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Around BIons

Pietro d'Avenia[1]

Abstract. We give a survey of the results obtained in [5] for the equation

$$(\mathcal{BI}) \qquad \begin{cases} -\operatorname{div}\left(\dfrac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right) = \rho \quad \text{in } \mathbb{R}^N \ , \\ \lim\limits_{|x|\to\infty} \phi(x) = 0 \ . \end{cases}$$

We analyze also other models involving the same operator in $(\mathcal{BI})$, as, for instance, the nonlinear Klein-Gordon-Born-Infeld system.

1. Introduction

In the classical Maxwell theory, the electric field $\mathbf{E}$, generated by a charge density ρ in the purely electrostatic case, satisfies the equations

$$\nabla \times \mathbf{E} = 0 \quad , \quad \text{in } \mathbb{R}^3 \ , \tag{1.1}$$

$$\operatorname{div} \mathbf{E} = \rho/\varepsilon_0 \quad , \quad \text{in } \mathbb{R}^3 \ , \tag{1.2}$$

where ε_0 is the permittivity of free space. Equation (1.1) implies the existence of an electrostatic potential ϕ such that $\mathbf{E} = -\nabla\phi$ and then, taking for simplicity $\varepsilon_0 = 1$, Equation (1.2) can be written as

$$-\Delta\phi = \rho \qquad \text{in } \mathbb{R}^3 \ . \tag{1.3}$$

If $\rho = \delta_{x_0}$, namely we consider a unitary point charge located in x_0, we get the well-known *infinity problem*: the solution of (1.3) is $\phi(x) = 1/(4\pi|x-x_0|)$ and the energy of the electric field, which is given by

$$\mathcal{H} = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{E}|^2\, dx = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2\, dx$$

is not finite.
Moreover if $\rho \in L^1(\mathbb{R}^3)$, namely if, for example, the total charge is finite, it is not clear if (1.3) admits a solution with finite energy $\mathcal{H}$. In fact, if $\rho \in L^1(\mathbb{R}^3) \cap L^{6/5}(\mathbb{R}^3)$

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[1]P. d'Avenia, Politecnico di Bari, Dipartimento di Meccanica, Matematica e Management, Via Orabona 4, 70125 Bari, Italy; pietro.davenia@poliba.it

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the existence of a finite energy solution is trivial (using the Hölder inequality), while if we consider

$$\rho(x) = \frac{1}{|x|^{5/2} + |x|^{7/2}} ,$$

(in this case $\rho \in L^1(\mathbb{R}^3) \setminus L^{6/5}(\mathbb{R}^3)$) we easily get that (1.3) has no finite energy and radial solutions.

To avoid these problems, and in particular the violation of the *principle of finiteness*, Max Born in 1933 (see [7, 8]) launches a new nonlinear theory, modifying the Maxwell Lagrangian density.

The idea is the following. When we pass from Newton to Einstein mechanics, we replace the Lagrangian $\mathcal{L}_{\mathrm{N}} = (1/2)mv^2$ with $\mathcal{L}_{\mathrm{E}} = mc^2(1-\sqrt{1 - v^2/c^2}\,)$ considering the simplest expression that admits the existence of a limit velocity c and that, for small velocities v, can be approximated by the classical one. Thus, instead of the Maxwell Lagrangian density

$$\mathcal{L}_{\mathrm{M}} = -\frac{F_{\mu\nu}F^{\mu\nu}}{4} ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor, $(A_0, A_1, A_2, A_3) = (\phi, -\mathbf{A})$ is the electromagnetic potential, $(x_0, x_1, x_2, x_3) = (t, x)$ and ∂_j denotes the partial derivative with respect to x_j, Born considers

$$\mathcal{L}_{\mathrm{B}} = b^2 \left(1 - \sqrt{1 + \frac{F_{\mu\nu}F^{\mu\nu}}{2b^2}}\right) \sqrt{-\det(g_{\mu\nu})}\,, \tag{1.4}$$

where b is a constant having the dimensions of e/r_0^2, e and r_0 being respectively the charge and the *effective radius* of the electron and $g_{\mu\nu}$ is the Minkowski metric tensor with signature $(+ - --)$.
But, the action obtained by $\mathcal{L}_{\mathrm{B}}$, as well as the Maxwell action, is invariant only for the Lorentz group of transformations. Thus, some months later, Born, in collaboration with Infeld, introduces the following modified version

$$\mathcal{L}_{\mathrm{BI}} = b^2 \left(\sqrt{-\det(g_{\mu\nu})} - \sqrt{-\det\left(g_{\mu\nu} + \frac{F_{\mu\nu}}{b}\right)}\right) , \tag{1.5}$$

whose action is now invariant for general transformations [9, 10].

In terms of the electromagnetic field $(\mathbf{E}, \mathbf{B})$, with $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}$, we have

$$\mathcal{L}_{\mathrm{M}} = \frac{|\mathbf{E}|^2 - |\mathbf{B}|^2}{2} \quad , \quad \mathcal{L}_{\mathrm{B}} = b^2 \left(1 - \sqrt{1 - \frac{|\mathbf{E}|^2 - |\mathbf{B}|^2}{b^2}}\right)$$

and

$$\mathcal{L}_{\mathrm{BI}} = b^2 \left(1 - \sqrt{1 - \frac{|\mathbf{E}|^2 - |\mathbf{B}|^2}{b^2} - \frac{(\mathbf{E}\cdot\mathbf{B})^2}{b^4}}\right) ,$$

and so, we can see as the Born Lagrangian density has the analogous expression of the Einsten's one.

In the electrostatic case

$$\mathcal{L}_{\mathrm{B}} = \mathcal{L}_{\mathrm{BI}} = b^2 \left(1 - \sqrt{1 - \frac{|\mathbf{E}|^2}{b^2}}\right)$$

and so, given a charge density ρ, we formally get the equation

$$-\mathrm{div}\left(\frac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2/b^2}}\right)=\rho\ , \tag{1.6}$$

which replaces the Poisson equation (1.3).

For simplicity, from now on, except in Section 6, we fix $b = 1$.

This equation can also be obtained observing that the Born-Infeld theory distinguishes between the electric field $\mathbf{E}$ and the electric induction $\mathbf{D}$: the fields $\mathbf{D}$ and $\mathbf{E}$ are related by

$$\mathbf{D}=\frac{\mathbf{E}}{\sqrt{1-|\mathbf{E}|^2}}\quad \text{or}\quad \mathbf{E}=\frac{\mathbf{D}}{\sqrt{1+|\mathbf{D}|^2}}$$

and $\mathbf{D}$ satisfies

$$\mathrm{div}\,\mathbf{D}=\rho\ .$$

In the case of a *delta*-source located in x_0 the field $\mathbf{D}$ blows up in the point x_0, while the electric field tends to a finite value and this maximal field strength has an interpretation in the string theory. Moreover, in this case, finite energy solutions have been called *BIons* (see [3, 14]).

