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Some results on the Chern-Simons-Schrödinger equation

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Abstract. In this review paper we present some results on the Chern-Simons-Schrödinger equation obtained recently in several papers.

1. INTRODUCTION

In this review paper we present some results on the following Chern-Simons-Schrödinger system

$$
\begin{aligned}
&iD_0\phi + (D_1D_1 + D_2D_2)\phi + |\phi|^{p-1}\phi = 0 \, ,\\
&\partial_0 A_1 - \partial_1 A_0 = \mathrm{Im}(\bar{\phi} D_2 \phi) \, ,\\
&\partial_0 A_2 - \partial_2 A_0 = -\mathrm{Im}(\bar{\phi} D_1 \phi) \, ,\\
&\partial_1 A_2 - \partial_2 A_1 = \frac{1}{2}\, |\phi|^2 \, .
\end{aligned}
\tag{1.1}
$$

This system describes nonrelativistic matter interacting with Chern-Simons gauge fields in the plane, under a suitable ansatz. For the reader convenience, in Section 2, we will give the physical motivations of this model, following [7].

This model was first proposed and studied in [15, 16], and sometimes has received the name of Chern-Simons-Schrödinger equation. The initial value problem, well-posedness, global existence and blow-up, scattering, etc. have been addressed in [2, 12, 14, 22, 24] for the case $p = 3$. See also [21] for a global existence result in the defocusing case, and [6] for a uniqueness result to the infinite radial hierarchy.

The existence of stationary states for (1.1) and general $p > 1$ has been studied in [5] for the regular case (see also [7, 13, 25, 26]). Very recently, in [4] the case with a vortex point has been taken into account. Consider the ansatz:

$$
\phi = u(r)e^{i(N\theta+\omega t)} \quad , \quad A_0 = A_0(r) \, ,
$$

$$
A_1 = -\frac{x_2}{r^2}\, h(r) \qquad , \quad A_2 = \frac{x_1}{r^2}\, h(r)
$$

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Here (r,θ) are the polar coordinates of $\mathbb{R}^2$, and $N\in\mathbb{N}\cup\{0\}$ is the order of the vortex at the origin ($N=0$ corresponds to the regular case).

In [4] it is found that u solves the equation:

$$-\Delta u(x)+\omega\, u+\frac{(h_u(|x|)-N)^2}{|x|^2}\,u+A_0(|x|)u(x)=|u(x)|^{p-1}u(x)\quad,\quad x\in\mathbb{R}^2\ ,$$

where

$$h_u(r)=\frac{1}{2}\int_0^r su^2(s)\,ds\ , \tag{1.2}$$

and

$$A_0(r)=\xi+\int_r^{+\infty}\frac{h_u(s)-N}{s}u^2(s)\,ds\quad,\quad \xi\in\mathbb{R}\ .$$

The value ξ above appears as an integration constant. Observe that, as usual in Chern-Simons theory, problem (1.1) is invariant under gauge transformation,

$$\phi\to\phi e^{i\chi}\quad,\quad A_\mu\to A_\mu-\partial_\mu\chi\ ,$$

for any arbitrary C^∞ function χ. Therefore, without loss of generality, we can assume $\xi=0$; otherwise it suffices to use the gauge invariance with $\chi=\xi t$. Then, our problem becomes:

$$\begin{aligned}&-\Delta u(x)+\omega\, u+\frac{(h_u(|x|)-N)^2}{|x|^2}\,u+\\ &+\left(\int_{|x|}^{+\infty}\frac{h_u(s)-N}{s}\,u^2(s)\,ds\right)u(x)=|u(x)|^{p-1}u(x)\ .\end{aligned}\tag{$\mathcal{CSS}$}$$

Observe that $(\mathcal{CSS})$ is a nonlocal equation. In [4] it is shown that $(\mathcal{CSS})$ is indeed the Euler-Lagrange equation of the energy functional $I_\omega:\mathcal{H}\to\mathbb{R}$,

$$\begin{aligned}I_\omega(u)=&\frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla u(x)|^2+\omega u^2(x)\right)\,dx+\\ &+\frac{1}{2}\int_{\mathbb{R}^2}\frac{u^2(x)}{|x|^2}\,(h_u(r)-N)^2\,dx-\frac{1}{p+1}\int_{\mathbb{R}^2}|u(x)|^{p+1}\,dx\ .\end{aligned}$$

The Hilbert space $\mathcal{H}$ is defined as:

$$\mathcal{H}=\left\{u\in H^1_r(\mathbb{R}^2)\ :\ \int_{\mathbb{R}^2}\frac{u^2(x)}{|x|^2}\,dx<+\infty\right\}\ , \tag{1.3}$$

endowed by the norm

$$\|u\|^2_{\mathcal{H}}=\int_{\mathbb{R}^2}|\nabla u(x)|^2+\left(1+\frac{1}{|x|^2}\right)u^2(x)\,dx\ .$$

Let us observe that the energy functional I_ω presents a competition between the nonlocal term and the local nonlinearity of power-type. The study of the behavior of the functional under this competition is one of the main motivations of this paper. For $p>3$, it is known that I_ω is unbounded from below, so it exhibits a mountain-pass geometry (see [5, 13] for the case $N=0$ and [4, Section 5] for $N\in\mathbb{N}$). In a certain sense, in this case the local nonlinearity dominates the nonlocal term. However the existence of a solution is not so direct, since for $p\in(3,5)$ the (PS) property is not known to hold. This problem is bypassed by combining the so-called monotonicity trick of Struwe [29] with a Pohozaev identity.

A special case in the above equation is $p = 3$: in this case, solutions have been explicitly found in [5, 4] as optimizers of a certain inequality. An alternative approach would be to pass to a self-dual equation, which leads to a Liouville equation in $\mathbb{R}^2$, singular if $N > 0$.

Whenever $p \in (1, 3)$, the nonlocal term prevails over the local nonlinearity, in a certain sense. This case has been considered in [25], for $N = 0$, and in [18], for $N \neq 0$, where in particular the authors have studied whether I_ω is bounded from below or not. The main purpose of this review paper is to present the conclusions of [18, 25]. As we shall see, the situation is quite rich and unexpected a priori, and very different from the usual Nonlinear Schrödinger Equation. We shall show, in particular, the existence of a threshold value ω_0 such that I_ω is bounded from below if $\omega \geq \omega_0$, and it is not for $\omega \in (0, \omega_0)$. But, in our opinion, what is most surprising is that ω_0 has an explicit expression, namely:

$$\omega_0 = \frac{3-p}{3+p}\, 3^{(p-1)/2(3-p)}\, 2^{2/(3-p)} \left(\frac{m^2(3+p)}{p-1}\right)^{-(p-1)/2(3-p)} ,$$

with

$$m = \int_{-\infty}^{+\infty} \left(\frac{2}{p+1}\, \cosh^2\left(\frac{p-1}{2}\, r\right)\right)^{2/(1-p)} dr\, .$$

Let us give an idea of the proofs. It is not difficult to show that I_ω is coercive when the problem is posed on a bounded domain. So, there exists a minimizer u_n on the ball $B(0, n)$ with Dirichlet boundary conditions. To prove boundedness of u_n, the problem is the possible loss of mass at infinity as $n \to +\infty$. The core of our proofs is a detailed study of the behavior of those masses. We are able to show that, if unbounded, the sequence u_n behaves as a soliton, if u_n is interpreted as a function of a single real variable. The proof uses a careful study of the level sets of u_n, which take into account the effect of the nonlocal term. Then, the energy functional I_ω admits a natural approximation through a convenient limit functional. Finally, the solutions of that limit functional, and their energy, can be found explicitly, so we can find ω_0. See Section 2 for an heuristic explanation of the proof and a derivation of the limit functional.

Regarding the existence of solutions, a priori, the global minimizer could correspond to the zero solution. And indeed this is the case for large ω. Instead, we show that $\inf I_\omega < 0$ if $\omega > \omega_0$ is close to the threshold value. Therefore, the global minimizer is not trivial, and corresponds to a positive solution. The mountain pass theorem will provide the existence of a second positive solution.

If $\omega < \omega_0$, I_ω is unbounded from below, and hence the geometric assumptions of the mountain-pass theorem are satisfied. However, the boundedness of (PS) sequences seems to be a hard question in this case. Solutions are found for almost all values of $\omega \in (0, \omega_0)$, by using the well-known monotonicity trick of Struwe [29] (see also [17]).

The main results are the following:

Theorem 1.1 ([18, 25])**.** *Let $N \in \mathbb{N} \cup \{0\}$. For ω_0 as given in* (4.11)*, there holds:*

(*i*) *if $\omega \in (0, \omega_0)$, then I_ω is unbounded from below;*

(*ii*) *if $\omega = \omega_0$, then I_{ω_0} is bounded from below, not coercive and* $\inf I_{\omega_0} < 0$*;*

(*iii*) *if $\omega > \omega_0$, then I_ω is bounded from below and coercive.*

Regarding the existence of solutions, we have the following results:

Theorem 1.2 ([18, 25])**.** *Let $N \in \mathbb{N} \cup \{0\}$. There exist $\bar{\omega} > \tilde{\omega} > \omega_0$ such that:*

- (i) *if $\omega > \bar{\omega}$, then $(\mathcal{CSS})$ has no solutions different from zero;*
- (ii) *if $\omega \in (\omega_0, \tilde{\omega})$, then $(\mathcal{CSS})$ admits at least two positive solutions: one of them is a global minimizer for I_ω and the other is a mountain-pass solution;*
- (iii) *for almost every $\omega \in (0, \omega_0)$, $(\mathcal{CSS})$ admits a positive solution.*

We finally recall a result contained in [7] which deals with the case $N = 0$ and shows that $(\mathcal{CSS})$ admits many solutions, as we want, if we take ω sufficiently small. This result can be obtained as a particular case of [7, Theorem 1.1], see also Remark 1.2 therein.

Theorem 1.3 ([7])**.** *Let $N = 0$. For every $n \in \mathbb{N}$ there exists $\omega_n > 0$ such that, for every $\omega \in (0, \omega_n)$, equation $(\mathcal{CSS})$ admits (at least) n distinct solutions.*

Remark 1.4. Hence, by means of Theorems 1.2 and 1.3, we have now a more precise picture on the existence of solutions of $(\mathcal{CSS})$, in the case $N = 0$. There exist, indeed, $\omega_0 < \tilde{\omega} < \bar{\omega}$ and a decreasing sequence $(\omega_n)_{n \geq 1}$ such that the following happens

for all $\omega \in (\omega_{n+1}, \omega_n)$	for a.e. $\omega \in (\omega_1, \omega_0)$	for all $\omega \in (\omega_0, \tilde{\omega})$	for all $\omega > \bar{\omega}$
at least n (possibly sign- changing) solutions	at least a positive solution	at least two positive solutions	no nontrivial solutions

The rest of the paper is organized as follows. In Section 2, we give the physical motivations of the Chern-Simon-Schrödinger model and we present a deduction of $(\mathcal{CSS})$, for simplicity in the case $N = 0$. Section 3 is devoted to some notations and preliminary results. In Section 4 we study the limit functional and, finally, in Section 5 we prove Theorems 1.1 and 1.2.

In this review paper, there is no new result and we just present main ideas of these topics and we sketch some proofs. We refer to [7, 18, 25, 26] for more details.

2. Physical motivation

Following [7], in this section we give the physical motivations of Chern-Simon-Schrödinger model and we derive the equation $(\mathcal{CSS})$, for simplicity, in the regular case, namely when $N = 0$.

Let us consider the three dimensional Lorentz space time $\mathbb{R}^{1,2}$ with metric tensor $\mathrm{diag}(+1, -1, -1)$ and coordinates $x^\mu = (ct, x^1, x^2)$, where c is the speed of light. As usual, we adopt the Einstein convention on repeated indices, where greek indices always vary in $\{0, 1, 2\}$ while latin ones run in $\{1, 2\}$. In particular we will always distinguish in the following between covariant and contravariant indices, which are obtained ones from the others by using the metric.

