

Theorem 4.3 ([30, Chapter VIII, Theorem I]). *Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solutions to (2.1) with Hamiltonians $H_1(t)$ and $H_2(t)$ satisfying $\mathbf{x}_1(0) = \mathbf{x}_2(0) \neq 0$ and let $\theta_{\mathbf{x}_i(t)}$ be a continuos branch of the argument of $\mathbf{x}_i$, if $(H_1(t)\mathbf{x}, \mathbf{x}) \geq (H_2(t)\mathbf{x}, \mathbf{x})$ for all vectors $\mathbf{x}$ and almost everywhere in $[0, T]$ then $\theta_{\mathbf{x}_1(t)} \geq \theta_{\mathbf{x}_2(t)}$. If $(H_1(t)\mathbf{x}, \mathbf{x}) > (H_2(t)\mathbf{x}, \mathbf{x})$ for all vectors $\mathbf{x}$ in a set of positive measure in the interval $(0, t_0)$ then $\theta_{\mathbf{x}_1(t)} > \theta_{\mathbf{x}_2(t)}$ for $t \geq t_0$.*

Proof of Theorem 2.2 completed. For $i = 1, 2$, let $H_i(t)$ be the Hamiltonian corresponding to $p_i(t) = n^2\pi^2/r_i^4 - r_i''/r_i$ with r_i as given in the statement of Theorem 2.2. From Proposition 4.2 it follows that $H_1(t) \in \Pi_n^{**}$ and $H_2(t) \in \Pi_{n+1}^{**}$. By assumption, $(H_1(t)\mathbf{x}, \mathbf{x}) \leq (H(t)\mathbf{x}, \mathbf{x}) \leq (H_2(t)\mathbf{x}, \mathbf{x})$ for all vectors $\mathbf{x}$ and almost everywhere in $[0, T]$ and each of these inequalities is strict on some set of positive measure. Let $\mathbf{x}, \mathbf{x}_1$ and $\mathbf{x}_2$ be solutions to (2.1) with Hamiltonians $H(t), H_1(t)$ and $H_2(t)$ satisfying the same initial condition $\mathbf{x}(0) = \mathbf{a}$. By Theorem 4.3, $\varphi_{\mathbf{x}_1} < \varphi_{\mathbf{x}} < \varphi_{\mathbf{x}_2}$. Since $H_1(t) \in \Pi_n^{**}$ and $H_2(t) \in \Pi_{n+1}^{**}$, from Proposition 4.1 it follows that $n\pi < \varphi_{\mathbf{x}} < (n+1)\pi$ and $H(t) \in \mathcal{O}_n$. This concludes the proof.

□

4.2. **Proof of Theorem 3.4.** A couple of preliminary remarks are needed. Set $\delta_l := \gamma\nu_l + \bar{a}_l$, $\rho_k := \gamma\mu_k + a_k$ and, for any $E > 0$,

$$\Lambda_{\pm}^k(E) = \frac{\sqrt{\rho_k^2 + 4b_kE} \pm \rho_k}{b_k} \,,$$

then (3.8) reads

$$(\overline{\varphi}_k')^2 = \frac{b_k}{2}(\Lambda_+^k(E) + \overline{\varphi}_k^2)(\Lambda_-^k(E) - \overline{\varphi}_k^2) .$$

Hence,

$$\|\overline{\varphi}_k\|_\infty = \sqrt{\Lambda_-^k(E)}$$

and, by standard ode's arguments, the period $T_k(E)$ of $\overline{\varphi}_k$ is explicitly given by

$$T_k(E) = \frac{4\sqrt{2}}{\sqrt{b_k}} \int_0^1 \frac{ds}{\sqrt{(\Lambda_+^k(E) + \Lambda_-^k(E)s^2)(1-s^2)}} .$$