The operator in (1.6) appears also in the study of D-branes and in classical relativity where it represents the mean curvature operator in Lorentz-Minkowski space.

Equations involving the operator

$$Q^-(\phi):=-\mathrm{div}\left(\frac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right)$$

have been studied in many situations (see [2, 4, 6, 16] and references therein).

In [5] we begin a rigorous study of the boundary value problem

$$(\mathcal{BI})\qquad \begin{cases} -\mathrm{div}\left(\dfrac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right)=\rho \quad \text{in } \mathbb{R}^N\ , \\ \\ \displaystyle\lim_{|x|\to\infty}\phi(x)=0\ , \end{cases}$$

for $N\geq 3$, working on

$$\mathcal{X}=D^{1,2}(\mathbb{R}^N)\cap\{\phi\in C^{0,1}(\mathbb{R}^N)\mid \|\nabla\phi\|_\infty\leq 1\}\ , \tag{1.7}$$

equipped with the norm

$$\|\phi\|_{\mathcal{X}}:=\left(\int_{\mathbb{R}^N}|\nabla\phi|^2\,dx\right)^{1/2}\ ,$$

where $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to above norm, and taking $\rho\in\mathcal{X}^*$, the dual space of $\mathcal{X}$.

In this paper we give a survey of the results in [5] and some ideas of their proofs.

Since $(\mathcal{BI})$ is, at least formally, the Euler equation of the action functional $I:\mathcal{X}\to\mathbb{R}$ defined by

$$I(\phi)=\int_{\mathbb{R}^N}\left(1-\sqrt{1-|\nabla\phi|^2}\right)dx-\langle\rho,\phi\rangle\ , \tag{1.8}$$

we want to derive existence and uniqueness of the solution from a variational principle. Furthermore the functional I is bounded from below in $\mathcal{X}$ and strictly convex. Thus it is natural to look for the solution as the minimizer of I in $\mathcal{X}$ by the direct methods of the Calculus of Variations. However, the functional is not regular if $\|\nabla\phi\|_\infty = 1$ and so we need to use some tools from the Calculus of Variations for non smooth functionals.

Hence, first we prove that for any $\rho \in \mathcal{X}^*$ there exists a unique ϕ_ρ which minimizes I and that such minimum is the unique critical point of the functional I in weak sense (see Definition 2.1).

Then, we want to prove that the minimum is a *weak* solution in the sense of the following

Definition 1.1. A *weak solution* of $(\mathcal{BI})$ is a function $\phi_\rho \in \mathcal{X}$ such that for all $\psi \in \mathcal{X}$, we have

$$\int_{\mathbb{R}^N} \frac{\nabla\phi_\rho \cdot \nabla\psi}{\sqrt{1-|\nabla\phi_\rho|^2}}\, dx = \langle \rho, \psi \rangle \ ,$$

where $\langle\ ,\ \rangle$ denotes the duality pairing between $\mathcal{X}^*$ and $\mathcal{X}$.

Observe that the boundary condition at infinity is encoded in the functional space.

We give a positive answer when $\rho \in \mathcal{X}^*$ is a radially distributed charge (see Section 3). Moreover, under stronger assumptions on ρ, still assuming the radial symmetry of the source, we investigate the regularity of the solution, partially recovering the regularity of the Poisson equation.

We are also able to consider the case $\rho \in L^\infty_{\rm loc}(\mathbb{R}^N) \cap \mathcal{X}^*$ (see Section 4).

The case of a superposition of charges, namely

$$\rho = \sum_{i=1}^k a_i \delta_{x_i} \ ,$$

where $a_i \in \mathbb{R}$ and $x_i \in \mathbb{R}^N$, for $i = 1, \cdots, k$, $k \in \mathbb{N}_0$, is more delicate (see Section 5). We are able to prove that the minimum ϕ_ρ of I is a distributional solution away from the charges and, if the intensities are small or if the charges are sufficiently far away from each other, it is *singular* only in the points x_i in which the charges are located, i.e.

$$\lim_{x\to x_i} |\nabla\phi_\rho(x)| = 1 \ .$$

Moreover, in some more general cases, even if we do not know that minimum ϕ_ρ of the functional I is actually a weak solution of $(\mathcal{BI})$, we prove that it is the *limit* of solutions of approximated problems obtained by considering our differential operator as the sum of p-Laplacians or mollifying the charge density (see Section 6).

Finally we give additional results and comments. In particular, we complete some previous studies [11, 17, 22, 21] on the nonlinear Klein-Gordon-Born-Infeld system (Section 7).

Notations. In the following C denotes a generic positive constants which can change from line to line, B_R is the ball centered in 0 with radius $R > 0$ and ω_N denotes the measure of the $(N-1)$-dimensional unitarian sphere.

2. Minimum as critical point in weak sense

The ambient space $\mathcal{X}$ defined in (1.7), satisfies the following properties:

(i) $\mathcal{X}$ is continuously embedded in $W^{1,p}(\mathbb{R}^N)$, for all $p \geq 2^* = 2N/(N-2)$;
(ii) $\mathcal{X}$ is continuously embedded in $L^\infty(\mathbb{R}^N)$;

(iii) if $\phi \in \mathcal{X}$, then $\lim_{|x|\to\infty} \phi(x) = 0$;
(iv) $\mathcal{X}$ is weakly closed;
(v) if $(\phi_n)_n \subset \mathcal{X}$ is bounded, there exists $\bar{\phi} \in \mathcal{X}$ such that, up to a subsequence, $\phi_n \rightharpoonup \bar{\phi}$ weakly in $\mathcal{X}$ and uniformly on compact sets.

Moreover the functional $I : \mathcal{X} \to \mathbb{R}$ is

(i) bounded from below,
(ii) coercive,
(iii) continuous,
(iv) strictly convex,
(v) weakly lower semi-continuous.

We look for critical points in the sense of the following classical definition (see [19]).

