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On a five critical points theorem

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Abstract. We present in a formal way a five critical points theorem, whose statement has never been given before, although it was applied several times.

1. Introduction

A very classical result attributed to Krasnoselskii states that

if $\Phi \in C^1(\mathbb{R}^N, \mathbb{R})$ *is such that* $\lim_{\|x\|\to\infty} \Phi(x) = \infty$ *and has a non degenerate critical point different from the global minimum, then* Φ *has a third critical point.*

Although Krasnoselskii didn't write this result precisely in this way, it is a consequence of [11, Lemma II.6.5]. After that, several developments and generalizations have been produced, mainly for showing multiplicity of solutions to partial differential equations. Of course, this implies a passage from $\mathbb{R}^N$ to a Hilbert or Banach setting, and also to a nonsmooth setting. There are several contributions in this directions, but surely some fundamental ones can be found in [1, 3, 4, 5, 6, 12, 14, 30, 35].

On the other hand, it is clear that the passage from $\mathbb{R}^N$ to an infinite–dimensional setting requires to add some *a priori* compactness condition, such as the Palais–Smale condition, or the request that some operators are of class $(S)_+$ (see [10, Definition 3.2.55(b)]). With conditions of this type, the established theorems read like

if X *is a Banach space and* $\Phi \in C^1(X, \mathbb{R})$ *has two local minima in* $X, +TA$, *then* Φ *has a third critical point*,

see [1, 3, 6, 12, 30], or like

if X *is a Banach (or Hilbert) space and* $\Phi + \lambda\Psi \in C^1(X, \mathbb{R})$ *is coercive*, $+TA$, *then there exists an open interval* $\Lambda \subseteq [0, \infty)$ *and* $\sigma > 0$ *such that* $\Phi + \lambda\Psi$ *has three critical points with norm less than* σ *for all* $\lambda \in \Lambda$,

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see [4, 5, 14, 34, 35]. Here, $+TA$ means that one needs some technical assumption, like the Palais–Smale condition, or like a set of minmax inequalities relating values of the functional in some subsets of X.

Main applications of theorems of these types are multiplicity results for problems of the form

$$\begin{cases} -\Delta_p u = g(\lambda, x, u) & \text{in } \Omega\,, \\ u = 0 \text{ or } \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$ is the p-Laplacian operator, $1 < p < \infty$, Ω is a bounded (sufficiently smooth) domain of $\mathbb{R}^N$, g satisfies suitable conditions and ν is the unit outward normal to Ω.

In this note we propose another abstract theorem which provides the existence of five critical points in a Hilbert setting, so that we can handle problems of the form

$$(1) \qquad \begin{cases} Lu = \lambda u + g(x, u) & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where Ω is a domain of $\mathbb{R}^N$, $N \geq 1$, $\lambda \in \mathbb{R}$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given superlinear function, L is an elliptic operator in a Hilbert space and B is a boundary one.

The first prototype we have in mind is

$$L = \Delta \quad \text{(the Laplace operator) and} \quad Bu = u\,,$$

treated in [19, 23, 24, 28, 38], but other ones can be handled, for instance

$$Lu = \Delta^2 u + c\Delta u \quad \text{and} \quad Bu = \{u, \Delta u\}\,,$$

see [25], or

$$Lu = (-\Delta)^{1/2} u \quad \text{and} \quad Bu = u\,,$$

considered in [26], or

$$Lu = -\mathcal{L}_K u \quad \text{and} \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\,,$$

where $\mathcal{L}_K u$ is a nonlocal operator defined as

$$\mathcal{L}_K u(x) = \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))\, K(y)\, dy \quad , \quad x \in \mathbb{R}^N\,,$$

and $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a function with the properties that

$mK \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^2, 1\}$,

there exists $\theta > 0$ such that $K(x) \geq \theta |x|^{-(N+2s)}$ for any $x \in \mathbb{R}^n \setminus \{0\}$,

see [21]. A model for K is given by $K(x) = |x|^{-(N+2s)}$. In this case $\mathcal{L}_K$ is the fractional Laplace operator $-(-\Delta)^s$.