The starting point for our equation is the Schrödinger Lagrangian density of the matter field

$$\mathcal{L}_{\mathrm{S}}(\psi) = i\hbar\overline{\psi}\partial_t\psi - \frac{\hbar^2}{2m}\,|\nabla\psi|^2 + 2W(\psi) \tag{2.1}$$

where $\psi : \mathbb{R}^{1,2} \to \mathbb{C}$, $m > 0$ is the mass parameter, $\hbar = h/2\pi$ is the normalized Planck constant and $W : \mathbb{C} \to \mathbb{R}$ is a nonlinearity of type $W(\psi) = W(|\psi|)$ which describes the interaction among many particles.

Let us define the electromagnetic tensor as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

where $A^\mu = (A^0, \mathbf{A}) = (A^0, A^1, A^2)$ is the gauge potential. We observe explicitly that, in virtue of the choice of the metric,

$$(A^0, A^1, A^2) = (A_0, -A_1, -A_2) \tag{2.2}$$

and indeed we will use, case by case, the notation which is more convenient. In vectorial notation the electromagnetic field is given by

$$\mathbf{E} = -\nabla A^0 - \frac{1}{c}\,\partial_t \mathbf{A} \quad , \quad B = \nabla \times \mathbf{A}\ . \tag{2.3}$$

To study the interaction between the matter (expressed by the wave function ψ) and the electromagnetic field $(\mathbf{E}, B)$ given by (2.3), we consider the gauge (or Weyl) covariant derivatives

$$D_t = \partial_t + \frac{ie}{\hbar}\,A^0 \quad , \quad \mathbf{D} = \nabla - \frac{ie}{\hbar c}\,\mathbf{A}\ ,$$

where e is a coupling constant. We point out that these operators are obtained by the so called *minimal coupling rule*, see e.g. [9, 23]. Then we substitute in (2.1) the derivatives with the covariant ones, getting

$$\tilde{\mathcal{L}}_{\mathrm{S}}(\psi, A^0, \mathbf{A}) = i\hbar\overline{\psi} D_t\psi - \frac{\hbar^2}{2m}\,|\mathbf{D}\psi|^2 + 2W(\psi)\ .$$

However to obtain the complete Lagrangian we need to consider also the term involving the gauge potentials, since they are unknown. In $\mathbb{R}^{1,2}$ one can take the more general term

$$\mathcal{L}_{\mathrm{MCS}} = \underbrace{-\frac{1}{4}\,F^{\mu\nu}F_{\mu\nu}}_{\mathcal{L}_{\mathrm{Max}}} + \underbrace{\frac{\kappa}{4}\,\varepsilon^{\mu\alpha\beta}A_\mu F_{\alpha\beta}}_{\mathcal{L}_{\mathrm{CS}}} \tag{2.4}$$

which involves not only the usual Maxwell term but also the so-called Chern-Simons term. Here ε is the Levi-Civita tensor and $\kappa \in \mathbb{R}$ is a parameter which controls the Chern-Simons term. Strictly speaking (2.4) is the Lagrangian of the gauge potentials in the vacuum.

Thus the total Lagrangian density is

$$\mathcal{L}_{\mathrm{tot}}(\psi, A^0, \mathbf{A}) := \mathcal{L}_{\mathrm{MCS}}(A^0, \mathbf{A}) + \tilde{\mathcal{L}}_{\mathrm{S}}(\psi, A^0, \mathbf{A}) =$$

$$-\frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} + \frac{\kappa}{4}\,\varepsilon^{\mu\alpha\beta}A_\mu F_{\alpha\beta} + i\hbar\overline{\psi} D_t\psi - \frac{\hbar^2}{2m}\,|\mathbf{D}\psi|^2 + 2W(\psi)\ .$$

Due to the presence of the Chern-Simon term, the total Lagrangian is not invariant with respect to the gauge transformations

$$A^\mu \to A^\mu + \partial^\mu\chi \quad , \quad \psi \to \psi e^{-(ie/\hbar c)\chi} \quad , \quad \chi \in C^\infty(\mathbb{R}^{1,2})\ , \tag{2.5}$$

nevertheless, its Euler-Lagrange equations are gauge invariant.

As discussed in [15], the Chern-Simons electrodynamics is obtained by taking the formal limit $|\kappa| \to \infty$ in the topologically massive model. Indeed at large distances and low energies the lower derivatives of the Chern-Simons term dominate the higher derivative appearing in the Maxwell term; hence this last term becomes negligible and the above total Lagrangian reduces to

$$\mathcal{L}(\psi, A^0, \mathbf{A}) = \frac{\kappa}{4}\, \varepsilon^{\mu\alpha\beta} A_\mu F_{\alpha\beta} + i\hbar\overline{\psi} D_t \psi - \frac{\hbar^2}{2m}\, |\mathbf{D}\psi|^2 + 2W(\psi)\ .$$

For this and further discussions see also [8, 11, 16, 27, 30]. This is the Lagrangian we are interested in. By taking the variations of the action functional $\mathcal{S} = \iint \mathcal{L}\, dxdt$ with respect to ψ, A_μ, recalling (2.2), we have the following Euler-Lagrange equations

$$\begin{aligned} & i\hbar D_t\psi + \frac{\hbar^2}{2m}\, \mathbf{D}^2\psi = -W'(\psi) \\ & \kappa(\partial_1 A_2 - \partial_2 A_1) = e|\psi|^2 \\ & \kappa(\partial_2 A_0 - \partial_0 A_2) = \frac{e}{c}\, \frac{\hbar}{m}\, \mathfrak{Im}(\overline{\psi}\, D_1\psi) \\ & \kappa(\partial_0 A_1 - \partial_1 A_0) = \frac{e}{c}\, \frac{\hbar}{m}\, \mathfrak{Im}(\overline{\psi}\, D_2\psi)\ . \end{aligned} \tag{2.6}$$

Note that these equations are invariant under the gauge transformations (2.5).

If we define

$$J^\mu = (c\rho, \mathbf{J}) := \left(c\overline{\psi}\,\psi, \frac{\hbar}{2mi}(\overline{\psi}\,\mathbf{D}\psi - \psi\overline{\mathbf{D}\psi}) \right) = \left(c|\psi|^2, \frac{\hbar}{m}\, \mathfrak{Im}(\overline{\psi}\,\mathbf{D}\psi) \right)\ ,$$

the last three equations in (2.6) can be written, since $\mathfrak{Im}(\overline{\psi}\, D^j\psi) = \mathfrak{Im}(\overline{\psi}\, D_j\psi)$, as

$$\frac{\kappa}{2}\, \varepsilon^{\mu\alpha\beta} F_{\alpha\beta} = \frac{e}{c}\, J^\mu \tag{2.7}$$

which are the gauge field equations of the Chern-Simons electrodynamics. Thus, J^μ can be interpreted as *a current density*. In particular, we have the continuity equation for the currents

$$\partial_\mu J^\mu = \partial_t\rho + \nabla\cdot\mathbf{J} = 0\ .$$

Moreover, in terms of the electromagnetic field $(\mathbf{E}, B)$, equations (2.7) are

$$\kappa c B = -eJ^0 \quad , \quad \kappa c E^1 = eJ^2 \quad , \quad \kappa c E^2 = -eJ^1\ .$$

The first equation yields the remarkable fact that $B = 0$ if and only if $\psi = 0$ and that any field configuration with total charge $Q(t) = e\int |\psi(t,x)|^2\, dx$ also carries a magnetic flux $\Phi(t) = \int B(t,x)\, dx$ given by

$$\Phi(t) = -\frac{1}{\kappa}\, Q(t)\ , \tag{2.8}$$

and indeed they are conserved quantities (i.e. constant in time) for "well behaved" ψ. Moreover the tying in (2.8) provides an explicit realization of anyons, see [19, 31]. On the other hand, the other two equations give the interesting identities

$$\nabla\cdot\mathbf{E} = \frac{e}{c\kappa}\, \nabla\times\mathbf{J} \quad \text{and} \quad \nabla\cdot\mathbf{J} = -\frac{c\kappa}{e}\, \nabla\times\mathbf{E}\ ,$$

so the matter and the electromagnetic field support each other.

However, it is convenient to write $\psi(t,x) = u(t,x)e^{iS(t,x)}$; hence by taking the variations of the action with respect to u, S, A_μ we get the following set of equations

$$
(2.9)\qquad
\begin{aligned}
&-\frac{\hbar^2}{2m}\,\Delta u + \Big(\hbar\partial_t S + eA^0 + \frac{\hbar^2}{2m}\,|\nabla S|^2 + \\
&\qquad + \frac{e^2}{2mc^2}\,|\mathbf{A}|^2 - \frac{\hbar e}{mc}\,\nabla S\cdot\mathbf{A}\Big)\,u = W'(u)\\
&\partial_t u^2 + \frac{\hbar}{m}\,\nabla\cdot\Big[\Big(\nabla S - \frac{e}{\hbar c}\,\mathbf{A}\Big)\,u^2\Big] = 0\\
&\kappa(\partial_1 A_2 - \partial_2 A_1) = eu^2\\
&\kappa(\partial_2 A_0 - \partial_0 A_2) = \frac{e\hbar}{cm}\Big(\partial_1 S + \frac{e}{\hbar c}\,A_1\Big)\,u^2\\
&\kappa(\partial_0 A_1 - \partial_1 A_0) = \frac{e\hbar}{cm}\Big(\partial_2 S + \frac{e}{\hbar c}\,A_2\Big)\,u^2\ .
\end{aligned}
$$

The second equation is of course a continuity equation (conservation law of currents), as $\mathcal{L}$ does not depend explicitly on S, but just on its derivatives.

In the static case $A^\mu = A^\mu(x)$, by using the Helmholtz decomposition and taking a suitable gauge, without loss of generality we can assume that $\lim_{|x|\to+\infty} A^0(x) = 0$ (if we suppose that such a limit exists) and that $\partial_j A^j = 0$ (Coulomb gauge). An interesting case appears when we take the ansatz of standing waves solutions $\psi(t,x) = u(x)e^{i\omega t}$. Then the previous equations (2.9) become

$$
(2.10)\qquad
\begin{aligned}
&-\frac{\hbar^2}{2m}\,\Delta u + \Big(\hbar\omega + eA^0 + \frac{e^2}{2mc^2}\,|\mathbf{A}|^2\Big)\,u = W'(u)\\
&\mathbf{A}\cdot\nabla u^2 = 0\\
&\kappa(\partial_1 A_2 - \partial_2 A_1) = eu^2\\
&\kappa\partial_2 A_0 = \frac{e^2}{mc^2}\,A_1 u^2\\
&\kappa\partial_1 A_0 = -\frac{e^2}{mc^2}\,A_2 u^2\ .
\end{aligned}
$$

Arguing as in [13], the Coulomb gauge and the third equation in (2.10) imply that A_1, A_2 are uniquely determined by u, since they satisfy

$$\Delta A_1 = -\frac{e}{\kappa}\,\partial_2 u^2 \quad , \quad \Delta A_2 = \frac{e}{\kappa}\,\partial_1 u^2\ .$$

Hence they are given by

$$A_1 = -\frac{e}{\kappa}\,G_2 * u^2 \quad , \quad A_2 = \frac{e}{\kappa}\,G_1 * u^2 \quad , \quad \text{where } G_i(x) = \frac{1}{2\pi}\,\frac{x^i}{|x|^2}\ .$$

Coming back to the last two equations in (2.10) we infer that

$$-\Delta A_0 = \frac{e^2}{\kappa mc^2}\,\big[\partial_1(A_2u^2) - \partial_2(A_1u^2)\big]$$

and hence

$$A_0 = \frac{e^2}{\kappa mc^2}\,\big[G_2 * (A_1u^2) - G_1 * (A_2u^2)\big]\ .$$

Observe also that from the second equation in (2.10) it follows that, up to the "trivial cases", the function u is radial if and only if $\mathbf{A}$ is a *tangential* vector field, i.e. of type

$$\mathbf{A} = \frac{e}{\kappa}\, h_u(x)\mathbf{t} \quad , \quad \text{where} \quad \mathbf{t} = (x^2/|x|^2, -x^1/|x|^2) \; .$$