Definition 2.1. Let X be a real Banach space and $\Psi : X \to (-\infty, +\infty]$ be a convex lower semicontinuous function. Let $D(\Psi) = \{u \in X \mid \Psi(u) < +\infty\}$ be the effective domain of Ψ. For $u \in D(\Psi)$, the set

$$\partial\Psi(u) = \{u^* \in X^* \mid \Psi(v) - \Psi(u) \geq \langle u^*, v - u\rangle \quad , \quad \forall v \in X\}$$

is called the *subdifferential* of Ψ at u. If, moreover, we consider a functional $I = \Psi + \Phi$, with Ψ as above and $\Phi \in C^1(X, \mathbb{R})$, then $u \in D(\Psi)$ is said to be *critical in weak sense* if $-\Phi'(u) \in \partial\Psi(u)$, that is

$$\langle \Phi'(u), v - u\rangle + \Psi(v) - \Psi(u) \geq 0 \quad , \quad \forall v \in X \ .$$

As a consequence of the previous properties and since, in our case, the critical points in weak sense for the functional I coincide with the minima for I, we get

Theorem 2.2. *For any $\rho \in \mathcal{X}^*$, there exists a unique ϕ_ρ which minimizes I. This is the unique critical point in weak sense of I.*

Moreover we have

Proposition 2.3. *Assume $\rho \in \mathcal{X}^*$. If $\phi \in \mathcal{X}$ is a weak solution of $(\mathcal{BI})$, then $\phi = \phi_\rho$.*

We would like to prove that the unique minimizer ϕ_ρ is weak solution of $(\mathcal{BI})$. We are able to show that it is true only in some cases even if we conjecture that it holds and the following results go in that direction.

Proposition 2.4. *Assume $\rho \in \mathcal{X}^*$ and let ϕ_ρ be the unique minimizer of I in $\mathcal{X}$. Then*

$$E = \{x \in \mathbb{R}^N \mid |\nabla\phi_\rho| = 1\}$$

is a null set (with respect to Lebesgue measure) and the function ϕ_ρ satisfies

$$\int_{\mathbb{R}^N} \frac{|\nabla\phi_\rho|^2}{\sqrt{1 - |\nabla\phi_\rho|^2}}\, dx \leq \langle \rho, \phi_\rho\rangle \ .$$

Moreover, for all $\psi \in \mathcal{X}$, we have the variational inequality

$$\int_{\mathbb{R}^N} \frac{|\nabla\phi_\rho|^2}{\sqrt{1 - |\nabla\phi_\rho|^2}}\, dx - \int_{\mathbb{R}^N} \frac{\nabla\phi_\rho \cdot \nabla\psi}{\sqrt{1 - |\nabla\phi_\rho|^2}}\, dx \leq \langle \rho, \phi_\rho - \psi\rangle \ . \tag{2.1}$$

Remark 2.5. If ϕ_ρ satisfies further

$$\int_{\mathbb{R}^N} \frac{|\nabla\phi_\rho|^2}{\sqrt{1-|\nabla\phi_\rho|^2}}\, dx = \langle \rho, \phi_\rho \rangle \, ,$$

then, by (2.1), it is easy to see that ϕ_ρ is a weak solution of $(\mathcal{BI})$.

3. The radial case

In this section we consider radially distributed charges.
Of course we first need to precise the meaning of a radially distributed charge density. For $\tau \in O(N)$, $\phi \in \mathcal{X}$ and $\rho \in \mathcal{X}^*$, we define $\phi^\tau \in \mathcal{X}$ as $\phi^\tau(x) = \phi(\tau x)$, for all $x \in \mathbb{R}^N$, and $\rho^\tau \in \mathcal{X}^*$ as $\langle \rho^\tau, \psi\rangle = \langle \rho, \psi^\tau\rangle$, for all $\psi \in \mathcal{X}$.

Definition 3.1. We say that $\rho \in \mathcal{X}^*$ is *radially distributed* if $\rho^\tau = \rho$, for any $\tau \in O(N)$.

Moreover, if $\phi \in \mathcal{X}_{\mathrm{rad}} := \{\phi \in \mathcal{X} \mid \phi^\tau = \phi \text{ for every } \tau \in O(N)\}$, we denote with the same letter ϕ the single real variable function $r = |x| \in \mathbb{R}_+ \mapsto \phi(r)$. We make furthermore a similar identification for the radially distributed maps $\rho \in \mathcal{X}^*$.

We have

Theorem 3.2. *If $\rho \in \mathcal{X}^*$ is radially distributed, then there exists a unique (radial) weak solution $\phi_\rho \in \mathcal{X}$ of $(\mathcal{BI})$.*

To prove this theorem, we use the following

Lemma 3.3. *Assume $\rho \in \mathcal{X}^*$ and let ϕ_ρ be the unique minimizer of I in $\mathcal{X}$. If $(\psi_n)_n \subset D^{1,2}(\mathbb{R}^N)$ is such that $\|\nabla\psi_n\|_\infty \le C$ for some $C > 0$ and $\psi_n \to \psi$ in $D^{1,2}(\mathbb{R}^N)$ then, up to a subsequence,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{\nabla\phi_\rho \cdot \nabla\psi_n}{\sqrt{1-|\nabla\phi_\rho|^2}}\, dx = \int_{\mathbb{R}^N} \frac{\nabla\phi_\rho \cdot \nabla\psi}{\sqrt{1-|\nabla\phi_\rho|^2}}\, dx \, .$$

Thus we can proceed as follows.

Sketch of the Proof. The argument is borrowed from [18].
First we prove that the minimum $\phi_\rho \in \mathcal{X}_{\mathrm{rad}}$.
Then we define, for $k \in \mathbb{N}^*$, the sets

$$E_k = \left\{ r \ge 0 \;\mid\; |\phi'_\rho(r)| \ge 1 - \frac{1}{k} \right\}$$

and, since by Proposition 2.4 we know that $|\{r \ge 0 \mid |\phi'_\rho(r)| = 1\}| = 0$, then we have that

$$\left| \bigcap_{k\ge 1} E_k \right| = 0 \, .$$

Thus we take $\psi \in \mathcal{X}_{\mathrm{rad}} \cap C_c^\infty(\mathbb{R}^N)$ with $\operatorname{supp}\psi \subset [0, R]$ and let

$$\psi_k(r) := -\int_r^{+\infty} \psi'(s)[1 - \chi_{E_k}(s)]\, ds \, .$$

We have that $\operatorname{supp}\psi_k \subset [0, R]$, for any $k \ge 1$ and that, if $|t|$ is sufficiently small, then $\phi_\rho + t\psi_k \in \mathcal{X}$.