Moreover, also a non smooth setting can be considered, and so one gets applications to variational inequalities of the form

$$\begin{cases} u \in K\,, \\ \langle u, v - u \rangle \geq \lambda \displaystyle\int_\Omega u(v-u)\, dx + \int_\Omega g(x,u)(v-u)\, dx \quad \forall\, v \in K\,, \end{cases}$$

where K is a closed convex set in a Hilbert space and $\langle\, ,\, \rangle$ denotes the inner product in H, see [13]. On the other hand, one can handle also *reversed* variational inequalities, i.e. problems of the form

$$\begin{cases} u \in K\ , \\ \langle u, v-u\rangle \leq \lambda \displaystyle\int_\Omega u(v-u)\, dx + \int_\Omega g(x,u)(v-u)\, dx \quad \forall\, v \in K\ , \end{cases}$$

see [15, 17, 25].

Without going into the technical assumptions on g, in the established results we have $g(x,t) \sim |t|^{p-2}t$, as in the seminal paper by Ambrosetti–Rabinowitz [2] for $L = -\Delta$ with Dirichlet boundary conditions, where the existence of two nontrivial solutions was proved for $\lambda = 0$ (but the same holds if $\lambda < \lambda_1$, λ_1 being the first eigenvalue of $-\Delta$ in $H^1_0(\Omega)$). Some twenty years later, Wang [39] proved the existence of three nontrivial solutions essentially under the same assumptions.

Since then, thousands of papers have established other multiplicity results for $\lambda = 0$ or $\lambda < \lambda_1$ (see also Mugnai–Papageorgiou [27] for the case of a (p,q)-Laplacian operator). However, not much was done for $\lambda > \lambda_1$. The first result in proving a result in the spirit of Wang's for $\lambda > \lambda_1$ and close to an eigenvalue can be found in Mugnai [24], whose result is complemented by Rabinowitz–Su–Wang in [33]. However, while in [33] $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is assumed to be of class C^1, in [24] less regularity conditions on $g : \Omega \times \mathbb{R} \to \mathbb{R}$ are required, with the "good" extended Ambrosetti–Rabinowitz condition (see Mugnai [22]).

The multiplicity result in [24] is proved by gluing a classical Linking Theorem with another Linking–type theorem of mixed type proved by Marino–Saccon in [19]. After that, this method, already used in [29], has been employed in [13, 15, 18, 17, 21, 23, 26, 25, 28, 36, 37, 38].

In order to state the promised abstract theorem, we need to recall some tools. First, some notations: by H we denote the underlying Hilbert space with $H \hookrightarrow\hookrightarrow L^2(\Omega)$, $B(r) = \{x \in H\ :\ \|x\| < r\}$ is the open ball centered at 0 with radius r, and $S(r) = \{x \in H\ :\ \|x\| = r\}$. We start from a special version of the Linking Theorem (see [31, Theorem 5.3] or [32, Theorem 1.1]), in which the *a priori* estimate on the critical value will be crucial to prove Theorem 1.4.

Theorem 1.1 (Linking Theorem)**.** *Suppose that $H = H_1 \oplus H_2$ with $\dim H_1 < \infty$, and let $f \in C^1(H,\mathbb{R})$. If*

i) $f(0) = 0$*;*

ii) there exist $\rho, \alpha > 0$ such that $\inf f(S(\rho) \cap H_2) \geq \alpha$ *and $\exists\ R > \rho$, $e \in H_2$ with $\|e\| = 1$ such that* $\sup f(\Sigma_R) \leq 0$*, where*

$$\Sigma_R = \partial_{H_1 \oplus \mathrm{span}(e)}\, \Delta_R \qquad \textit{and}$$

$$\Delta_R = \{u + te : u \in H_1, t > 0, \|u + te\| \leq R\}\ .$$

iii) $(PS)_\beta$ *holds for*

$$\beta = \inf_{h\in\Gamma} \sup_{u\in\Delta_R} f(h(u))$$

and

$$\Gamma = \{h \in C(\Delta_R, H) : h_{|\Sigma_R} = Id\}\ .$$

Then β is a critical value for f with

$$\alpha \leq \beta \leq \sup_{u\in\Delta_R} f(u)\ .$$

As usual, here $(PS)_\beta$ stands for the Palais–Smale condition at level $\beta \in \mathbb{R}$, namely: every sequence $(u_n)_n$ such that $f(u_n) \to \beta$ and $f'(u_n) \to 0$ as $n \to \infty$ has a converging subsequence.