Moreover, since the problem is invariant by translations, to avoid the related difficulties, we look for radial solutions u. Thus, from this choice, arguing as in [13, Lemma 3.3], it follows that $\mathbf{A}$ has to be invariant for the group action

$$\mathrm{T}_g \mathbf{A}(x) = g^{-1} \cdot \mathbf{A}(g(x)) \quad , \quad g \in O(2) \; ,$$

and this readily implies that h_u has to be a radial function. Summing up, whenever u is radial, the magnetic potential has to be necessarily written as

$$A^1(x) = \frac{e}{\kappa}\, \frac{x^2}{|x|^2}\, h_u(|x|) \quad , \quad A^2(x) = -\frac{e}{\kappa}\, \frac{x^1}{|x|^2}\, h_u(|x|) \; . \tag{2.11}$$

Finally, by (2.10) we see that

$$\nabla A^0 = -\frac{e^3}{mc^2\kappa^2}\, u^2(|x|) h_u(|x|)\mathbf{n} \quad , \quad \text{where } \mathbf{n} = (x^1/|x|^2, x^2/|x|^2) \; ;$$

in other words, the electric potential is radial and so can be written as

$$A^0(x) = A^0(|x|) \; . \tag{2.12}$$

Now we can find the explicit dependence of A^0 and h_u from the function u; indeed, by substituting (2.11) and (2.12) into (2.10) one find (assuming $h_u(0) = 0$, which is necessary to have $\mathbf{A}$ smooth)

$$h_u(|x|) = \int_0^{|x|} \tau u^2(\tau)\, d\tau \quad , \quad A^0(|x|) = \frac{e^3}{mc^2\kappa^2} \int_{|x|}^{\infty} \frac{u^2(\tau)}{\tau}\, h_u(\tau)\, d\tau \; .$$

With these expressions in hands the first equation in (2.10), the equation of the matter field, is

$$-\frac{\hbar^2}{2m}\, \Delta u + \hbar\omega u + \frac{e^4}{mc^2\kappa^2}\, u \int_{|x|}^{\infty} \frac{u^2(s)}{s}\, h_u(s)\, ds + \frac{e^4}{2mc^2\kappa^2}\, u \frac{h_u^2(|x|)}{|x|^2} = W'(u) \; ,$$

which is nothing but $(\mathcal{CSS})$ with $N = 0$, "normalizing" the constants $\hbar$ and $2m$ and taking $W'(u) = |u|^{p-1}u$.

3. PRELIMINARIES

Let us first fix some notations. We denote by $H^1_r(\mathbb{R}^2)$ the Sobolev space of radially symmetric functions, and $\|\cdot\|$ its usual norm. We denote by $\|u\|_{L^p}$ the usual Lebesgue norm in $\mathbb{R}^2$. Moreover, we will write $\|\cdot\|_{H^1(\mathbb{R})}$, $\|\cdot\|_{H^1(a,b)}$ to indicate the norms of the Sobolev spaces of dimension 1.

However our functional I_ω is defined in the space $\mathcal{H}$, defined in (1.3). Its norm will be denoted by $\|\cdot\|_{\mathcal{H}}$. In [4, Proposition 3.1] it is shown that

$$\mathcal{H} \subset \{u \in C(\mathbb{R}^2) \, : \, u(0) = 0\} \cap L^\infty(\mathbb{R}^2) \; .$$

If nothing is specified, strong and weak convergence of sequences of functions are assumed in the space $H^1(\mathbb{R}^2)$.

In our estimates, we will frequently denote by $C > 0$, $c > 0$ fixed constants, that may change from line to line, but are always independent of the variable under consideration. We also use the notations $O(1), o(1), O(\varepsilon), o(\varepsilon)$ to describe the asymptotic behaviors of quantities in a standard way. Finally the letters x, y indicate two-dimensional variables and r, s denote one-dimensional variables.

Let us start with the following proposition, proved in [5, 4]:

Proposition 3.1. *I_ω is a C^1 functional, and its critical points correspond to classical solutions of* $(\mathcal{CSS})$*.*

The next result is contained in [4, Proposition 3.4], and deals with the behavior of I_ω under weak limits.

Proposition 3.2. *Recalling the definition of h_u,* (1.2)*, let us define:*

$$K(u) = \frac{1}{2} \int_{\mathbb{R}^2} h_u^2(x) \frac{u^2(x)}{|x|^2} - 2N h_u(x) \, \frac{u^2(x)}{|x|^2} \, dx \, . \tag{3.1}$$

Then K and K' are weakly continuous in $\mathcal{H}$. As a consequence, I_ω is weak lower semicontinuous, and I'_ω is weakly continuous in $\mathcal{H}$.

Next lemma relates boundedness of sequences in $H^1(\mathbb{R}^2)$ and in $\mathcal{H}$, and will be very useful in Section 5.

Lemma 3.3. *The map K defined in* (3.1) *is actually well defined in $H^1(\mathbb{R}^2)$ and $K(u_n)$ is bounded if $\|u_n\|$ is bounded. As a consequence, for any sequence $u_n \in \mathcal{H}$ such that $I_\omega(u_n)$ is bounded from above, $\|u_n\|$ is bounded if and only if $\|u_n\|_{\mathcal{H}}$ is bounded.*

Proof. By [5], we only need to consider the term:

$$\int_{\mathbb{R}^2} h_u(x) \, \frac{u^2(x)}{|x|^2} \, dx = \pi \int_0^{+\infty} \frac{u^2(r)}{r} \left(\int_0^r s u^2(s) \, ds \right) dr \leq$$

$$\leq \pi \int_0^{+\infty} u^2(r) \left(\int_0^r u^2(s) \, ds \right) \, dr = \frac{\pi}{2} \left(\int_0^{+\infty} u^2(r) \, dr \right)^2 .$$

Observe now that:

$$2\pi \int_0^{+\infty} u^2(r) \, dr = \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|} \, dx \leq \int_{B(0,1)} \frac{u^2(x)}{|x|} \, dx + \int_{\mathbb{R}^2 \setminus B(0,1)} u^2(x) \, dx \leq$$

$$\leq C(\|u\|_{L^2}^2 + \|u\|_{L^p}^2) \quad , \quad p > 4 \, ,$$

by Holder inequality. The first assertion of the Lemma follows then from the Sobolev embedding.

Suppose that u_n is bounded in $H^1(\mathbb{R}^2)$; then

$$I_\omega(u_n) = O(1) + \frac{N^2}{2} \int_{\mathbb{R}^2} \frac{u_n^2(x)}{|x|^2} \, dx \, ,$$

and by hypothesis u_n is bounded in $\mathcal{H}$. The reverse is trivial.

□

The following is a Pohozaev-type identity for problem $(\mathcal{CSS})$, see (2.11), (5.6) in [4]:

Proposition 3.4. *For any $u \in \mathcal{H}$ solution of* $(\mathcal{CSS})$*, the following identity holds:*

$$\int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \, (h_u(|x|) - N)^2 \, dx - \frac{p-1}{p+1} \int_{\mathbb{R}^2} |u|^{p+1} \, dx = 0 \, .$$

We now state an inequality which will prove to be fundamental in our analysis. This inequality is proved in [4, Proposition 3.5], where also the maximizers are found.

Proposition 3.5. *For any $u \in \mathcal{H}$,*

$$\int_{\mathbb{R}^2} |u(x)|^4 \, dx \leq$$

$$\leq 4 \left(\int_{\mathbb{R}^2} |\nabla u(x)|^2 \, dx \right)^{1/2} \left(\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \left(h_u(|x|) - N\right)^2 \, dx \right)^{1/2} . \tag{3.2}$$

4. The limit problem

As commented in the introduction, this paper is concerned with the boundedness from below of I_ω. First of all, let us give a heuristic derivation of the limit energy functional. Consider $u(r)$ a fixed function, and define $u_\rho(r) = u(r - \rho)$. Let us now estimate $I_\omega(u_\rho)$ as $\rho \to +\infty$; after the change of variables $r \to r + \rho$, we obtain:

$$\frac{I_\omega(u_\rho)}{2\pi} = \frac{1}{2} \int_{-\rho}^{+\infty} (|u'|^2 + \omega u^2)(r+\rho) \, dr +$$

$$+ \frac{1}{8} \int_{-\rho}^{\infty} \frac{u^2(r)}{r+\rho} \left(\int_{-\rho}^{r} (s+\rho) u^2(s) \, ds - 2N \right)^2 dr -$$

$$- \frac{1}{p+1} \int_{-\rho}^{\infty} |u|^{p+1}(r+\rho) \, dr \; .$$

We estimate the above expression by simply replacing the expressions $(r + \rho)$, $(s + \rho)$ with the constant ρ; observe that the estimate is independent of N:

$$(2\pi)^{-1} I_\omega(u_\rho) \sim$$

$$\sim \rho \left[\frac{1}{2} \int_{-\infty}^{+\infty} (|u|'^2 + \omega u^2) \, dr + \frac{1}{8} \int_{-\infty}^{+\infty} u^2(r) \left(\int_{-\infty}^{r} u^2(s) \, ds \right)^2 dr - \right.$$

$$\left. - \frac{1}{p+1} \int_{-\infty}^{+\infty} |u|^{p+1} \, dr \right] =$$

$$= \rho \left[\frac{1}{2} \int_{-\infty}^{+\infty} (|u|'^2 + \omega u^2) \, dr + \frac{1}{24} \left(\int_{-\infty}^{+\infty} u^2 dr \right)^3 - \frac{1}{p+1} \int_{-\infty}^{+\infty} |u|^{p+1} \, dr \right] .$$

Therefore, it is natural to consider the limit functional $J_\omega : H^1(\mathbb{R}) \to \mathbb{R}$,

$$J_\omega(u) = \frac{1}{2} \int_{-\infty}^{+\infty} \left(|u'|^2 + \omega u^2 \right) \, dr +$$

$$+ \frac{1}{24} \left(\int_{-\infty}^{+\infty} u^2 \, dr \right)^3 - \frac{1}{p+1} \int_{-\infty}^{+\infty} |u|^{p+1} \, dr \; . \tag{4.1}$$

Clearly, the Euler-Lagrange equation of (4.1) is the following problem:

$$-u'' + \omega u + \frac{1}{4}\left(\int_{-\infty}^{+\infty} u^2(s)\, ds\right)^2 u = |u|^{p-1}u \quad , \quad \text{in } \mathbb{R} \, . \tag{4.2}$$

As we shall see later, we will find the explicit solutions of (4.2) later. But, first, let us study it from a variational point of view: this study will give us some further information on the solutions.

Before going on, we need a technical result proved in [25].

Lemma 4.1. *Let* $u_n \in H^1(\mathbb{R})$ *a sequence of even non-negative functions which are decreasing in* $r > 0$*, and assume that* $u_n \rightharpoonup u_0$ *weakly in* $H^1(\mathbb{R})$*. Then* u_0 *is also even, non-negative and decreasing in* $r > 0$*, and* $u_n \to u_0$ *in* $L^q(\mathbb{R})$ *for any* $q \in (2, +\infty)$*.*

Some of the properties of the functional J_ω are discussed below:

Proposition 4.2. *Consider the functional* J_ω *with* $p \in (1, 3)$ *and* $\omega > 0$*. The following properties hold:*

- *a)* J_ω *is coercive and attains its infimum.*
- *b)* 0 *is a local minimum of* J_ω*. Indeed, there exists* $r_0 > 0$ *with the following property:*
  for any $r \in (0, r_0)$*, there exists* $\alpha > 0$ *satisfying that* $J_\omega(u) > \alpha$*, for any* $u \in H^1(\mathbb{R})$ *with* $\|u\|_{H^1(\mathbb{R})} = r$*.*
- *c) There exists* $\omega_0 > 0$ *such that* $\min J_\omega < 0$ *if and only if* $\omega \in [0, \omega_0)$*.*

Proof. Proof of a). To prove coercivity, we use Gagliardo-Nirenberg inequality:

$$\|u\|_{L^4} \leq C \|u'\|_{L^2}^{1/4} \, \|u\|_{L^2}^{3/4} \, .$$

Hence

$$\int_{-\infty}^{+\infty} u^4 \, dr \leq \frac{C}{2}\left[\int_{-\infty}^{+\infty} |u'|^2 \, dr + \left(\int_{-\infty}^{+\infty} u^2 \, dr\right)^3\right] \, .$$

Then,

$$J_\omega(u) \geq \frac{1}{4} \int_{-\infty}^{+\infty} |u'|^2 \, dr +$$

$$\frac{1}{48}\left(\int_{-\infty}^{+\infty} u^2 \, dr\right)^3 + c\int_{-\infty}^{+\infty} u^4 \, dr - \frac{1}{p+1} \int_{-\infty}^{+\infty} |u|^{p+1} \, dr \, . \tag{4.3}$$

Observe that for any $C > 0$ we can choose $D > 0$ so that $t^3 \geq Ct - D$ for every $t \geq 0$. Applying this with $t = \int_{-\infty}^{+\infty} u^2 \, dr$ into (4.3), and renaming C, we obtain:

$$J_\omega(u) \geq \frac{1}{4} \int_{-\infty}^{+\infty} |u'|^2 \, dr + \int_{-\infty}^{+\infty} \left(Cu^2 + cu^4 - \frac{1}{p+1} \, |u|^{p+1}\right) dr - D \, .$$

Now, it suffices to take C so that the function $Cu^2 + cu^4 - (1/(p+1))|u|^{p+1} \geq 0$ for any $u \in \mathbb{R}$.