Since ϕ_ρ is the minimizer of I, then

$$\begin{aligned} 0 &= \lim_{t\to 0} \frac{I(\phi_\rho + t\psi_k) - I(\phi_\rho)}{t} = \\ &= \omega_N \int_0^{+\infty} \frac{\phi'_\rho \psi'}{\sqrt{1-|\phi'_\rho|^2}}[1-\chi_{E_k}]r^{N-1}\,dr - \langle \rho, \psi_k\rangle \ . \end{aligned}$$

But, since $\chi_{E_k} \to 0$ a.e. in $\mathbb{R}^N$, by Lebesgue's Dominated Convergence Theorem we have

$$\int_0^{+\infty} \frac{\phi'_\rho \psi'}{\sqrt{1-|\phi'_\rho|^2}}[1-\chi_{E_k}]r^{N-1}\,dr \to \int_0^{+\infty} \frac{\phi'_\rho \psi'}{\sqrt{1-|\phi'_\rho|^2}}\, r^{N-1}\,dr \ .$$

Moreover, by $\psi_k \to \psi$ in $\mathcal{X}$, we have

$$\langle \rho, \psi_k\rangle \to \langle \rho, \psi\rangle \ .$$

Thus for any $\psi \in \mathcal{X}_{\text{rad}} \cap C_c^\infty(\mathbb{R}^N)$, we conclude that

$$\int_{\mathbb{R}^N} \frac{\nabla\phi_\rho \cdot \nabla\psi}{\sqrt{1-|\nabla\phi_\rho|^2}}\,dx = \langle \rho, \psi\rangle \ . \tag{3.1}$$

To show that (3.1) holds for any $\psi \in \mathcal{X}_{\text{rad}}$, we construct $(\psi_n)_n \subset C_c^\infty(\mathbb{R}^N)$, ψ_n radially symmetric such that $\psi_n \to \psi$ in $D^{1,2}(\mathbb{R}^N)$ and with $\|\nabla\psi_n\|_\infty \le C$ and we apply Lemma 3.3.
Finally we take $\psi = \phi_\rho$ in (3.1) and we have that (3.1) holds for any $\psi \in \mathcal{X}$. This fact allows us to conclude using Remark 2.5.

□

Assuming further hypotheses on ρ, we prove that $|\phi'_\rho(r)| = 1$ can only happen at $r = 0$ and

Theorem 3.4. *Assume that ρ is a radially symmetric function such that $\rho \in L^s(\mathbb{R}^N) \cap L^\sigma(B_\delta(0))$, for some $s \ge 1$, $\sigma \ge N$ and $\delta > 0$. Then the weak solution ϕ_ρ of $(\mathcal{BI})$ is $C^1(\mathbb{R}^N;\mathbb{R})$.*

4. Bounded charges

In this section we consider bounded charge distributions.
First of all we give the following definition which is standard in classical relativity.

Definition 4.1. Let $\phi \in C^{0,1}(\Omega)$, with $\Omega \subset \mathbb{R}^N$. We say that ϕ is

- *weakly spacelike* if $|\nabla\phi| \le 1$ a.e. in Ω;
- *spacelike* if $|\phi(x) - \phi(y)| < |x-y|$ whenever $x, y \in \Omega$, $x \ne y$ and the line segment $\overline{xy} \subset \Omega$;
- *strictly spacelike* if ϕ is spacelike, $\phi \in C^1(\Omega)$ and $|\nabla\phi| < 1$ in Ω.

Using some fundamental tools in [4] we get

Theorem 4.2. *If $\rho \in L^\infty_{\text{loc}}(\mathbb{R}^N) \cap \mathcal{X}^*$, then ϕ_ρ is a (locally strictly) space-like weak solution of $(\mathcal{BI})$.*

Sketch of the Proof. Let Ω be an arbitrary bounded domain with smooth boundary in $\mathbb{R}^N$. We set

$$C^{0,1}_{\phi_\rho}(\Omega) = \left\{\phi \in C^{0,1}(\Omega) \mid \phi|_{\partial\Omega} = \phi_\rho|_{\partial\Omega}, |\nabla\phi| \le 1\right\} \ ,$$

$$K = \{\overline{xy} \subset \Omega \mid x, y \in \partial\Omega, x \neq y, |\phi_\rho(x) - \phi_\rho(y)| = |x - y|\} \ ,$$

and we define $I_\Omega : C^{0,1}_{\phi_\rho}(\Omega) \to \mathbb{R}$ by

$$I_\Omega(\phi) = \int_\Omega \left(1 - \sqrt{1 - |\nabla\phi|^2}\right) dx - \int_\Omega \rho\phi \, dx \ .$$

It is easy to see that $\phi_\rho|_\Omega$ is a minimizer for I_Ω in $C^{0,1}_{\phi_\rho}(\Omega)$.
By [4, Corollary 4.2] we have that ϕ_ρ is strictly spacelike in $\Omega \setminus K$ and $Q^-(\phi_\rho) = \rho$ in $\Omega \setminus K$. Furthermore,

$$\phi_\rho(tx + (1-t)y) = t\phi_\rho(x) + (1-t)\phi_\rho(y) \ , \qquad 0 < t < 1$$

for every $x, y \in \partial\Omega$ such that $|\phi_\rho(x) - \phi_\rho(y)| = |x - y|$ and $\overline{xy} \subset \Omega$.
If $K = \emptyset$, then ϕ_ρ is strictly spacelike in Ω.
Assume by contradiction that $K \neq \emptyset$.
Then there exist $x, y \in \partial\Omega$ such that $x \neq y$, $\overline{xy} \subset \Omega$ and $|\phi_\rho(x) - \phi_\rho(y)| = |x - y|$.
Without loss of generality we can assume that $\phi_\rho(x) > \phi_\rho(y)$.
It is easy to see that for all $t \in (0,1)$

$$\phi_\rho(tx + (1-t)y) = \phi_\rho(y) + t|x - y| \ . \tag{4.1}$$

Since, for any $R > 0$ such that $\Omega \subset B_R$, $\phi_\rho|_{B_R}$ is a minimizer of I_{B_R} in $C^{0,1}_{\phi_\rho}(B_R)$, then, by [4, Theorem 3.2], we have that (4.1) holds for all $t \in \mathbb{R}$ such that $tx + (1-t)y \in B_R$. Hence we reach a contradiction with the boundedness of ϕ_ρ, for an R sufficiently large.

□

Remark 4.3. As observed in [4, Remark p. 147], if $\rho \in C^k(\mathbb{R}^N)$, then $\phi_\rho \in C^{k+1}(\mathbb{R}^N)$.

5. k POINT CHARGES

In this section we consider

$$\rho = \sum_{i=1}^k a_i \delta_{x_i} \ ,$$

where $a_i \in \mathbb{R}$ and $x_i \in \mathbb{R}^N$, for $i = 1, \cdots, k$, $k \in \mathbb{N}_0$ and then the problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla\phi}{\sqrt{1 - |\nabla\phi|^2}}\right) = \displaystyle\sum_{i=1}^k a_i \delta_{x_i} & , \quad \text{in } \mathbb{R}^N \ , \\ \phi(x) \to 0 & , \quad \text{as } x \to \infty \ . \end{cases} \tag{5.1}$$

In the recent contribution [15], the author claims the existence [15, Proposition 2.1] of a weak solution v_∞ of (5.1) under the assumption that $x_i \in \mathbb{R}^3$ and $a_i \in \mathbb{R}$ for $i = 1, \cdots, k$. However, the proof of [15, Step 2.6, page 515] is incomplete (see [5] for the details).