In particular,

(4.2) the map $E \mapsto T_k(E)$ is strictly decreasing and

$$\lim_{E\to 0} T_k(E) = T_k(0) = \frac{2\pi}{\sqrt{\rho_k}} .$$

Since the function $\overline{\varphi}_k^2$ is $T_k/2$-periodic, then $A_{l,k}(t)$ is a positive $T_k/2$-periodic function and we may apply the Zhukovskii's stability criterion given in Corollary 2.1. Namely, the trivial solution of (3.9) is stable provided that there exists an integer $n \geq 0$ such that

$$\frac{4n^2\pi^2}{T_k(E)^2} \leq A_{l,k}(t) \leq \frac{4(n+1)^2\pi^2}{T_k(E)^2} \qquad \forall\, t \in \mathbb{R} . \tag{4.3}$$

Remembering the definition of $A_{l,k}(t)$ and what previously remarked, the above condition follows if

$$\frac{4n^2\pi^2}{\delta_l} \leq T_k(E)^2 \leq \frac{4(n+1)^2\pi^2}{\delta_l + d_{l,k}\Lambda_-^k(E)}$$

Let

$$n = \max\left\{ i \in \mathbb{N};\ i < \sqrt{\frac{\delta_l}{\rho_k}} \right\} \tag{4.4}$$

so that, by (4.2), $\frac{4n^2\pi^2}{\delta_l} < T_k(0)^2$. We then infer that there exists $E_1(l,k) > 0$ such that

$$\frac{4n^2\pi^2}{\delta_l} \leq T_k(E)^2 \qquad \forall\, E \leq E_1(l,k) . \tag{4.5}$$

In view of (4.4) and of the assumption (3.11) we know that

$$T_k(0)^2 = \frac{4\pi^2}{\rho_k} < \frac{4(n+1)^2\pi^2}{\delta_l} = \frac{4(n+1)^2\pi^2}{\delta_l + d_{l,k}\Lambda_-^k(0)} .$$

Then, the continuity of the maps $E \mapsto T_k(E)$ and $E \mapsto \Lambda_-^k(E)$ implies that there exists $E_2(l,k) > 0$ such that

$$T_k(E)^2 \leq \frac{4(n+1)^2\pi^2}{\delta_l + d_{l,k}\Lambda_-^k(E)} \qquad \forall\, E \leq E_2(l,k) . \tag{4.6}$$

Let now n as in (4.4), $E_1(l,k)$ and $E_2(l,k)$ as in (4.5)-(4.6), and

$$E \leq E_k^l := \min\{E_1(l,k), E_2(l,k)\} .$$

Then, (4.3) is satisfied and $\overline{\varphi}_k$ is stable with respect to the l-th torsional mode.

References

[1] O.H. Ammann, T. von Kármán & G.B. Woodruff, *The failure of the Tacoma Narrows Bridge*, Federal Works Agency, Washington D.C., 1941.

[2] G. Arioli & F. Gazzola, *A new mathematical explanation of what triggered the catastrophic torsional mode of the Tacoma Narrows Bridge collapse*, Appl. Math. Modelling, 39(2015), 901–912.

[3] D. Baccarin, *Is internal parametric resonance a potential failure mode for suspension bridges?*, Master Thesis in Civil Engineering, Politecnico di Milano, Italy (2013).

[4] G. Bartoli & P. Spinelli, *The stochastic differential calculus for the determination of structural response under wind*, J. Wind Engineering and Industrial Aerodynamics, 48(1993), 175–188.

[5] E. Berchio, D. Buoso & F. Gazzola, *On the variation of longitudinal and torsional frequencies in a partially hinged rectangular plate*, Preprint

[6] E. Berchio & F. Gazzola, *A qualitative explanation of the origin of torsional instability in suspension bridges*, Nonlin. Anal. TMA, 121(2015), 54–72.

[7] E. Berchio, A. Ferrero & F. Gazzola, *Structural instability of nonlinear plates modelling suspension bridges: mathematical answers to some long-standing questions*, Nonlin. Anal. Real World Appl., 28(2016), 91–125.