Take now u_n such that $J_\omega(u_n) \to \inf J_\omega$. From the coercivity, it follows that u_n is bounded. Consider now the sequence $v_n = |u_n|^*$ of non-negative symmetrized functions. Clearly, v_n is also bounded, and it is easy to observe that $\inf J_\omega \leq J_\omega(v_n) \leq J_\omega(u_n) \to \inf J_\omega$.

Assume, passing to a subsequence, that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R})$. By Lemma 3.3, $v_n \to v$ in $L^{p+1}(\mathbb{R})$. The weak lower semicontinuity of the norm allows us to conclude that u is a minimizer of J_ω.

Proof of b). This is quite standard. Indeed, by using Sobolev inequality,

$$J_\omega(u) \geq \frac{1}{2}\, \min\{1, \omega\} \|u\|^2_{H^1(\mathbb{R})} - C\|u\|^{p+1}_{H^1(\mathbb{R})} \,.$$

Proof of c). Let us define the map $\phi : [0, +\infty) \to \mathbb{R}$, $\phi(\omega) = \min J_\omega$. It is easy to check that ϕ is increasing and continuous. Moreover, $\phi(\omega) \leq 0$ for all ω (observe that $J_\omega(0) = 0$).

We claim that $\phi(\omega) = 0$ for large ω. Indeed, by the same arguments of the proof of *a)*:

$$J_\omega(u) \geq \int_{-\infty}^{+\infty} \left(\frac{\omega}{2}\, u^2 + cu^4 - \frac{1}{p+1}\, |u|^{p+1} \right) dr \,.$$

For ω sufficiently large, $(\omega/2)u^2 + cu^4 - (1/(p+1))|u|^{p+1} \geq 0$ for any $u \in \mathbb{R}$. Then $J_\omega(u) \geq 0$ for any $u \in H^1(\mathbb{R})$, proving the claim.

We now show that $\phi(0) < 0$. On that purpose, fix $u \in H^1(\mathbb{R})$ and define $u_\lambda(r) = \lambda^{2/(p-1)} u(\lambda r)$. There holds:

$$J_0(u_\lambda) = \frac{1}{2}\, \lambda^{(p+3)/(p-1)} \int_{-\infty}^{+\infty} |u'|^2\, dr + \frac{1}{24}\, \lambda^{3(5-p)/(p-1)} \left(\int_{-\infty}^{+\infty} u^2\, dr \right)^3 -$$

$$-\frac{1}{p+1}\, \lambda^{(p+3)/(p-1)} \int_{-\infty}^{+\infty} |u|^{p+1}\, dr \,.$$

Therefore, for λ sufficiently small, $J_0(u_\lambda)$ has the sign of the term

$$\frac{1}{2} \int_{-\infty}^{+\infty} |u'|^2\, dr - \frac{1}{p+1} \int_{-\infty}^{+\infty} |u|^{p+1}\, dr \,.$$

It suffices to take u such that this quantity is negative to conclude.

So, we can define $\omega_0 = \min\{\omega \geq 0 :\ \phi(\omega) = 0\} > 0$.

□

As a consequence of the previous result, for $\omega \in [0, \omega_0)$ there exists a nontrivial solution for (4.2), which corresponds to a global minimum of J_ω. As announced in the introduction, the expression for ω_0 will found later on.

We now pass to finding the explicit solutions of problem (4.2). For any $k > 0$ we denote by $w_k \in H^1(\mathbb{R})$ the unique positive radial solution of:

$$-w_k'' + k w_k = w_k^p \quad , \quad \text{in } \mathbb{R} \,. \tag{4.4}$$

Let us state some well-known properties of this equation. First, the Hamiltonian of w_k is equal to 0, that is,

$$-\frac{1}{2}\, |w_k'(r)|^2 + \frac{k}{2}\, w_k^2(r) - \frac{1}{p+1}\, w_k^{p+1}(r) = 0 \quad , \quad \text{for all } r \in \mathbb{R} \,. \tag{4.5}$$

It is also known that any solution of (4.4) is of the form $u(x) = \pm w_k(x - y)$, for some $y \in \mathbb{R}$. Moreover,

$$w_k(r) = k^{1/(p-1)} w_1(\sqrt{k}\, r) \quad , \quad \text{where} \tag{4.6}$$

$$w_1(r) = \left(\frac{2}{p+1} \cosh^2 \left(\frac{p-1}{2}\, r \right) \right)^{1/(1-p)} .$$

In what follows we define

$$m = \int_{-\infty}^{+\infty} w_1^2 \, dr . \tag{4.7}$$

The following relations are also well known, and can be deduced from (4.5):

$$\int_{-\infty}^{+\infty} |w_1'|^2 \, dr = \frac{p-1}{p+3}\, m \quad , \quad \int_{-\infty}^{+\infty} w_1^{p+1} \, dr = \frac{2(p+1)}{p+3}\, m . \tag{4.8}$$

Proposition 4.3. *Let us consider the equation:*

$$k = \omega + \frac{1}{4}\, m^2 k^{(5-p)/(p-1)} \quad , \quad k > 0 . \tag{4.9}$$

Then, u is a nontrivial solution of (4.2) *if and only if $u(r) = w_k(r-\xi)$ for some $\xi \in \mathbb{R}$ and k a root of* (4.9)*. Define:*

$$\omega_1 = \left(\frac{(5-p)m^2}{4(p-1)} \right)^{-(p-1)/2(3-p)} - \frac{m^2}{4} \left(\frac{(5-p)m^2}{4(p-1)} \right)^{-(5-p)/2(3-p)} . \tag{4.10}$$

The following holds:

1. *if $\omega > \omega_1$, equation* (4.9) *has no solution and there is no nontrivial solution of* (4.2)*;*
2. *if $\omega = \omega_1$, equation* (4.9) *has only one solution k_0 and $w_{k_0}(r)$ is the only non-trivial solution of* (4.2) (*apart from translations*)*;*
3. *if $\omega \in (0, \omega_1)$, equation* (4.9) *has two solutions $k_1(\omega) < k_2(\omega)$ and $w_{k_1}(r)$, $w_{k_2}(r)$ are the only two non-trivial solutions of* (4.2) (*apart from translations*)*.*

Proof. Let u be a nontrivial solution of (4.2), and define $k = \omega + (1/4)\left(\int_{-\infty}^{+\infty} u^2 dr\right)^2$. Then, u is a solution of $-u'' + ku = u^p$, so $u(r) = w_k(r-\xi)$ for some $\xi \in \mathbb{R}$. By using (4.6), we obtain:

$$k = \omega + \frac{1}{4} \left(\int_{-\infty}^{+\infty} w_k^2(r) \, dr \right)^2 = \omega + \frac{1}{4}\, k^{4/(p-1)} \left(\int_{-\infty}^{+\infty} w_1^2(\sqrt{k} r) \, dr \right)^2 .$$

A change of variables leads us to equation (4.9). Moreover,

$$1 < p < 3 \;\Rightarrow\; \frac{5-p}{p-1} > 1 .$$

Therefore, the function $(0, +\infty) \ni k \mapsto k^{(5-p)/(p-1)}$ is convex. Therefore, there exists $\omega_1 > 0$ with the properties indicated.

In order to get the exact value of ω_1, observe that the function $k \mapsto \omega_1 + (1/4)m^2 k^{(5-p)/(p-1)} - k$ has a degenerate 0. Then, ω_1 solves the system:

$$\begin{cases} \omega + \dfrac{m^2}{4}\, k^{(5-p)/(p-1)} = k , \\ \dfrac{5-p}{4(p-1)}\, m^2 k^{(5-p)/(p-1)-1} = 1 . \end{cases}$$

From this one obtains formula (4.10).

□

In our next result, we obtain information from Proposition 4.3.

Proposition 4.4. *Let ω_0, ω_1 be the values defined in Propositions 4.2 and 4.3 . Then:*

(1) *$\omega_0 < \omega_1$, and ω_0 has the expression:*

$$\omega_0 = \frac{3-p}{3+p}\, 3^{(p-1)/2(3-p)}\, 2^{2/(3-p)} \left(\frac{m^2(3+p)}{p-1}\right)^{-(p-1)/2(3-p)}, \tag{4.11}$$

where m is as in (4.7).

(2) *For any $\omega \in (0, \omega_1)$, $J_\omega(w_{k_1}) > J_\omega(w_{k_2})$. In particular, for any $\omega \in (0, \omega_0)$, w_{k_2} is a global minimizer of J_ω.*

Proof. We consider the energy functional J_ω evaluated on the curve $k \mapsto w_k$. In the computations that follow we use (4.6) and change of variables. We have

$$\psi(k) := J_\omega(w_k) = \frac{k^{(3+p)/2(p-1)}}{2} \int_{-\infty}^{+\infty} |w_1'(r)|^2\, dr +$$

$$+\, \omega\, \frac{k^{(5-p)/2(p-1)}}{2} \int_{-\infty}^{+\infty} w_1^2(r)\, dr +$$

$$+\, \frac{k^{3(5-p)/2(p-1)}}{24} \left(\int_{-\infty}^{+\infty} w_1^2(r)\, dr\right)^3 - \frac{k^{(3+p)/(2(p-1)}}{p+1} \int_{-\infty}^{+\infty} |w_1(r)|^{p+1}\, dr.$$

Plugging (4.8) into that expression,

$$\psi(k) = m\left[\frac{p-5}{2(3+p)}\, k^{(3+p)/(2(p-1)} + \frac{\omega}{2}\, k^{(5-p)/(2(p-1)} + \frac{m^2}{24}\, k^{3(5-p)/2(p-1)}\right].$$

Then:

$$\frac{d}{dk}\psi(k) = m\, k^{(7-3p)/(2(p-1)}\, \frac{5-p}{4(p-1)} \left[-k + \omega + \frac{1}{4}\, m^2 k^{(5-p)/(p-1)}\right].$$

In particular, the roots of (4.9) are exactly the critical points of ψ. Observe that:

$$\frac{5-p}{2(p-1)} < \frac{3+p}{2(p-1)} < \frac{3(5-p)}{2(p-1)}.$$

Then ψ is increasing near 0 (for $\omega > 0$) and near infinity. Therefore, for $\omega \in (0, \omega_1)$, its first root corresponds to a local maximum of ψ and the second one to a local minimum, so $J(w_{k_1}) > J(w_{k_2})$. Take now $\omega \in (0, \omega_0)$. Since in this case the minimizer is nontrivial, it must correspond to w_{k_2}. Moreover, $\omega_0 < \omega_1$.