Of course there exists a unique minimizer ϕ_ρ of the associated energy functional

$$I(\phi) = \int_{\mathbb{R}^N} \left(1 - \sqrt{1 - |\nabla\phi|^2}\right) dx - \sum_{i=1}^k a_i \phi(x_i)$$

(see [15] or Theorem 2.2).
In particular we are interested in proving that this minimizer solves (5.1) in a weak or a strong sense. Let

$$\Gamma = \bigcup_{i\neq j} \overline{x_i x_j} \ .$$

We have

Theorem 5.1. *The minimizer ϕ_ρ is a distributional solution of the Euler-Lagrange equation in $\mathbb{R}^N \setminus \{x_1, \cdots, x_k\}$. Namely, for every $\psi \in C_c^\infty(\mathbb{R}^N \setminus \{x_1, \cdots, x_k\})$, we have*

$$\int_{\mathbb{R}^N} \frac{\nabla\phi_\rho \cdot \nabla\psi}{\sqrt{1-|\nabla\phi_\rho|^2}}\, dx = 0 \ .$$

It is a classical solution of the equation in $\mathbb{R}^N \setminus \Gamma$, namely $\phi_\rho \in C^\infty(\mathbb{R}^N \setminus \Gamma)$ and

$$-\mathrm{div}\left(\frac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right) = 0$$

in the classical sense in $\mathbb{R}^N \setminus \Gamma$. Moreover

(1) *for any fixed $x_i \in \mathbb{R}^N$, $i = 1, \cdots, k$, there exists $\sigma = \sigma(x_1, \cdots, x_k) > 0$ such that if*

$$\max_{i=1,\cdots,k} |a_i| < \sigma \ ,$$

then ϕ_ρ is a classical solution in $\mathbb{R}^N \setminus \{x_1, \cdots, x_k\}$;

(2) *for any $a_i \in \mathbb{R}$, $i = 1, \cdots, k$, there exists $\tau = \tau(a_1, \cdots, a_k) > 0$ such that if*

$$\min_{i,j=1,\cdots,k,\ i\neq j} |x_i - x_j| > \tau \ ,$$

then ϕ_ρ is a classical solution in $\mathbb{R}^N \setminus \{x_1, \cdots, x_k\}$.

In these last cases, $\phi_\rho \in C^\infty(\mathbb{R}^N \setminus \{x_1, \cdots, x_k\})$, it is strictly spacelike on $\mathbb{R}^N \setminus \{x_1, \cdots, x_k\}$ and

$$\lim_{x\to x_i} |\nabla\phi_\rho(x)| = 1 \ .$$

To show this last result, first we prove that ϕ_ρ satisfies strongly (5.1) in $\mathbb{R}^N \setminus \Gamma$ and this is true whithout any restriction on the coefficients a_i and the location of the charges. We have

Lemma 5.2. *The minimum ϕ_ρ of I satisfies strongly*

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right) = 0 & , \quad \text{in } \mathbb{R}^N \setminus \Gamma \ , \\ \phi(x) \to 0 & , \quad \text{as } x \to \infty \ . \end{cases}$$

Furthermore, we have that

(i) *$\phi_\rho \in C^\infty(\mathbb{R}^N \setminus \Gamma) \cap C(\mathbb{R}^N)$;*
(ii) *ϕ_ρ is strictly spacelike on $\mathbb{R}^N \setminus \Gamma$;*
(iii) *for $i \neq j$, either ϕ_ρ is a classical solution on $\overline{x_i x_j}$, or*

$$\phi_\rho(tx_i + (1-t)x_j) = t\phi_\rho(x_i) + (1-t)\phi_\rho(x_j) \quad , \quad 0 < t < 1 \ .$$

Sketch of the Proof. Let Ω be an arbitrary bounded open domain with smooth boundary in $\mathbb{R}^N \setminus \{x_1, \cdots, x_k\}$.
Here we repeat the same arguments of the Proof of Theorem 4.2 whit the same notations.
The main difference now is that $\rho = 0$, since $\Omega \subset \mathbb{R}^N \setminus \{x_1, \cdots, x_k\}$.
As before we infer that ϕ_ρ is strictly spacelike and $Q^-(\phi_\rho) = 0$ in $\Omega \setminus K$.
Furthermore, we have

$$\phi_\rho(tx + (1-t)y) = t\phi_\rho(x) + (1-t)\phi_\rho(y) \ ,$$

for every $0 < t < 1$, where $x, y \in \partial\Omega$ are such that $|\phi_\rho(x) - \phi_\rho(y)| = |x - y|$ and $\overline{xy} \subset \Omega$. Again, if $K = \emptyset$, then ϕ_ρ is strictly spacelike.
We now show that K contains at most Γ.
Assume by contradiction that there exist $x, y \in \partial\Omega$ such that $x \neq y$, $\overline{xy} \subset \Omega$ and $|\phi_\rho(x) - \phi_\rho(y)| = |x - y|$ and such that the straight line spanned by $\overline{xy}$ intersects Γ at a finite number of points (possibly zero). Without loss of generality, we can assume that $\phi_\rho(x) > \phi_\rho(y)$. It is easy to see that for all $t \in (0, 1)$

$$\phi_\rho(tx + (1-t)y) = \phi_\rho(y) + t|x - y| \ . \tag{5.2}$$

Observe also that, since the line spanned by x and y intersects Γ at a finite number of points only, we can arbitrarily stretch Ω in at least one direction of $\overline{xy}$ to build new open sets $\Omega' \subset \mathbb{R}^N \setminus \{x_1, \cdots, x_k\}$ with smooth boundaries and such that $\Omega \subset \Omega'$. Observe that $\phi_\rho|_{\Omega'}$ is a minimizer for $I_{\Omega'}$ in $C^{0,1}_{\phi_\rho}(\Omega')$ and, by [4, Theorem 3.2], we have that (5.2) holds for all $t \in \mathbb{R}$ such that $tx + (1-t)y \in \Omega'$. Now, we reach a contradiction with the boundedness of ϕ_ρ by choosing Ω' long enough in the direction $\overline{xy}$.
Arguing in a similar way, we see that on each edge of Γ, either $Q^-(\phi_\rho) = 0$ or the full edge belongs to K, namely (iii) holds.
Assertion (i) follows from [4, Remark p. 147], assertion (ii) from [4, Corollary 4.2]. □

Then we prove the following preliminary results.

Lemma 5.3. *For any $\varepsilon > 0$ there exists $\sigma > 0$ such that, if $\max_{i=1,\cdots,k} |a_i| < \sigma$, then $\|\phi_\rho\|_\infty < \varepsilon$.*

Lemma 5.4. *There exists $C = C(a_1, \cdots, a_k) > 0$ such that, for all $x_i \in \mathbb{R}^N$, $i = 1, \cdots, k$, $\|\phi_\rho\|_\infty < C$.*

Thus we can complete the proof of Theorem 5.1.