Now, let us recall the ∇-condition and the ∇-Theorem from [19].

Definition 1.2 (∇-condition)**.** Let $f : H \to \mathbb{R}$ be a C^1 function, M a closed subspace of H, $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$.

We say that *condition* (∇)-(f, M, a, b) *holds* if there exists $\gamma > 0$ such that

$$\inf \{\|P_M \nabla f(u)\| \,:\, a \le f(u) \le b,\ d(u, M) \le \gamma\} > 0 \ ,$$

where $P_M : H \to M$ is the orthogonal projection of H onto M and $d(u, M) = \inf_{z \in M} d(u, z)$ denotes the distance of u from the subspace M.

Theorem 1.3 (∇-Theorem)**.** *Let $H = H_1 \oplus H_2 \oplus H_3$ with $\dim H_i < \infty$ for $i = 1, 2$. Denote with $P_i : H \to H_i$ the orthogonal projection of H onto H_i and let $f \in C^1(H, \mathbb{R})$.*

Let ρ, ρ', ρ'', ρ_1 be such that $\rho_1 > 0$, $0 \le \rho' < \rho < \rho''$ and define

$$\Delta = \{u \in H_1 \oplus H_2 \,:\, \rho' \le \|P_2 u\| \le \rho'',\ \|P_1 u\| \le \rho_1\} \ ,$$

$$T = \partial_{H_1 \oplus H_2} \Delta \quad , \quad S_{23}(\rho) = \{u \in H_2 \oplus H_3 \,:\, \|u\| = \rho\}$$

and

$$B_{23}(\rho) = \{u \in H_2 \oplus H_3 \,:\, \|u\| \le \rho\} \ .$$

Suppose that

$$a' := \sup f(T) < \inf f(S_{23}(\rho)) =: a'' \ .$$

Let a and b be such that $a\prime < a < a''$ and $b > \sup f(\Delta)$. Suppose that condition (∇)-$(f, H_1 \oplus H_3, a, b)$ holds and $(PS)_c$ holds for all $c \in [a, b]$. Then, f has at least two critical points in $f^{-1}([a, b])$. Moreover, if

$$\inf f(B_{23}(\rho)) > -\infty$$

and $(PS)_c$ holds for all $c \in [a_1, b]$ with

$$a_1 < \inf f(B_{23}(\rho)) \ ,$$

then f has another critical level in $[a_1, a\prime]$.

Now, let us go back to problem (1), and let us suppose that

(H): • L is a differential operator associated to a quadratic form $Q : H \times H \to \mathbb{R}$ in such a way that

$$\langle Lu, v\rangle_{H', H} = Q(u, v) \qquad \text{for every } \ u, v \in H \ .$$

• L^{-1} is a self-adjoint compact operator.

This implies that L has a sequence $(\lambda_i)_{i \in \mathbb{N}}$ of (possibly negative) eigenvalues diverging to $+\infty$ with associated eigenfunctions $(e_i)_{i \in \mathbb{N}}$ which are a Hilbert basis for $L^2(\Omega)$. Moreover, for every $i \in$, set

$$\mathcal{H}_i = \operatorname{span}(e_1, \cdots, e_i) \quad \text{ and } \quad \mathcal{H}_i^\perp = \overline{\operatorname{span}(e_{i+1}, \cdots)} \ .$$

Then, by a Fourier decomposition, we immediately have that

$$Q(u, u) \le \lambda_i \int_\Omega u^2 \, dx \qquad \text{for all } \ u \in H_i$$

and

$$Q(u,u) \geq \lambda_{i+1} \int_\Omega u^2\,dx \qquad \text{for all } \; u \in H_i^\perp \; .$$

At this point, it is clear that problem (1) is variational and its solutions are critical points of the C^1 functional $f_\lambda : H \to \mathbb{R}$ defined by

$$f_\lambda(u) = \frac{1}{2}\, Q(u,u) - \frac{\lambda}{2} \int_\Omega u^2\,dx - \int_\Omega G(x,u)\,dx \; ,$$

where $G(x,s) = \int_0^s g(x,t)\,dt$.