In order to get the value of ω_0, observe that $J_{\omega_0}(w_{k_2}) = 0$. Therefore, $\omega_0 > 0$ solves:

$$\begin{cases} \omega + \frac{1}{4}\, m^2 k^{(5-p)/(p-1)} = k, \\ \frac{p-5}{2(3+p)}\, k^{(3+p)/(2(p-1))} + \frac{\omega}{2}\, k^{(5-p)/2(p-1)} + \frac{m^2}{24}\, k^{3(5-p)/2(p-1)} = 0. \end{cases}$$

From there, expression (4.11) follows.

□

Remark 4.5. Observe that the map ψ defined in the proof of Proposition 4.4 gives us a quite clear interpretation of the functional J_ω. Indeed, k is a critical point of ψ if and only if w_k is a critical point of J_ω. Moreover, the following holds.

(1) If $\omega > \omega_1$, ψ is positive and increasing without critical points.
(2) If $\omega = \omega_1$, ψ is still positive and increasing, but it has an inflection point at $k = k_0$.
(3) If $\omega \in (0, \omega_1)$, ψ has a local maximum and minimum attained at k_1 and k_2, respectively.
(4) If $\omega = \omega_0$, $\psi(k_2) = 0$. Observe then, in this case, the minimum of J_{ω_0} is 0, and is attained at 0 and w_{k_2}.
(5) If $\omega \in [0, \omega_0)$, $\psi(k_2) < 0$ and then w_{k_2} is the unique global minimizer, with $J_\omega(w_{k_2}) < 0$.

Remark 4.6. In general, we cannot obtain a more explicit expression of m depending on p, but it can be easily approximated by using some software. In Figure 1 the maps $\omega_0(p)$ and $\omega_1(p)$ have been plotted.

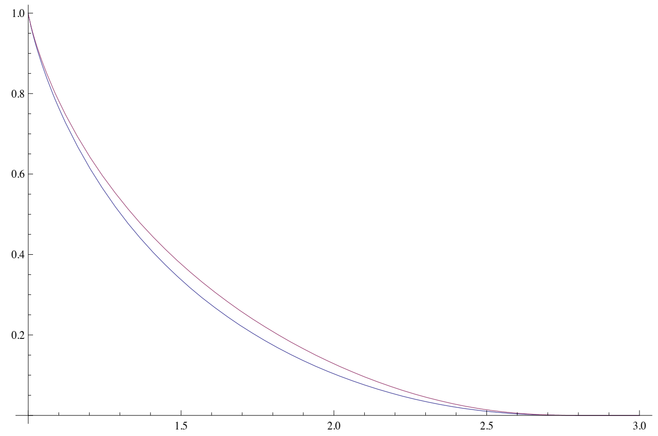

FIGURE 1. The values $\omega_0(p) < \omega_1(p)$, for $p \in (1, 3)$.

For some specific values of p, m can be explicitly computed, and hence ω_0 and ω_1. For instance, if $p = 2$, $m = 6$, $\omega_1 = 2/9\sqrt{3}$ and $\omega_0 = 2/5\sqrt{15}$.

We finish this section with a technical result that will be of use later in the proof of Theorem 1.1.

Proposition 4.7. *Assume* $\omega \geq \omega_0$, *and* $u_n \in H^1(\mathbb{R})$ *such that* $J_\omega(u_n) \to 0$. *There holds*

(1) *if* $\omega > \omega_0$, *then* $u_n \to 0$ *in* $H^1(\mathbb{R})$*;*
(2) *if* $\omega = \omega_0$, *then, up to a subsequence, either* $u_n \to 0$ *or* $u_n(\cdot - x_n) \to w_{k_2}$ *in* $H^1(\mathbb{R})$, *for some sequence* $x_n \in \mathbb{R}$.

Proof. Since J_ω is coercive, we have that u_n is bounded. If $u_n \to 0$ in $H^1(\mathbb{R})$, we are done. Otherwise, we have that:

$$o_n(1) = J_\omega(u_n) \geq \frac{1}{2} \int_{-\infty}^{+\infty} \left(|u_n'(r)|^2 + \omega u_n^2(r)\right) dr - \frac{1}{p+1} \int_{-\infty}^{+\infty} |u_n(r)|^{p+1} \, dr \ .$$

Then, $u_n \nrightarrow 0$ in $L^{p+1}(\mathbb{R})$. We can apply concentration-compactness lemma (see [20, Lemma I.1]), and there exists $\xi_n \in \mathbb{R}$ such that $\int_{\xi_n - 1}^{\xi_n + 1} u_n^2 \geq \varepsilon > 0$. Therefore, $\tilde{u}_n(r) = u_n(r - \xi_n) \rightharpoonup u \neq 0$ weakly in $H^1(\mathbb{R})$. Define $v_n = \tilde{u}_n - u$, which clearly converges weakly to 0 in $H^1(\mathbb{R})$.

Step 1: $v_n \to 0$ in $L^2(\mathbb{R})$.
We just compute

$$\begin{aligned} o_n(1) &= J_\omega(u_n) = J_\omega(\tilde{u_n}) = J_\omega(v_n + u) = \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} \left(|v_n'|^2 + |u'|^2 + 2v_n' u'\right) dr + \frac{\omega}{2} \int_{-\infty}^{+\infty} \left(v_n^2 + u^2 + 2v_n u\right) dr + \\ &+ \frac{1}{8} \left[\left(\int_{-\infty}^{+\infty} v_n^2 \, dr\right)^3 + \left(\int_{-\infty}^{+\infty} u^2 \, dr\right)^3 + 3 \left(\int_{-\infty}^{+\infty} v_n^2 \, dr\right)^2 \left(\int_{-\infty}^{+\infty} u^2 \, dr\right) + \right. \\ &\left. + 3 \left(\int_{-\infty}^{+\infty} v_n^2 \, dr\right) \left(\int_{-\infty}^{+\infty} u^2 \, dr\right)^2 \right] - \frac{1}{p+1} \int_{-\infty}^{+\infty} |v_n + u|^{p+1} \, dr + o_n(1) \ . \end{aligned}$$

Here the mixed products converge to zero, since $v_n \rightharpoonup 0$. Passing to a subsequence, we can assume that $v_n \to 0$ almost everywhere. Then, the well-known Brezis-Lieb lemma ([3]) implies that

$$\int_{-\infty}^{+\infty} |v_n + u|^{p+1} \, dr - \int_{-\infty}^{+\infty} (|v_n|^{p+1} + |u|^{p+1}) \, dr \to 0 \ .$$

Then,

$$\begin{aligned} o_n(1) = J_\omega(u_n) = J_\omega(v_n) + J_\omega(u) + \frac{3}{8} &\left[\left(\int_{-\infty}^{+\infty} v_n^2 \, dr\right)^2 \left(\int_{-\infty}^{+\infty} u^2 \, dr\right) + \right. \\ &\left. + \left(\int_{-\infty}^{+\infty} v_n^2 \, dr\right) \left(\int_{-\infty}^{+\infty} u^2 \, dr\right)^2 \right] + o_n(1) \ . \end{aligned}$$

It is here that the assumption $\omega \geq \omega_0$ is crucial. Indeed, it implies that $J_\omega(v_n) \geq 0$, $J_\omega(u) \geq 0$. Recall that $u \neq 0$, to conclude the proof of Step 1.

Step 2: conclusion.
By interpolation,

$$\|v_n\|_{L^{p+1}} \leq \|v_n\|_{L^2}^{\alpha} \|v_n\|_{L^{p+2}}^{1-\alpha} \ ,$$

for some $\alpha \in (0,1)$. Since v_n is bounded in $H^1(\mathbb{R})$, then all norms above are bounded. Then, by Step 1, $\|v_n\|_{L^{p+1}} \to 0$. In other words, $\tilde{u}_n \to u$ in $L^{p+1}(\mathbb{R})$.

From this it is easy to conclude. Indeed,

$$o_n(1) = J_\omega(\tilde{u}_n) = \frac{1}{2}\int_{-\infty}^{+\infty} \left(|\tilde{u}_n'|^2 + \omega \tilde{u}_n^2\right) dr + \frac{1}{8}\left(\int_{-\infty}^{+\infty} \tilde{u}_n^2 \, dr\right)^3 -$$

$$- \frac{1}{p+1}\int_{-\infty}^{+\infty} |\tilde{u}_n|^{p+1} \, dr \ ,$$

$$0 \le J_\omega(u) = \frac{1}{2}\int_{-\infty}^{+\infty} \left(|u'|^2 + \omega u^2\right) dr + \frac{1}{8}\left(\int_{-\infty}^{+\infty} u^2 \, dr\right)^3 -$$

$$- \frac{1}{p+1}\int_{-\infty}^{+\infty} |u|^{p+1} \, dr \ .$$

Then, $\|\tilde{u}_n\|_{H^1(\mathbb{R})} \to \|u\|_{H^1(\mathbb{R})}$. And this implies that $\tilde{u}_n \to u$ in $H^1(\mathbb{R})$, finishing the proof.

□

4.1. **The non-degeneracy.** In[26], the authors study the non-degeneracy of the solution of the limit problem (4.2). Even if this is not necessary for our purpose in this review paper, we present their conclusions here for the sake of completeness in the study of the limit problem.

We say that a solution u of (4.2) is degenerate if the problem:

$$\begin{aligned} &-\phi'' + \omega\phi + \frac{1}{4}\left(\int_{-\infty}^{+\infty} u^2(s)\, ds\right)^2 \phi + \\ &+ \left(\int_{-\infty}^{+\infty} u^2(s)\, ds\right)\left(\int_{-\infty}^{+\infty} u(s)\phi(s)\, ds\right) u - p u^{p-1}\phi = 0 \end{aligned} \tag{4.12}$$

has a solution $\phi \in H^1(\mathbb{R})$ different from $c\, u'(r)$, $c \in \mathbb{R}$.

Next proposition is devoted to this question.

Proposition 4.8. *Let ω_1, k_0, k_1, k_2 as in Proposition 4.3. Then:*

1. *If $\omega = \omega_1$, w_{k_0} is a degenerate solution of* (4.2).
2. *If $\omega \in (0, \omega_1)$, both w_{k_1} and w_{k_2} are non-degenerate solutions of* (4.2).

Proof. Take w_k a solution of (4.2), and ϕ a solution of (4.12). Define

$$\alpha = \left(\int_{-\infty}^{+\infty} w_k^2(s)\, ds\right)\left(\int_{-\infty}^{+\infty} w_k(s)\phi(s)\, ds\right) \in \mathbb{R} \ .$$

Then, ϕ is a solution of the problem:

$$-\phi'' + \omega\phi + \frac{1}{4}\left(\int_{-\infty}^{+\infty} w_k^2(s)\, ds\right)^2 \phi - p w_k^{p-1}\phi = -\alpha w_k \ .$$

Write $\phi(r) = \psi(r) w_k'(r)$: then,

$$\psi''(r) = -2\psi'(r)\,\frac{w_k''(r)}{w_k'(r)} + \alpha\frac{w_k(r)}{w_k'(r)} \ .$$

This is a linear equation of first order in $\psi'(r)$, which can be solved by variation of constants:

$$\psi'(r) = \frac{C}{(w_k'(r))^2} + \frac{\alpha}{2}\,\frac{w_k^2(r)}{(w_k'(r))^2} \quad , \quad C \in \mathbb{R}\ .$$

Observe that if $C \neq 0$, then $\phi(r) = \psi(r) w_k'(r)$ tends to $+\infty$ at infinity, which is not possible if $\phi \in H^1(\mathbb{R})$. So, C must be equal to 0, and we are led to:

$$\psi'(r) = \frac{\alpha}{2}\,\frac{w_k^2(r)}{(w_k'(r))^2}\ .$$

By using (4.5), the above equation can be explicitly integrated:

$$\psi(r) = \frac{\alpha}{2k}\left(\frac{2}{(p-1)}\,\frac{w_k(r)}{w_k' r)} + r\right) + \beta \quad , \quad \text{with } \beta \in \mathbb{R}\ .$$

Therefore,

$$\phi(r) = \frac{\alpha}{2k}\left(\frac{2}{p-1}\, w_k(r) + r w_k'(r)\right) + \beta w_k'(r)\ .$$

Clearly, w_k is degenerate if and only if α can take values different from zero. Take $\beta = 0$, and recall now the definition of α:

$$\alpha = \frac{\alpha}{2k}\left(\int_{-\infty}^{+\infty} w_k^2(s)\,ds\right)\left(\int_{-\infty}^{+\infty} \frac{2}{p-1}\, w_k^2(s) + s w_k'(s) w_k(s)\,ds\right)\ .$$

By integration by parts,

$$\int_{-\infty}^{+\infty} s w_k'(s) w_k(s)\,ds = -\frac{1}{2}\int_{-\infty}^{+\infty} w_k^2(s)\,ds = -\frac{1}{2}\, k^{(5-p)/2(p-1)} m\ .$$

Then, either $\alpha = 0$ or

$$1 = m^2 k^{(6-2p)/(p-1)}\,\frac{5-p}{4(p-1)}\ .$$

This, together with (4.9), implies that $\omega = \omega_1$ and $k = k_0$.

