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A survey on pseudorelativistic Hartree equation

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To Francesca, once more

Abstract. We present a survey on recent results about the pseudorelativistic Hartree equation governed by the non-local operator $\sqrt{\Delta + m^2}$ in $\mathbb{R}^N$.

1. Introduction

Let us consider a system of N identical bosons with relativistic dispersion relation and with a mean field Coulomb interaction. This system is described in the subspace $L^2_s(\mathbb{R}^{3N})$ of $L^2(\mathbb{R}^{3N})$ consisting of all functions symmetric with respect to permutations. The Hamiltonian function of the system is

$$H_N = \sum_{j=1}^{N} \sqrt{-\hbar^2 c^2 \Delta_{x_j} + m^2 c^4} + \frac{\nu}{N} \sum_{i<j} |x_i - x_j|^{-1} .$$

The constant ν can be either positive or negative, corresponding to a repulsive or an attractive interaction, respectively. We refer to [21] for more information on the physical motivation. As proved in [14], the macroscopic dynamics of this system can be described, in the limit $N \to +\infty$, by a single nonlinear relativistic Hartree equation,

$$i \frac{\partial \varphi}{\partial t} = \sqrt{-\hbar^2 c^2 \Delta + m^2 c^4}\, \varphi - m\varphi + V_{\mathrm{eff}}(\varphi)\varphi , \tag{1.1}$$

where

$$V_{\mathrm{eff}}(\varphi) = \nu \int_{\mathbb{R}^3} \Phi(|x-y|)|\varphi(t,y)|^2 \, dy$$

and $\Phi(|x-y|) = |x-y|^{-1}$. From time to time we will assume without loss of generality that $\hbar = 1 = c$, and we will focus on the attractive case $\nu < 0$.

It was proved in [19] that equation (1.1) is locally well-posed in the fractional Sobolev space $H^s(\mathbb{R}^3)$ with $s \geq 1/2$, and it is global in time for L^2–small initial data. Blow-up results were proved in [16].

Solitary wave solutions are functions that solve the stationary equation

$$\sqrt{-\Delta + m^2}\, \varphi - m\varphi + V_{\mathrm{eff}}(\varphi)\varphi = \lambda\varphi$$

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for some $\lambda \in \mathbb{R}$. In the paper [22], Lieb and Yau proved that stationary solutions with prescribed "mass" $\int_{\mathbb{R}^3} |\varphi|^2 = M$ exist, provided that $M < M_c$, a suitable threshold called *Chandrasekhar limit mass*. By [15][Theorem 2.1], [18, Theorem 1], such solutions lie in $H^s(\mathbb{R}^3)$ for all $s \geq 1/2$, are strictly positive and decay exponentially fast at infinity: for every given $0 < \delta < \min\{m, \nu\}$, there exists a constant $C > 0$ such that $|\varphi(x)| \leq Ce^{-\delta|x|}$ for all $x \in \mathbb{R}^3$. All these results depend strongly on the choice of the interaction $\Phi(|x-y|) = |x-y|^{-1}$, which makes many computations explicit.

More general cases were studied only in recent years. In this survey we want to offer the reader some insight about the pseudo-relativistic Hartree equation from a modern viewpoint.

2. A common variational framework

The Hamiltonian

$$H_0 = \sqrt{-\Delta + m^2}$$

can be defined for all $f \in L^2(\mathbb{R}^3)$ as the inverse Fourier transform of the L^2-function

$$p \mapsto \sqrt{|p|^2 + m^2}\hat{f}(p) \ ,$$

where $\hat{f}$ denotes the Fourier transform of f. To H_0 we associate a quadratic form $\mathcal{Q}$ by setting

$$\mathcal{Q}(f,g) = \int_{\mathbb{R}^3} \sqrt{|p|^2 + m^2}\ \hat{f}(p)\hat{g}(p)\, dp \ .$$

The domain of $\mathcal{Q}$ is the fractional Sobolev space

$$H^{1/2}(\mathbb{R}^3) = \left\{ f \in L^2(\mathbb{R}^3) \ \mid \ \int_{\mathbb{R}^3} (1 + |p|)\, |\hat{f}(p)|^2 \, dp < +\infty \right\} \ .$$

This setting is very elegant, but it soon becomes too abstract if we want to do some *hard analysis* in the spirit of PDE theory. A more convenient approach was suggested by the seminal paper [4], where a Dirichlet-to-Neumann extension was proposed for the homogeneous fractional Laplacian $(-\Delta)^s$, $0 < s < 1$.

We take any function $u \in \mathcal{S}(\mathbb{R}^3)$, the space of rapidly decreasing functions, and we introduce

$$v(x,y) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ip\cdot y} \hat{u}(p) e^{-\sqrt{|p|^2+m^2}\ x}\, dp$$

for $x > 0$ and $y \in \mathbb{R}^3$. It is readily seen that v solves the Dirichlet boundary value problem

$$\begin{cases} -\partial_x^2 v - \Delta_y v + m^2 v = 0 & \text{in } \mathbb{R}^4_+ = (0,+\infty) \times \mathbb{R}^3 \\ v(0,y) = u(y) & \text{for } y \in \mathbb{R}^3 = \partial\mathbb{R}^4_+ \ . \end{cases}$$

We point out that $\partial_x^2 + \Delta_y$ is the Laplacian in the variables $(x,y) \in \mathbb{R}^4_+$.