Hence, the existence of solutions to problem (1) can be showed by applying the following abstract theorem, whose statement was hidden in the references cited so far.

Theorem 1.4 (Five critical points Theorem)**.** *Assume* **(H)** *and suppose that for some $j \geq i \geq 2$ we have $\lambda_{i-1} < \lambda < \lambda_i = \cdots = \lambda_j < \lambda_{j+1}$. Suppose that the hypotheses of Theorem 1.3 hold with $H_1 = \mathcal{H}_{i-1}$, $H_2 = \mathrm{span}(e_i, \cdots, e_j)$, $H_3 = \mathcal{H}_j^\perp$ and that there exist $\delta_i > 0$, $\rho_1 > 0$ such that $\forall\, \lambda \in (\lambda_i - \delta_i, \lambda_i)$*

$$\inf f_\lambda\left(S_{i-1}^\perp(\rho_1)\right) > \sup f_\lambda(H_j) \; .$$

Then f_λ has at least four critical points for all $\lambda \in (\lambda_i - \delta_i, \lambda_i)$.

Moreover, if in addition there exists $R_1 > \rho_1$ such that

$$\inf f_\lambda\left(S_j^\perp(\rho_1)\right) > \sup f_\lambda(\mathcal{T}_{j,j+1}(R_1)) \; ,$$

$\sup f_\lambda(\mathcal{T}_{j,j+1}(R_1)) < \mathfrak{a} < \inf f_\lambda\left(S_j^\perp(\rho_1)\right)$, $\mathfrak{b} > \sup f_\lambda\left(\overline{B_{j+1}(R_1)}\right)$ *and* (∇)-$(f_\lambda, H_j \oplus H_{j+1}^\perp, \mathfrak{a}, \mathfrak{b})$ *holds, then f has at least five critical points.*

Here, for any $h \leq k$ in $\mathbb{N}$, we have set

$$S_k^\perp(\rho_1) \; = \{u \in H_k^\perp \,:\, \|u\| = \rho_1\} \; ,$$
$$B_k(R_1) = \{u \in H_k \,:\, \|u\| < R_1\}$$

and

$$T_{h,k}(R_1) = \{u \in H_h \,:\, \|u\| \leq R_1\} \cup \{u \in H_k \,:\, \|u\| = R_1\} \; .$$

Proof. By Theorem 1.3, there exist two critical points u_1, u_2 for f_λ with $f_\lambda(u_i) \leq \sup f_\lambda(H_j)$ and a third critical point u_0 with $f_\lambda(u_0) < f_\lambda(u_i)$, $i = 1, 2$.

Moreover, Theorem 1.1 ensure the existence of a critical point u_3 with $f_\lambda(u_3) \geq \inf f_\lambda\left(S_j^\perp(\rho_1)\right)$, so that $u_3 \neq u_i$, $i = 0, 1, 2$.

In addition, if (∇)-$(f_\lambda, H_j \oplus H_{j+1}^\perp, \mathfrak{a}, \mathfrak{b})$ holds, applying again Theorem 1.3 in place of Theorem 1.1, jointly with u_3 we find another critical point u_4.

□

Unfortunately, in spite of the rich theoretical result established in the previous theorem, in the situations studied so far, the second (∇)-condition cannot be verified and the critical level in $[a_1, a']$ (see Theorem 1.3) could be $0 = f_\lambda(0)$, so that we cannot guarantee the existence of another nontrivial solution. So, the usual result which has been proved sounds like the following

Theorem 1.5 (Typical result for three nontrivial solutions)**.** *Under some technical assumptions on g, there exists $\lambda_i > 0$ such that problem* (1) *has at least three nontrivial solutions for all $\lambda \in (\lambda_i - \delta_i, \lambda_i)$.*

Here, "technical assumptions on g" mean classical superlinear and subcritical growth conditions.

A "nonsmooth" version of this theorem is available, as well, replacing the notion of gradient with the one of weak slope introduced in [8], or the one of Frechét sub/super–differential, see [7]. The related multiplicity results are established in [13] and [15], respectively.

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