Sketch of Proof of Theorem 5.1 Using a general result of Trudinger on divergence elliptic operators with measurable coefficients, see [20, Theorem 3.2] we have that for every bounded domain Ω such that $\bar{\Omega} \subset \mathbb{R}^N \setminus \{x_1, \cdots, x_k\}$, there exists a unique distributional solution $\bar{\phi}_\rho$ of the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla\phi}{\sqrt{1 - |\nabla\phi_\rho|^2}}\right) = 0 & , \quad \text{in } \Omega \ , \\ \phi = \phi_\rho & , \quad \text{on } \partial\Omega \ . \end{cases}$$

Then by the arbitrariness of Ω in $\mathbb{R}^N \setminus \{x_1, \cdots, x_k\}$ we deduce that $\bar{\phi}_\rho$ is the unique distributional solution of

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla\phi}{\sqrt{1-|\nabla\phi_\rho|^2}}\right) = 0 & , \quad \text{in } \mathbb{R}^N \setminus \{x_1, \cdots, x_k\} \ , \\ \phi(x) \to 0 & , \quad \text{as } x \to \infty \ . \end{cases}$$

Thus Lemma 5.2 and the uniqueness of $\bar{\phi}_\rho$ then imply that $\bar{\phi}_\rho = \phi_\rho$ a.e. in $\mathbb{R}^N$.
To prove (1), using again the Bartnik–Simon trick, we want to get that $K = \emptyset$. Assume by contradiction that $K \neq \emptyset$. Then there exist $x, y \in \partial\Omega$ such that $x \neq y$, $\overline{xy} \subset \Omega$ and $|\phi_\rho(x) - \phi_\rho(y)| = |x - y|$. Without loss of generality, we can assume that $\phi_\rho(x) > \phi_\rho(y)$. It is easy to see that for all $t \in (0, 1)$

$$\phi_\rho(tx + (1-t)y) = \phi_\rho(y) + t|x - y| \ .$$

Two possibilities occur: either $\overline{xy}$ intersects Γ in a finite number of points (possibly zero), or $\overline{xy}$ intersects Γ in an infinite number of points.
In the first case, we conclude as in the proof of Lemma 5.2.
In the second case, without loss of generality, we can assume that $\overline{xy}$ coincide with a piece of $\overline{x_1x_2}$ and, applying if necessary again the [4, Theorem 3.2], we can consider any extension of $\overline{xy}$ in $\overline{x_1x_2}$.
We apply Lemma 5.3 and so, fixing $\varepsilon > 0$ such that $2\varepsilon < \min_{i,j=1,\cdots,k,\ i\neq j} |x_i - x_j|$, there exists $\sigma > 0$ such that, if $\max_{i=1,\cdots,k} |a_i| < \sigma$, then $\|\phi_\rho\|_\infty < \varepsilon$. Since we can find $x', y' \in \overline{x_1x_2}$ with $|x' - y'| > 2\varepsilon$ and $\overline{x'y'} \neq \overline{x_1x_2}$, we reach the contradiction

$$2\varepsilon < |x' - y'| = |\phi_\rho(x') - \phi_\rho(y')| < 2\varepsilon \ .$$

Moreover, (2) is a consequence of Bartnik–Simon trick and of Lemma 5.4 Finally, the behavior of the gradient of ϕ_ρ near the singularities x_i is a consequence of [12, Theorem 1.5].

□

We conclude this section by showing what happens in some particular cases. Assume that we have two point charges with equal coefficients, namely

$$\rho = a(\delta_{x_1} + \delta_{x_2}) \ .$$

By uniqueness of the minimizer and since the functional is now invariant under the orthogonal transformations that exchanges x_1 and x_2, we infer that ϕ_ρ is symmetric and therefore cannot be affine with slope 1 on the segment $\overline{x_1x_2}$. Therefore, the assertion (iii) of Lemma 5.2 allows to conclude.
The same argument can be used when we have a symmetric configuration of charges with equal coefficient.
In some special situations, we can argue without assuming any symmetry to prove that the minimizer is not affine with slope 1 on some of the edges of Γ. As an example, take three charges located at x_1, x_2 and x_3 and suppose that those points are not colinear. One can order the value of ϕ_ρ and assume without loss of generality that $\phi_\rho(x_1) \leq \phi_\rho(x_2) \leq \phi_\rho(x_3)$. Then, ϕ_ρ cannot be affine with slope 1 on $\overline{x_1x_2}$ and $\overline{x_2x_3}$ since otherwise we have

$$\phi_\rho(x_3) - \phi_\rho(x_1) = \phi_\rho(x_3) - \phi_\rho(x_2) + \phi_\rho(x_2) - \phi_\rho(x_1) = |x_3 - x_2| + |x_2 - x_1| > |x_3 - x_1| \ .$$

Other similar situations can be ruled out with the same argument but this is clearly an incomplete and unsatisfactory approach towards the understanding of the general case.

6. Approximations

In this section, we provide some ways to approximate the minimizer ϕ_ρ by a sequence of solutions of some approximating PDEs.

Let us still denote by ϕ_ρ the minimum associated to $\rho \in \mathcal{X}^*$.

In [13] Fortunato, Orsina and Pisani consider $N = 3$, observe that, for b large,

$$\mathcal{L}_{\mathrm{BI}} = b^2\left(1 - \sqrt{1 - \frac{|\nabla\phi|^2}{b^2}}\right) \sim \frac{|\nabla\phi|^2}{2} + \frac{|\nabla\phi|^4}{8b^2} \ ,$$

and so they get the Euler equation

$$-\mathrm{div}\left(\left(1 + \frac{1}{2b^2}|\nabla\phi|^2\right)\nabla\phi\right) = \rho \quad , \quad \text{in } \mathbb{R}^3 \ .$$

Setting $b = 1$, for $n \geq 1$, we define $\mathcal{X}_{2n}$ as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm defined by

$$\|\phi\|^2_{\mathcal{X}_{2n}} := \int_{\mathbb{R}^N} |\nabla\phi|^2 + \left(\int_{\mathbb{R}^N} |\nabla\phi|^{2n}\, dx\right)^{1/n} \ .$$

Formally

$$Q^-(\phi) = -\sum_{h=1}^{\infty} \alpha_h \Delta_{2h}\phi \ ,$$

where for all $h \geq 1$, $\Delta_{2h}\phi := \mathrm{div}(|\nabla\phi|^{2h-2}\nabla\phi)$ and $\alpha_h > 0$ (the exact values of the coefficient α_h are given in [22, 15], they are not important for our purpose). The operator Q^- is formally the Gateaux derivative of the functional

$$\int_{\mathbb{R}^N} \left(1 - \sqrt{1 - |\nabla\phi|^2}\right) dx = \int_{\mathbb{R}^N} \sum_{h=1}^{\infty} \frac{\alpha_h}{2h} |\nabla\phi|^{2h}\, dx \ ,$$

where the power series in the right hand side converges pointwise when $|\nabla\phi(x)| \leq 1$. Assuming $\rho \in \mathcal{X}^*_{2n}$, let us denote the nth approximation of the functional (1.8) by

$$I_n := \phi \in \mathcal{X}_{2n} \mapsto \sum_{h=1}^{n} \frac{\alpha_h}{2h} \int_{\mathbb{R}^N} |\nabla\phi|^{2h}\, dx - \langle\rho, \phi\rangle_{\mathcal{X}_{2n}} \ .$$

This functional is C^1 and we have this first approximation result.