[8] E. Berchio, F. Gazzola & C. Zanini, *Which residual mode captures the energy of the dominating mode in second order Hamiltonian systems?*, SIAM J. Appl. Dyn. Syst., 15(1)(2016), 338–355.

[9] P. Bergot & L. Civati, *Dynamic structural instability in suspension bridges*, Master Thesis in Civil Engineering, Politecnico di Milano, Italy (2014).

[10] H. Broer & M. Levi, *Geometrical aspects of stability theory for Hill's equations*, Arch. Rational Mech. Anal., 131(1995). 225–240.

[11] C. Chicone, *Ordinary differential equations with applications*, (2nd Edition) Springer 2006.

[12] M. Como, S. Del Ferraro & A. Grimaldi, *A parametric analysis of the flutter instability for long span suspension bridges*, Wind and Structures, 8(2005), 1–12.

[13] A. Ferrero & F. Gazzola, *A partially hinged rectangular plate as a model for suspension bridges*, Disc. Cont. Dyn. Syst. A., 35(2015), 5879–5908.

[14] F. Gazzola, *Mathematical models for suspension bridges*, MS & A, Vol.15, Springer, 2015.

[15] G.W. Hill, *On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and the moon*, Acta Math., 8(1886), 1–36.

[16] H.M. Irvine, *Cable structures*, MIT Press Series in Structural Mechanics, Massachusetts, 1981.

[17] J.A. Jurado, S. Hernandez, F. Nieto & A. Mosquera, *Bridge aeroelasticity: sensitivity analysis and optimum design (high performance structures and materials)*, WIT Press - Computational Mechanics, Southampton, 2011.

[18] W. Magnus & S. Winkler, *Hill's equation*, Dover, New York, 1979.

[19] E. Mathieu, *Mémoire sur le mouvement vibratoire d'une membrane de forme elliptique*, J. Math. Pure Appl., 13(1868), 137–203.

[20] M. Matsumoto, H. Matsumiya, S. Fujiwara & Y. Ito, *New consideration on flutter properties based on step-by-step analysis*, Wind Engineering Indust. Aerodynamics, 98(2010), 429–437.

[21] P.J. McKenna, *Torsional oscillations in suspension bridges revisited: fixing an old approximation*, Amer. Math. Monthly, 106(1999), 1–18.

[22] P.J. McKenna, *Oscillations in suspension bridges, vertical and torsional*, Disc. Cont. Dynam. System S, 7(2014), 785–791.

[23] P.J. McKenna & C.Ó Tuama, *Large torsional oscillations in suspension bridges visited again: vertical forcing creates torsional response*, Amer. Math. Monthly, 108(2001), 738–745.

[24] N.W. McLachlan, *Theory and application of Mathieu functions*, Dover Publications, Inc., New York, 1964.

[25] R.H. Plaut & F.M. Davis, *Sudden lateral asymmetry and torsional oscillations of section models of suspension bridges*, J. Sound and Vibration, 307(2007), 894–905.

[26] H. Poincaré, *Introduction to the collected mathematical works of George William Hill*, Carnegie Institution of Washington, Vol. I, 1905, pp.vii-xviii.

[27] J.A. Sanders, F. Verhulst & J. Murdock, *Averaging methods in nonlinear dynamical systems*, 2nd Ed. Applied Mathematical Sciences 59, Springer, New York, 2007.

[28] J.J. Stoker, *Nonlinear vibrations in mechanical and electrical systems*, John Wiley and Sons, New York, 1950.

[29] F. Verhulst, *Perturbation analysis of parametric resonance*, *Encyclopedia of Complexity and Systems Science*, Springer, 2009, 6625–6639.

[30] V.A. Yakubovich & V.M. Starzhinskii, *Linear differential equations with periodic coefficients*, J. Wiley & sons, New York, 1975; (Russian original in Izdat. Nauka, Moscow, 1972).