□

Observe that, due to the nonlocal character of our problem, the linearized problem (4.12) does not have the form of a Schrödinger operator. Then, some more work is needed in order to accomplish a Lyapunov-Schmidt reduction. Let us define the following operators:

$$\begin{aligned} T[\phi] &= -\phi'' + \left[\omega + \frac{1}{4}\left(\int_{-\infty}^{+\infty} u^2(s)\,ds\right)^2 - p u^{p-1}\right]\phi\ ,\\ K[\phi] &= \left(\int_{-\infty}^{+\infty} u^2(s)\,ds\right)\left(\int_{-\infty}^{+\infty} u(s)\phi(s)\,ds\right) u\ ,\\ L[\phi] &= T[\phi] + K[\phi]\ . \end{aligned}$$

Observe that L, T, $K : H^1(\mathbb{R}) \cap \{u'\}^\perp \to H^{-1}(\mathbb{R}) \cap \{u'\}^\perp$. Of course, here orthogonality is understood for the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{H^1(\mathbb{R})}$.

Corollary 4.9. *Assume $\omega \in (0, \omega_1)$. Then the operator L defined above is a bijection for $u = w_{k_1}$ or $u = w_{k_2}$. Moreover, if $\omega < \omega_0$ and $u = w_{k_2}$, there exists $c > 0$ so that:*

$$\langle L[\phi], \phi\rangle_{H^1(\mathbb{R})} \geq c\|\phi\|^2_{H^1(\mathbb{R})} \qquad \text{for any } \phi \in H^1(\mathbb{R}) \cap \{u'\}^\perp\ . \tag{4.13}$$

Proof. It is well-known that T is a bijection. Since the image of K is of dimension 1, K is obviously a compact operator. Moreover, by Proposition 4.8, L is injective. By Fredholm alternative, L is a bijection.

Moreover, by Proposition 4.3, we know that w_{k_2} is a minimizer of J with non-degenerate second derivative. From this, (4.13) follows.

□

5. PROOFS OF MAIN RESULTS

Our first lemma makes rigorous the heuristic derivation of the limit functional made in Section 3. This estimate has been accomplished in [25, Lemma 4.1] for $N = 0$ and in [18, Lemma 3.1] for $N \neq 0$. Since the functions in $\mathcal{H}$ must vanish at 0, we need to truncate our sequence around the origin. For that purpose, take a Lipschitz continuous function $\phi_0 : \mathbb{R} \to \mathbb{R}$ such that

$$\phi_0(r) = \begin{cases} 0 \quad , \quad \text{if } |r| \leq 1 \ , \\ 1 \quad , \quad \text{if } |r| \geq 2 \ , \end{cases} \qquad |\phi_0'(r)| \leq 1 \ . \tag{5.1}$$

Lemma 5.1. *Let $U \in H^1(\mathbb{R})$ be an even function which decays to zero exponentially at infinity, and $\phi_0(r)$ as in* (5.1)*. Let us denote $U_\rho(r) = \phi_0(r) U(r-\rho)$. Then there exists $C > 0$ such that:*

$$I_\omega(U_\rho) = 2\pi\rho J_\omega(U(r)) - C + o_\rho(1) \ .$$

In the next proposition we make use of the fundamental inequality (3.2) to study the behavior of unbounded sequences with energy bounded from above.

Proposition 5.2. *Assume $\omega > 0$, and $u_n \in \mathcal{H}$ such that $\|u_n\|$ is unbounded but $I_\omega(u_n)$ is bounded from above. Then, there exists a subsequence (still denoted by u_n) such that:*

i) for all $\varepsilon > 0$, $\displaystyle\int_{\varepsilon\|u_n\|^2}^{+\infty} \left(|u_n'|^2 + u_n^2\right) dr \leq C$;

ii) there exists $\delta \in (0,1)$ such that $\displaystyle\int_{\delta\|u_n\|^2}^{\delta^{-1}\|u_n\|^2} \left(|u_n'|^2 + u_n^2\right) dr \geq c > 0$;

iii) $\|u_n\|_{L^2(\mathbb{R}^2)} \to +\infty$.

Proof. By inequality (3.2) and Cauchy-Schwartz inequality, we can estimate:

$$\begin{aligned} I_\omega(u) \geq \frac{\pi}{2} \int_0^{+\infty} \left(|u'|^2 + \omega u^2\right) r\, dr + \\ + \frac{\pi}{8} \int_0^{+\infty} \frac{u^2(r)}{r} \left(\int_0^r s u^2(s)\, ds - 2N\right)^2 dr + \\ + 2\pi \int_0^{+\infty} \left(\frac{\omega}{4}\, u^2 + \frac{1}{8}\, u^4 - \frac{1}{p+1}\, |u|^{p+1}\right) r\, dr \ . \end{aligned} \tag{5.2}$$

Define

$$f : \mathbb{R}_+ \to \mathbb{R} \quad , \quad f(t) = \frac{\omega}{4}\, t^2 + \frac{1}{8}\, t^4 - \frac{1}{p+1}\, t^{p+1} \ .$$

Then, the set $\{t > 0 : f(t) < 0\}$ is of the form (α, β), where α, β are positive constants depending only on p, ω. Moreover, we denote by $-c_0 = \min f < 0$.

For each function u_n, we define:

$$A_n = \{x \in \mathbb{R}^2 \,:\, u_n(x) \in (\alpha, \beta)\} \quad , \quad \rho_n = \sup\{|x| \,:\, x \in A_n\} \; .$$

With these definitions, we can rewrite (5.2) in the form

$$\begin{aligned} I_\omega(u_n) \geq \frac{\pi}{2} \int_0^{+\infty} \left(|u_n'|^2 + \omega u_n^2\right) r \, dr + \\ + \frac{\pi}{8} \int_0^{+\infty} \frac{u_n^2(r)}{r} \left(\int_0^r s u_n^2(s) \, ds - 2N\right)^2 dr - c_0 |A_n| \; . \end{aligned} \tag{5.3}$$

In particular this implies that $|A_n|$ must diverge, and hence ρ_n. This already proves (iii).

By Strauss Lemma [28], we have

$$\alpha \leq u_n(\rho_n) \leq \frac{\|u_n\|}{\sqrt{\rho_n}} \quad \Rightarrow \quad \|u_n\|^2 \geq \alpha^2 \rho_n \; . \tag{5.4}$$

We now estimate the nonlocal term. For that, define

$$B_n = A_n \cap B(0, \gamma_n) \quad , \quad \text{for } \gamma_n \in (0, \rho_n) \text{ such that } |B_n| = \frac{1}{2} \, |A_n| \; . \tag{5.5}$$

Then $\int_{B_n} u_n^2(x) \, dx \geq \alpha^2 |B_n|$ diverges, indeed

$$\int_{B_n} u_n^2(x) \, dx - 2N > c|A_n| \; .$$

Following [18], we have

$$\int_0^{+\infty} \frac{u_n^2(r)}{r} \left(\int_0^r s u_n^2(s) \, ds - 2N\right)^2 dr \geq c \, \frac{|A_n|^3}{\rho_n^2} \; . \tag{5.6}$$

Hence, by (5.2), (5.4) and (5.6), we get

$$I_\omega(u_n) \geq c\rho_n + c \, \frac{|A_n|^3}{\rho_n^2} - c_0 |A_n| = \rho_n \left(c + c \, \frac{|A_n|^3}{\rho_n^3} - c_0 \, \frac{|A_n|}{\rho_n} \right) \; .$$

Observe that $t \mapsto c + ct^3 - c_0 t$ is strictly positive near zero and goes to $+\infty$, as $t \to +\infty$. Then we can assume, passing to a subsequence, that $|A_n| \sim \rho_n$. In other words, there exists $m > 0$ such that $\rho_n |A_n|^{-1} \to m$ as $n \to +\infty$.
Taking into account (5.3) and (5.4), we conclude that up to a subsequence, $\|u_n\|^2 \sim \rho_n$. Moreover, for any fixed $\varepsilon > 0$, we have:

$$C\rho_n \geq \|u_n\|_{L^2}^2 \geq \int_{\varepsilon\rho_n}^{+\infty} u_n^2 r \, dr \geq \varepsilon\rho_n \int_{\varepsilon\rho_n}^{+\infty} u_n^2 \, dr \; .$$

An analogous estimate works also for $\int_{\varepsilon\rho_n}^{+\infty} |u_n'|^2 \, dr$. This proves (i).

We now show that for some $\delta > 0$, $\|u_n\|_{H^1(\delta\rho_n, \rho_n)} \nrightarrow 0$, which implies assertion (ii).
First, recall the definition of B_n and γ_n in (5.5). Then,

$$\begin{aligned} \int_{\gamma_n}^{\rho_n} u_n^2(r) \, dr \geq \rho_n^{-1} \int_{\gamma_n}^{\rho_n} u_n^2(r) r \, dr \geq \\ \geq \rho_n^{-1} \int_{A_n \setminus B_n} u_n^2(x) \, dx \geq \rho_n^{-1} |A_n \setminus B_n| \alpha^2 > c > 0 \; . \end{aligned}$$

To conclude it suffices to show that $\gamma_n \sim \rho_n$. Define

$$C_n = B_n \cap B(0,\tau_n) \quad , \quad \text{for } \tau_n \in (0,\gamma_n) \text{ such that } |C_n| = \frac{1}{2}\, |B_n| \, .$$

We can repeat the estimate (5.6) with A_n, B_n replaced with B_n, C_n respectively, to obtain that

$$\int_0^{+\infty} \frac{u_n^2(r)}{r} \left(\int_0^r s u_n^2(s)\, ds - 2N \right)^2 dr \geq c\, \frac{|B_n|^3}{\gamma_n^2} \, .$$

Hence,

$$I_\omega(u_n) \geq c\rho_n + c\frac{|A_n|^3}{\gamma_n^2} - c_0|A_n| = \gamma_n \left(c\frac{\rho_n}{\gamma_n} + c\frac{|A_n|^3}{\gamma_n^3} - c_0\frac{|A_n|}{\gamma_n} \right) \, .$$

And we are done since $I_\omega(u_n)$ is bounded from above.

□

Proof of Theorem 1.1. If $\omega \in (0,\omega_0)$, then $J_\omega(w_{k_2}) < 0$ (see Proposition 4.3): applying Lemma 5.1 with $U = w_{k_2}$ we conclude assertion (i).

We now prove (ii) and (iii). We denote by $H^1_{0,r}(B(0,R))$ the Sobolev space of radial functions with zero boundary value and

$$\mathcal{H}(B(0,R)) = \left\{ u \in H^1_{0,r}(B(0,R)) \, : \int_{B(0,R)} \frac{u^2(x)}{|x|^2}\, dx < +\infty \right\} \, ,$$

endowed by the norm $\| \cdot \|_{\mathcal{H}}$.
Fixed $n \in \mathbb{N}$ and given a sequence $v_i \in \mathcal{H}(B(0,n))$ unbounded with respect to the norm $\| \cdot \|$, (5.3) implies that $I_\omega(v_i) \to +\infty$. By Lemma 3.3, we conclude that $I_\omega|_{\mathcal{H}(B(0,n))}$ is coercive.
So, there exists u_n a minimizer for $I_\omega|_{\mathcal{H}(B(0,n))}$. By taking absolute value, we can assume that $u_n \geq 0$. Moreover,

$$I_\omega(u_n) \to \inf I_\omega \quad , \quad \text{as } n \to +\infty \, .$$

In the following, u_n may be extended as functions in $\mathcal{H}$ by setting $u_n(x) = 0$ for $x \in \mathbb{R}^2 \setminus B(0,n)$. If u_n is bounded in $H^1(\mathbb{R}^2)$, Lemma 3.3 implies that u_n is bounded in $\mathcal{H}$ and then $I_\omega(u_n)$ is bounded. In such case we conclude that $\inf I_\omega$ is finite. In what follows we assume that u_n is an unbounded sequence in $H^1(\mathbb{R}^2)$, and we shall show that $I_\omega(u_n)$ is still bounded for $\omega \geq \omega_0$.