Theorem 6.1. *Given $n_0 \geq 1$ and $\rho \in \mathcal{X}^*_{2n_0}$, then, for all $n \geq n_0$, the functional $I_n : \mathcal{X}_{2n} \to \mathbb{R}$ has one and only one critical point ϕ_n. Moreover ϕ_n tends to ϕ_ρ weakly in $\mathcal{X}_{2m}$ for all $m \geq n_0$ and uniformly on compact sets.*

Other approximation schemes can be used.

Another truncation was successfully proposed in [6] to deal with a related problem and could have been used here as well. Let us set $a_0(s) = (1 - s)^{-1/2}$ for all $s < 1$. Then

$$I(\phi) = \frac{1}{2} \int_{\mathbb{R}^N} A_0(|\nabla\phi|^2)\, dx - \langle\rho, \phi\rangle \ ,$$

where $A_0(t) = \int_0^t a_0(s)\, ds$. Take $\theta \in (0,1)$ and define $a_\theta : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$a_\theta(s) = \begin{cases} a_0(s) & \text{for } s \in [0, 1-\theta] \\ \gamma s^{n-1} + \delta & \text{for } s \in (1-\theta, +\infty) \end{cases}$$

where γ and δ are chosen in such a way that a_θ is C^1. The truncated functional $I_{\theta,n} : \mathcal{X}_{2n} \to \mathbb{R}$ defined by

$$I_{\theta,n}(\phi) := \frac{1}{2} \int_{\mathbb{R}^N} A_\theta(|\nabla\phi|^2)\, dx - \langle \rho, \phi \rangle \ ,$$

where $A_\theta(t) = \displaystyle\int_0^t a_\theta(s)\, ds$ gives another lower estimate of $I(\phi)$. Then we can show that given $n \geq 1$ and $\rho \in \mathcal{X}_{2n}^*$, the functional $I_{\theta,n}$ has one and only one critical point which is a weak solution of

$$-\text{div}\left(a_\theta(|\nabla\phi|^2)\nabla\phi\right) = \rho \ .$$

In this approach, n is fixed which makes the functional setting easier than in the finite order approximation of the power series. It is chosen in such a way that $\rho \in \mathcal{X}_{2n}^*$. Taking a sequence $\theta_k \to 1$, we can show that the sequence of minimizers of $I_{\theta_k,n}$ converge to the minimizer of I, giving yet another way to approach the minimizer by a sequence of solutions of approximating problems.

Finally, if we *mollify* the charge (we recall that a smooth charge yields a smooth minimizer that solves the Euler-Lagrange equation associated to the minimization problem) we get another approximation result. Indeed we can prove

Theorem 6.2. *Let $\rho \in \mathcal{X}^*$ and suppose that there exist $(\rho_n)_n \subset \mathcal{X}^*$ and $\tilde\rho \in \mathcal{X}^*$ such that $\tilde\rho \geq 0$, $\rho_n \to \rho$ in $\mathcal{X}^*$ and $-\tilde\rho \leq \rho_n \leq \tilde\rho$. Then ϕ_{ρ_n} converges to ϕ_ρ weakly in $\mathcal{X}$ and uniformly in $\mathbb{R}^N$.*

Here, if $\bar\rho, \hat\rho \in \mathcal{X}^*$, $\bar\rho \leq \hat\rho$ means that $\langle \bar\rho, \varphi \rangle \leq \langle \hat\rho, \varphi \rangle$ for any $\varphi \in \mathcal{X}$ with $\varphi \geq 0$.

Thus, by Theorem 4.2 and Theorem 6.2, we get the following approximation of ϕ_ρ as uniform limit of smooth solutions of a sequence of approximated problems.

Corollary 6.3. *Let $\rho \in \mathcal{X}^*$ and suppose that there exist $(\rho_n)_n \subset \mathcal{X}^* \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$ and $\tilde\rho \in \mathcal{X}^*$ such that $\rho_n \to \rho$ in $\mathcal{X}^*$ and $-\tilde\rho \leq \rho_n \leq \tilde\rho$. Then the sequence $(\phi_{\rho_n})_n$ of (locally strictly) spacelike solutions of $(\mathcal{BI})$ with ρ_n converges to ϕ_ρ weakly in $\mathcal{X}$ and uniformly in $\mathbb{R}^N$.*

Remark 6.4. If $(\rho_n)_n \subset L^p(\mathbb{R}^N)$, with $1 \leq p < +\infty$, is such that $\rho_n \to \rho$ in $L^p(\mathbb{R}^N)$, we can immediately conclude that ϕ_{ρ_n} converges to ϕ_ρ weakly in $\mathcal{X}$ and uniformly in $\mathbb{R}^N$. In particular, for a datum $\rho \in L^p(\mathbb{R}^N)$, the approximating sequence $(\phi_{\rho_n})_n$, where $(\rho_n)_n$ is a standard sequence of mollifications of ρ, is made of smooth strictly spacelike solutions of $(\mathcal{BI})$ with the data ρ_n.

7. The Born-Infeld-Klein-Gordon equation and other extensions

Another interesting problem which involves the Born-Infeld theory appears when we couple a field, governed by the nonlinear Klein-Gordon equation, with the electromagnetic field whose Lagrangian density is given by (1.4) or (1.5), by means of the Weil covariant derivatives.