Our sequence u_n satisfies the hypotheses of Proposition 5.2, so let $\delta > 0$ be given by that proposition.

The proof will be divided in several steps.

Step 1 : $\displaystyle\int_{(\delta/2)\|u_n\|^2}^{(2/\delta)\|u_n\|^2} |u_n|^{p+1}\, dr \nrightarrow 0.$

By Proposition 5.2, i), we have that:

$$\sum_{k=1}^{[(\delta/2)\|u_n\|^2]} \int_{(\delta/2)\|u_n\|^2+k-1}^{(\delta/2)\|u_n\|^2+k} \left(|u_n'|^2 + u_n^2 \right) dr \leq \int_{(\delta/2)\|u_n\|^2}^{\delta\|u_n\|^2} \left(|u_n'|^2 + u_n^2 \right) dr \leq C \, .$$

Taking the smaller summand in the left hand side we find x_n,

$$\frac{\delta}{2}\, \|u_n\|^2 \leq x_n \leq \delta\|u_n\|^2 - 1 \text{ such that } \|u_n\|^2_{H^1(x_n,x_n+1)} \leq \frac{C}{\|u_n\|^2} \, .$$

Reasoning in an analogous way, we can choose y_n,

$$\delta^{-1}\|u_n\|^2+1\le y_n\le 2\delta^{-1}\|u_n\|^2 \text{ such that } \|u_n\|^2_{H^1(y_n,y_n+1)}\le \frac{C}{\|u_n\|^2}\ .$$

Observe that if $\delta^{-1}\|u_n\|^2\ge n$, the choice of y_n can be arbitrary, but it is unnecessary. Take $\phi_n:[0,+\infty]\to[0,1]$ be a C^∞-function such that

$$\phi_n(r)=\begin{cases}0 & , \quad \text{if } r\le x_n\ ,\\ 1 & , \quad \text{if } x_n+1\le r\le y_n\ ,\\ 0 & , \quad \text{if } r\ge y_n+1\ ,\end{cases} \qquad |\phi_n'(r)|\le 2\ .$$

By the choice of x_n, y_n and Proposition 5.2, i),

$$0=I'_\omega(u_n)[\phi_n u_n]\ge 2\pi\int_{x_n}^{y_n}\left(|u_n'|^2+\omega u_n^2\right) r\,dr-2\pi\int_{x_n}^{y_n}|u_n|^{p+1}r\,dr+O(1)\ge$$

$$\ge\|u_n\|^2\left(\frac{\delta}{2}\int_{x_n}^{y_n}\left(|u_n'|^2+\omega u_n^2\right)\,dr-\frac{2}{\delta}\int_{x_n}^{y_n}|u_n|^{p+1}\,dr\right)+O(1)\ .$$

This, together with the fact that $\|u_n\|_{H^1(x_n,y_n)}$ does not tend to zero, allows us to conclude the proof of Step 1.

Step 2: exponential decay.

At this point we can apply the concentration-compactness principle (see [20, Lemma 1.1]); there exists $\sigma>0$ such that

$$\sup_{\xi\in[x_n,\ y_n]}\int_{\xi-1}^{\xi+1}u_n^2\,dr\ge 2\sigma>0\ .$$

Let us define:

$$D_n=\left\{\xi>0\ :\ \int_{\xi-1}^{\xi+1}\left(|u_n'|^2+u_n^2\right)\,dr\ge\sigma\right\}\ne\emptyset\ , \tag{5.7}$$

and

$$\xi_n=\max D_n\in[x_n,n+1)\ .$$

Let us observe that $\xi_n\sim\|u_n\|^2$; indeed $\xi_n\ge x_n\ge c\|u_n\|^2$ and, moreover,

$$\|u_n\|^2\ge c\int_{\xi_n-1}^{\xi_n+1}\left(|u_n'|^2+u_n^2\right)r\,dr\ge c(\xi_n-1)\int_{\xi_n-1}^{\xi_n+1}\left(|u_n'|^2+u_n^2\right)\,dr\ge c(\xi_n-1)\ .$$

By definition, $\int_{\zeta-1}^{\zeta+1}(|u_n'|^2+u_n^2)\,dr<\sigma$ for all $\zeta>\xi_n$. By embedding of $H^1(\zeta-1,\zeta+1)$ in L^∞, $0\le u_n(\zeta)<C\sqrt{\sigma}$ for any $\zeta>\xi_n$. From this we will get exponential decay of u_n. Indeed, u_n is a solution of

$$-u_n''(r)-\frac{u_n'(r)}{r}+\omega u_n(r)+f_n(r)u_n(r)=u_n^p(r)\ ,$$

with

$$f_n(r)=\frac{(h_n(r)-N)^2}{r^2}+\int_r^n\frac{h_n(s)-N}{s}\,u_n^2(s)\,ds\quad ,\quad h_n(r)=\frac{1}{2}\int_0^r s u_n^2(s)\,ds\ .$$

If $r > \delta\|u_n\|^2$, again by Proposition 5.2, i), we see that $\int_r^n (N/s)\, u_n^2(s)\, ds = o(1)$. Then, by taking smaller σ, if necessary, we can conclude that there exists $C > 0$ such that

$$|u_n(r)| < C \exp\left(-\sqrt{\frac{\omega}{2}}\,(r - \xi_n)\right) \quad , \quad \text{for all } r > \xi_n \ .$$

The local C^1 regularity theory for the Laplace operator (see [10, Section 3.4]) implies a similar estimate for $u_n'(r)$. In other words,

$$|u_n(r)| + |u_n'(r)| < C \exp\left(-\sqrt{\frac{\omega}{2}}\,(r - \xi_n)\right) \quad , \quad \text{for all } r > \xi_n \ .$$

Step 3: splitting of $I_\omega(u_n)$.

Reasoning as in the beginning of Step 1, we can take z_n:

$$\xi_n - 3\|u_n\| \leq z_n \leq \xi_n - 2\|u_n\| \quad \text{with} \quad \|u_n\|^2_{H^1(z_n, z_n+1)} \leq \frac{C}{\|u_n\|} \ .$$

Define $\psi_n : [0, +\infty] \to [0, 1]$ be a smooth function such that

$$\psi_n(r) = \begin{cases} 0 & , \quad \text{if } r \leq z_n \ , \\ 1 & , \quad \text{if } r \geq z_n + 1 \ , \end{cases} \qquad |\psi_n'(r)| \leq 2 \ .$$

Arguing as in [18, 25], we can prove that

$$I_\omega(u_n) \geq I_\omega(u_n \psi_n) + I_\omega\left(u_n(1 - \psi_n)\right) + c\|u_n(1 - \psi_n)\|^2_{L^2(\mathbb{R}^2)} + O(\|u_n\|) \ . \tag{5.8}$$

Step 4. The following estimate holds:

$$I_\omega(u_n \psi_n) = 2\pi \xi_n J_\omega(u_n \psi_n) + O(\|u_n\|) \ . \tag{5.9}$$

For the proof, see [18, 25].

Step 5: conclusion for $\omega > \omega_0$.

By (5.8) and (5.9), we have

$$I_\omega(u_n) \geq 2\pi \xi_n J_\omega(u_n \psi_n) + I_\omega(u_n(1 - \psi_n)) + c\|u_n(1 - \psi_n)\|^2_{L^2(\mathbb{R}^2)} + O(\|u_n\|) \ . \tag{5.10}$$

Recall that $\|u_n \psi_n\|^2_{H^1(\mathbb{R})} \geq \sigma > 0$. By Proposition 4.7, we have that $J_\omega(u_n \psi_n) \to c > 0$, up to a subsequence. Since $\xi_n \sim \|u_n\|^2$, it turns out from (5.10) that $I_\omega(u_n) > I_\omega(u_n(1 - \psi_n))$, which is a contradiction with the definition of u_n. Therefore, u_n needs to be a bounded sequence and, in particular, $\inf I_\omega > -\infty$.

Let us now show that I_ω is coercive. Indeed, take $u_n \in \mathcal{H}$ an unbounded sequence, and assume that $I_\omega(u_n)$ is bounded from above. By Lemma 3.3, $\|u_n\|$ is unbounded, so that Proposition 5.2, (iii), shows us that $I_{\hat{\omega}}(u_n) \to -\infty$ for any $\omega_0 < \hat{\omega} < \omega$, a contradiction.

Step 6: conclusion for $\omega = \omega_0$.

As above, (5.10) gives a contradiction unless $J_\omega(u_n \psi_n) \to 0$. Proposition 4.7 now implies that $\psi_n u_n(\cdot - t_n) \to w_{k_2}$ up to a subsequence, for some $t_n \in (0, +\infty)$. Since $\xi_n \in D_n$ (recall its definition in (5.7)), we have that $|t_n - \xi_n|$ is bounded. With this extra information, we have a better estimate of the decay of the solutions: indeed,

$$|u_n(r)| + |u_n'(r)| < C \exp\left(-\sqrt{\frac{\omega}{2}}\,|r - \xi_n|\right) \quad , \quad \text{for all } r > \xi_n - 2\|u_n\| \ . \tag{5.11}$$

This allows us to do the cut-off procedure in a much more accurate way. Indeed, take $\tilde{z}_n = \xi_n - \|u_n\|$. Then, (5.11) implies that

$$\|u_n\|^2_{H^1(\tilde{z}_n, \tilde{z}_n+1)} \leq C \exp(-\sqrt{\frac{\omega}{2}}\, \|u_n\|) \,. \tag{5.12}$$

Define $\tilde{\psi}_n : [0, +\infty] \to [0, 1]$ accordingly:

$$\tilde{\psi}_n(r) = \begin{cases} 0 \quad , & \text{if } r \leq \tilde{z}_n \,, \\ 1 \quad , & \text{if } r \geq \tilde{z}_n + 1 \,, \end{cases} \qquad |\tilde{\psi}_n'(r)| \leq 2 \,.$$

The advantage is that, in the estimate of $I_\omega(u_n)$, now the errors are exponentially small. Indeed, by repeating the estimates of Step 3 with the new information (5.12), we obtain:

$$I_\omega(u_n) \geq I_\omega(u_n\tilde{\psi}_n) + I_\omega(u_n(1-\tilde{\psi}_n)) + c\|u_n(1-\tilde{\psi}_n)\|^2_{L^2(\mathbb{R}^2)} + O(1) \,,$$

Then,

$$\begin{aligned} I_\omega(u_n) &\geq I_\omega(u_n\tilde{\psi}_n) + I_\omega\left(u_n(1-\tilde{\psi}_n)\right) + c\|u_n(1-\tilde{\psi}_n)\|^2_{L^2(\mathbb{R}^2)} + O(1) = \\ &= 2\pi\xi_n J_\omega(u_n\tilde{\psi}_n) + I_\omega\left(u_n(1-\tilde{\psi}_n)\right) + c\|u_n(1-\tilde{\psi}_n)\|^2_{L^2(\mathbb{R}^2)} + O(1) \geq \\ &\geq I_{(\omega+2c)}\left(u_n(1-\tilde{\psi}_n)\right) + O(1) \,. \end{aligned}$$

But, by Step 5, we already know that $I_{(\omega+2c)}$ is bounded from below, and hence $\inf I_{\omega_0} > -\infty$.

Finally, by applying Lemma 5.1 to $U = w_{k_2}$ we readily get that I_{ω_0} is not coercive.