In the wake of [11, 17], Yu, in [21], deals with the system

$$(7.1)\qquad \begin{cases} \operatorname{div}\left(\dfrac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right) = u^2(\omega+\phi) & , \quad x\in\mathbb{R}^3 , \\ \Delta u = \left(m^2-(\omega+\phi)^2\right)u - |u|^{p-2}u & , \quad x\in\mathbb{R}^3 . \end{cases}$$

Fixing u in a convenient space of radial functions, Yu considers the functional

$$E_u(\phi) = \int_{\mathbb{R}^3}\left[\left(1-\sqrt{1-|\nabla\phi|^2}\right) + \omega u^2\phi + \frac{1}{2}\,\phi^2u^2\right]dx ,$$

and proves that E_u possesses a minimizer ϕ_u without proving that the minimum ϕ_u is a critical point of E_u. Then the second equation of (7.1) is solved with ϕ_u in place of ϕ. Yu's conclusion is then that (u,ϕ_u) is a solution of (7.1) in a generalized sense, meaning that the second equation is classically satisfied while ϕ_u is a minimizer of E_u.

Our aim here is to show that the minimizer of E_u and of similar functionals is actually a solution of the corresponding equation. This leads us to consider more general equations of the form

$$(7.2)\qquad \begin{cases} -\operatorname{div}\left(\dfrac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right) + f(x,\phi) = 0 & , \quad x\in\mathbb{R}^N , \\ \phi(x)\to 0 & , \quad \text{as } |x|\to\infty . \end{cases}$$

We assume that $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that

(f1) there exists $p\geq 2^*-1$ such that for all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$
$$|f(x,t)|\leq C|t|^p ;$$
(f2) $f(\cdot,t)$ is radially symmetric, for all $t\in\mathbb{R}$;
(f3) the functional $I_F:\mathcal{X}\to\mathbb{R}$ defined by
$$I_F(\phi) = \int_{\mathbb{R}^N}\left(1-\sqrt{1-|\nabla\phi|^2}\right)dx + \int_{\mathbb{R}^N}F(x,\phi)\,dx ,$$
where $F(x,t)=\int_0^t f(x,s)\,ds$, has a nontrivial radial local minimum ϕ_f in $\mathcal{X}$.

Remark 7.1. The existence of a local minimum of I_F in $\mathcal{X}$ follows, for example, by standard assumptions such as the coercivity of I_F and the convexity of the function $F(x,\cdot)$.

Similar arguments to that used in Proposition 2.4 and Theorem 3.2 allow to get

Theorem 7.2. *Suppose that* (f1)-(f3) *hold, then ϕ_f is a nontrivial weak solution of* (7.2).

Arguing as in the proof of Theorem 7.2 we can complete the arguments of [21] concerning the existence of a nontrivial solution of (7.1). In that precise case, one can even conclude that the solution is classical and even smooth.

Finally, if I_F has a nontrivial *local minimizer* in the generalized sense of Morse, i.e. the function is minimal with respect to compactly supported variations, see for example [1], then it is a solution of (7.2). Of course, any local or global minimizer is a local minimizer in the sense of Morse. We emphasize that we do not require any radial symmetry here. We have

Theorem 7.3. *Suppose that* (f1) *holds and that there exists* $\phi_0 \in \mathcal{X}$ *such that for any bounded open set* $\Omega \subset \mathbb{R}^N$*, and for any* $\phi \in \mathcal{X}$ *with* $\phi = \phi_0$ *in* $\mathbb{R}^N \setminus \Omega$*,* $I_F(\phi_0) \leq I_F(\phi)$*. Then* ϕ_0 *is a weak solution of* (7.2)*.*

References

[1] G. Alberti, L. Ambrosio & X. Cabré, *On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property*, Acta Appl. Math., 65(2001), 9–33.

[2] A. Azzollini, *Ground state solution for a problem with mean curvature operator in Minkowski space*, J. Funct. Anal., 266(2014), 2086–2095.

[3] C. Bachas & M. Poratti, *Pair creation of open strings in an electric field*, Phys. Lett. B, 296(1992), 77–84.

[4] R. Bartnik & L. Simon, *Spacelike hypersurfaces with prescribed boundary values and mean curvature*, Comm. Math. Phys., 87(1982), 131–152.

[5] D. Bonheure, P. d'Avenia & A. Pomponio, *On the electrostatic Born-Infeld equation with extended charges*, to appear on Comm. Math. Phys.

[6] D. Bonheure, A. Derlet & C. De Coster, *Infinitely many radial solutions of a mean curvature equation in Lorentz-Minkowski space*, Rend. Istit. Mat. Univ. Trieste, 44(2012), 259–284.

[7] M. Born, *Modified field equations with a finite radius of the electron*, Nature, 132(1933), 282.

[8] M. Born, *On the quantum theory of the electromagnetic field*, Proc. Roy. Soc. London, Ser. A, 143(1934), 410–437.

[9] M. Born & L. Infeld, *Foundations of the new field theory*, Nature, 132(1933), 1004.

[10] M. Born & L. Infeld, *Foundations of the new field theory*, Proc. Roy. Soc. London, Ser. A, 144(1934), 425–451.

[11] P. d'Avenia & L. Pisani, *Nonlinear Klein-Gordon equations coupled with Born-Infeld type equations*, Electron. J. Differential Equations 2002, No. 26, 13 pp.

[12] K. Ecker, *Area maximizing hypersurfaces in Minkowski space having an isolated singularity*, Manuscr. Math., 56(1986), 375–397.

[13] D. Fortunato, L. Orsina & L. Pisani, *Born-Infeld type equations for electrostatic fields*, J. Math. Phys., 43(2002), 5698–5706.

[14] G.W. Gibbons, *Born-Infeld particles and Dirichlet p-branes*, Nuclear Phys. B, 514(1998), 603–639.

[15] M.K.-H. Kiessling, *On the quasi-linear elliptic PDE* $-\nabla \cdot (\nabla u/\sqrt{1-|\nabla u|^2}\,) = 4\pi \sum_k a_k \delta_{s_k}$ *in physics and geometry*, Comm. Math. Phys., 314(2012), 509–523.

[16] J. Mawhin, *Nonlinear boundary value problems involving the extrinsic mean curvature operator*, Math. Bohem., 139(2014), 299–313.

[17] D. Mugnai, *Coupled Klein-Gordon and Born-Infeld type equations: looking for solitary waves*, Proc. Roy. Soc. London, Ser. A, Math. Phys. Eng. Sci, 460(2004), 1519–1527.

[18] E. Serra & P. Tilli, *Monotonicity constraints and supercritical Neumann problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 28(2011), 63–74.

[19] A. Szulkin, *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 3(1986), 77–109.

[20] N.S. Trudinger, *Linear elliptic operators with measurable coefficients*, Ann. Scuola Norm. Sup. Pisa, 27(3)(1973), 265–308.

[21] Y. Yu, *Solitary waves for nonlinear Klein-Gordon equations coupled with Born-Infeld theory*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27(2010), 351–376.

[22] F. Wang, *Solitary waves for the coupled nonlinear Klein-Gordon and Born-Infeld type equations*, Electron. J. Differential Equations, 2012, no. 82, 12 pp.