□

Proof of Theorem 1.2. We shall prove each assessment separately.

Proof of (*ii*). First, we observe that since $\inf I_{\omega_0} < 0$, there exists $\tilde{\omega} > \omega_0$ such that $\inf I_\omega < 0$ if and only if $\omega \in (\omega_0, \tilde{\omega})$. Since, by Theorem 1.1 and Proposition 3.2, I_ω is coercive and weakly lower semicontinuous, we infer that the infimum is attained at a negative value. This gives the first solution u_1.
Clearly, 0 is a local minimum for I_ω, and $I_\omega(u_1) < 0$. Then, the functional satisfies the geometrical assumptions of the Mountain Pass Theorem, see [1]. Since I_ω is coercive, (PS) sequences are bounded. By the compact embedding of $H^1_r(\mathbb{R}^2)$ into $L^{p+1}(\mathbb{R}^2)$ and Proposition 3.2, standard arguments show that I_ω satisfies the Palais-Smale condition and so we find a second solution which is at a positive energy level.

Proof of (*iii*). Let now consider $\omega \in (0, \omega_0)$. Performing the rescaling $u \mapsto u_\omega = \sqrt{\omega}\, u(\sqrt{\omega}\, \cdot)$, we get

$$I_\omega(u_\omega) = \omega \left[\frac{1}{2} \int_{\mathbb{R}^2} \left(|\nabla u|^2 + u^2\right) dx + \right.$$
$$\left. + \frac{1}{8} \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \left(\int_0^{|x|} s u^2(s)\, ds - 2N \right)^2 dx - \frac{\omega^{(p-3)/2}}{p+1} \int_{\mathbb{R}^2} |u|^{p+1}\, dx \right] \,.$$

Define $\lambda = \omega^{(p-3)/2}$ and $\mathcal{I}_\lambda : H(\mathbb{R}^2) \to \mathbb{R}$ as

$$\mathcal{I}_\lambda(u) = \Phi(u) - \frac{\lambda}{p+1} \int_{\mathbb{R}^2} |u|^{p+1}\, dx \,,$$

with

$$\Phi(u) = \frac{1}{2}\,\|u\|^2 + \frac{1}{2}\,\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}(h_u(|x|) - N)^2\,dx =$$

$$= \frac{1}{2}\,\|u\|^2 + K(u) + \frac{N^2}{2}\,\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\,dx\ ,$$

where K is as defined in (3.1). Then $\mathcal{I}_\lambda$ satisfies the geometrical assumptions of the Mountain Pass Theorem. The main problem here is that we do not know whether a (PS) sequence could be unbounded.

By Lemma 3.3, the functional $\Phi : \mathcal{H} \to \mathbb{R}$ is coercive. Then we can use [17, Theorem 1.1] to obtain a bounded Palais-Smale sequence $u_n \in \mathcal{H}$ for almost every λ. Passing to a subsequence, we can assume that $u_n \rightharpoonup u$; Proposition 3.2 and standard arguments imply that u is a critical point of $\mathcal{I}_\lambda$. Making the change of variables back we obtain a solution of $(\mathcal{CSS})$ for almost every $\omega \in (0, \omega_0)$.
Finally, in order to find positive solutions of $(\mathcal{CSS})$, we simply observe that the above arguments apply to the functional $I^+_\omega : \mathcal{H} \to \mathbb{R}$

$$I^+_\omega(u) = \frac{1}{2}\,\int_{\mathbb{R}^2} \left(|\nabla u|^2 + \omega u^2\right)\,dx +$$

$$+\frac{1}{8}\,\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(\int_0^{|x|} s u^2(s)\,ds - 2N\right)^2 dx - \frac{1}{p+1}\,\int_{\mathbb{R}^2} (u^+)^{p+1}\,dx\ .$$

Due to the maximum principle, the critical points of I^+_ω are positive solutions of $(\mathcal{CSS})$.

Proof of (i). Let u be a solution of $(\mathcal{CSS})$. If we multiply $(\mathcal{CSS})$ by u and integrate, we get

$$0 = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \omega u^2\right)\,dx + 3\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\,(h_u(|x|) - N)^2\,dx +$$

(5.13)

$$+2N\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\,(h_u(|x|) - N)\,dx - \int_{\mathbb{R}^2} |u|^{p+1}\,dx\ .$$

From (5.13) and the Pohozaev identity (Proposition 3.4), we obtain that, for any $l > 0$,

$$0 = (l+1)\int_{\mathbb{R}^2} |\nabla u|^2\,dx + \omega\int_{\mathbb{R}^2} u^2\,dx + (l+3)\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\,(h_u(|x|) - N)^2\,dx +$$

$$+2N\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\,(h_u(|x|) - N)\,dx - \left(\frac{p-1}{p+1}\,l + 1\right)\int_{\mathbb{R}^2} |u|^{p+1}\,dx\ . \tag{5.14}$$

By using (3.2) in (5.14),

$$0 \geq \int_{\mathbb{R}^2} \left(|\nabla u|^2 + \omega u^2\right)\,dx + 3\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\,(h_u(|x|) - N)^2\,dx +$$

$$+2N\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\,(h_u(|x|) - N)\,dx - \tag{5.15}$$

$$-\left(\frac{p-1}{p+1}\,l + 1\right)\int_{\mathbb{R}^2} |u|^{p+1}\,dx + \frac{l}{2}\,\int_{\mathbb{R}^2} |u|^4\,dx\ .$$

We can estimate

$$
\begin{aligned}
&3\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(h_u(|x|)-N\right)^2 + 2N\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(h_u(|x|)-N\right)\,dx \geq \\
&\geq -\frac{N^2}{3}\int_{\{N/3\leq h_u\leq N\}} \frac{u^2(x)}{|x|^2}\,dx\,,
\end{aligned}
\tag{5.16}
$$

where $\{N/3 \leq h_u \leq N\} = \{r_0 \leq |x| \leq r_1\}$ with $h_u(r_0) = N/3$ and $h_u(r_1) = N$ (here we have used that h_u is increasing in r). For any $r > 0$, by the definition of h_u we have

$$
4\pi h_u(r) = \int_{B_r} u^2(x)\,dx \leq Cr\left(\int_{B_r} u^4(x)\,dx\right)^{1/2}.
$$

Then

$$
\int_{B_r} h_u^2(|x|)\frac{u^2(x)}{|x|^2}\,dx \leq C h_u(r)\int_{B_r} u^4(x)\,dx\,. \tag{5.17}
$$

We now apply (5.17) to estimate

$$
\int_{\{N/3\leq h_u\leq N\}} \frac{u^2(x)}{|x|^2}\,dx \leq C\int_{\mathbb{R}^2} u^4\,dx\,. \tag{5.18}
$$

We apply (5.16) and (5.18) in (5.15):

$$
0 \geq \int_{\mathbb{R}^2} |\nabla u|^2\,dx + \omega\int_{\mathbb{R}^2} u^2\,dx - c\int_{\mathbb{R}^2} u^4\,dx + \frac{l}{2}\int_{\mathbb{R}^2} u^4 - \left(\frac{p-1}{p+1}\,l+1\right)\int_{\mathbb{R}^2} |u|^{p+1}\,dx\,.
$$

Therefore it suffices to take l so that $-c + (l/2) = 1$, and then to take ω so that the function

$$
s \to \omega s^2 + s^4 - \left(\frac{p-1}{p+1}\,l+1\right)|s|^{p+1}
$$

is non-negative for any s. Therefore u must be identically equal to zero.

□

References

[1] A. Ambrosetti & P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., 14(1973), 349–381.

[2] L. Bergé, A. de Bouard & J.C. Saut, *Blowing up time-dependent solutions of the planar Chern-Simons gauged nonlinear Schrödinger equation*, Nonlinearity, 8(1995), 235–253.

[3] H. Brézis & E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc., 88(1983), 486–490.

[4] J. Byeon, H. Huh & J. Seok, *On standing waves with a vortex point of order N for the nonlinear Chern-Simons-Schrödinger equations*, J. Differential Equations, 261(2016), 1285–1316.

[5] J. Byeon, H. Huh & J. Seok, *Standing waves of nonlinear Schrödinger equations with the gauge field*, J. Funct. Anal., 263(2012), 1575–1608.

[6] X. Chen & P. Smith, *On the unconditional uniqueness of solutions to the infinite radial Chern-Simons-Schrödinger hierarchy*, Preprint arXiv:1406.2649, to appear in Analysis and PDE's.

[7] P.L. Cunha, P. d'Avenia, A. Pomponio & G. Siciliano, *A multiplicity result for Chern-Simons-Schrödinger equation with a general nonlinearity*, NoDEA Nonlinear Differential Equations Appl., 22(2015), 1831–1850.

[8] S. Deser, R. Jackiw & S. Templeton, *Topologically massive gauge theories*, Ann. Physics, 140(1982), 372–411.

[9] B. Felsager, *Geometry, particles, and fields*, Springer-Verlag, New York, 1998.

[10] D. Gilbarg & N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer, 3rd edition, 1998.

[11] C. Hagen, *A new gauge theory without an elementary photon*, Ann. Physics, 157(1984), 342–359.

[12] H. Huh, *Blow-up solutions of the Chern-Simons-Schrödinger equations*, Nonlinearity, 22(2009) 967-974.

[13] H. Huh, *Standing waves of the Schrödinger equation coupled with the Chern-Simons gauge field*, J. Math. Phys., 53(2012), 063702, 8 pp.

[14] H. Huh, *Energy solution to the Chern-Simons-Schrödinger equations*, Abstract and Applied Analysis, 2013, Article ID 590653, 7 pp.

[15] R. Jackiw & S.-Y. Pi, *Soliton solutions to the gauged nonlinear Schrödinger equation on the plane*, Phys. Rev. Lett., 64(1990), 2969–2972.

[16] R. Jackiw & S.-Y. Pi, *Classical and quantal nonrelativistic Chern-Simons theory*, Phys. Rev. D (3), 42(1990), 3500–3513.

[17] L. Jeanjean, *On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on* $\mathbb{R}^n$, Proc. Roy. Soc. Edinburgh Sect. A, 129(1999), 787–809.

[18] Y. Jiang, A. Pomponio & D. Ruiz, *Standing waves for a gauged nonlinear Schrödinger equation with a vortex point*, to appear on Communications in Contemporary Mathematics, doi: 10.1142/S0219199715500741.

[19] A. Lerda, *Anyons: quantum mechanics of particles with fractional statistics*, Springer-Verlag, Berlin, Eidelberg, 1992.

[20] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, Part 1 and 2*, Ann. Inst. Henri Poincaré Anal. Mom linéaire, 1(2)(1984), 109–145 and 1(4)(1984), 223–283.

[21] B. Liu & P. Smith, *Global wellposedness of the equivariant Chern-Simons-Schrödinger equation*, Preprint, arXiv:1312.5567.

[22] B. Liu, P. Smith & D. Tataru, *Local wellposedness of Chern-Simons-Schrödinger*, Int. Math. Res. Notices, doi: 10.1093/imrn/rnt161.

[23] G.L. Naber, *Topology, geometry, and gauge fields: foundations-interactions*, Springer, New York, 2011.

[24] S.-J. Oh & F. Pusateri, *Decay and scattering for the Chern-Simons-Schrödinger equations*, Preprint arXiv:1311.2088.

[25] A. Pomponio & D. Ruiz, *A variational analysis of a gauged nonlinear Schrödinger equation*, J. Eur. Math. Soc., 17(2015), 1463–1486.

[26] A. Pomponio & D. Ruiz, *Boundary concentration of a gauged nonlinear Schrödinger equation on large balls*, Calc. Var. Partial Differential Equations, 53(2015), 289–316.

[27] J.F. Schonfeld, *A mass term for three-dimensional gauge fields*, Nuclear Physics B, 185(1981), 157–171.

[28] W.-A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys., 55(1977), 149–162.

[29] M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helv., 60(1985) 558–581.

[30] G. Tarantello, *Self-dual gauge field vortices: an analytical approach*, Birkäuser, Boston, 2007.

[31] F. Wilczek, *Fractional statistics and anyon superconductivity*, World Scientific, Teaneck, 1990.