If we set

$$Tu(y) = -\frac{\partial v}{\partial x}(0,y) \ ,$$

i.e. Tu is the *outer normal derivative* of v at the boundary, then

$$Tu(y) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{ip\cdot y} \sqrt{|p|^2 + m^2}\ \hat{u}(p)\, dp \ ,$$

namely $T = H_0$ on the dense domain $\mathcal{S}(\mathbb{R}^3)$. As a consequence, the (formal) equation

$$H_0 u = g \qquad \text{in } \mathbb{R}^3$$

turns out to be equivalent to the Neumann problem

$$\begin{cases} -\partial_x^2 v - \Delta_y v + m^2 v = 0 & \text{in } \mathbb{R}^4_+ = (0,+\infty) \times \mathbb{R}^3 \\ v(0,y) = g(y) & \text{for } y \in \mathbb{R}^3 = \partial \mathbb{R}^4_+ \ , \end{cases}$$

whose solution can be (formally) identified as critical points of the Euler functional

$$I : H^1(\mathbb{R}^4_+) \to \mathbb{R}$$

$$v \mapsto \frac{1}{2} \int_{\mathbb{R}^4_+} \left(|\nabla v(x,y)|^2 + |v(x,y)|^2 \right) \, dx \, dy - \int_{\mathbb{R}^3} g(y) \gamma(v)(y) \, dy \ .$$

Here $\gamma : H^1(\mathbb{R}^4_+) \to H^{1/2}(\mathbb{R}^3)$ is the Sobolev trace operator, see for example [28] for more details on the theory of traces. We collect here only the most relevant results.

Proposition 2.1. *The map $\gamma : H^1(\mathbb{R}^4_+) \to H^{1/2}(\mathbb{R}^3)$ satisfies the estimate*

$$\|\gamma(v)\|^p_{L^p(\mathbb{R}^N)} \leq p \|v\|^{p-1}_{L^{2p-2}(\mathbb{R}^{N+1}_+)} \left\| \frac{\partial v}{\partial x} \right\|_{L^2(\mathbb{R}^{N+1}_+)} \tag{2.1}$$

for all $v \in H^1(\mathbb{R}^4_+)$.

Proof. It suffices to prove the estimate for $v \in H^1(\mathbb{R}^4_+) \cap C_0^\infty(\mathbb{R}^{N+1}_+)$. We compute

$$\int_{\mathbb{R}^N} |v(0,y)|^p \, dy = \int_{\mathbb{R}^N} dy \int_{-\infty}^0 \frac{\partial}{\partial x} |v(x,y)|^p \, dx \leq$$

$$\leq p \int_{\mathbb{R}^{N+1}_+} |v(x,y)|^{p-1} \left| \frac{\partial v}{\partial x}(x,y) \right| \, dx \, dy \leq$$

$$\leq p \left(\int_{\mathbb{R}^{N+1}_+} |v(x,y)|^{p-1} \, dx \, dy \right)^{1/2} \left(\int_{\mathbb{R}^{N+1}_+} \left| \frac{\partial v}{\partial x}(x,y) \right|^2 \, dx, dy \right)^{1/2}$$

and deduce that

$$\|v(0,\cdot)\|^p_{L^p(\mathbb{R}^N)} \leq p \|v\|^{p-1}_{L^{2p-2}(\mathbb{R}^{N+1}_+)} \left\| \frac{\partial v}{\partial x} \right\|_{L^2(\mathbb{R}^{N+1}_+)} \ .$$

The right-hand side is finite as long as $2 \leq 2p-2 \leq 2(N+1)/[(N+1)-2] = (2N+2)/(N-1)$, or $2 \leq p \leq 2N/(N-1)$. We conclude by density of $H^1(\mathbb{R}^4_+) \cap C_0^\infty(\mathbb{R}^{N+1}_+)$ that (2.1) holds for all $v \in H^1(\mathbb{R}^{N+1}_+)$.

□

As a consequence of (2.1) we record the following.

Corollary 2.2. *For all $\lambda > 0$, all $2 \leq p \leq 2N/(N-1)$ and all $v \in H^1(\mathbb{R}^{N+1}_+)$ there results*

$$\int_{\mathbb{R}^N} |\gamma(v)|^p \leq \frac{\lambda p^2}{4} \int_{\mathbb{R}^{N+1}_+} |v|^{2(p-1)} + \frac{1}{\lambda} \int_{\mathbb{R}^{N+1}_+} \left| \frac{\partial v}{\partial x} \right|^2 \, dx \, dy$$

3. Existence of ground states

In this section we will present a survey of recent results on the existence of ground states for the more general equation

$$\sqrt{-\Delta + m^2}\, u = \mu u + \nu |u|^{p-2} u + \sigma \left(W * u^2 \right) u \qquad \text{in } \mathbb{R}^N , \tag{3.1}$$

with $N \geq 2$. The symbol $*$ denotes the standard convolution of two functions: $f * g(x) = \int_{\mathbb{R}^N} f(x-y) g(y)\, dy$. The following result was obtained in [9].

Theorem 3.1 (Coti Zelati and Nolasco, 2011)**.** *Let*

$$2 < p < \frac{2N}{N-1} \quad , \quad \mu < m \quad , \quad \min\{\nu, \sigma\} \geq 0$$

but $\nu + \sigma > 0$. Assume that

$$W \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$$

for some $r > N/2$, $W \geq 0$, $\lim_{|x| \to +\infty} W(x) = 0$ and W is radially symmetric. Then equation (3.1) *has at least a positive solution $u \in C^\infty(\mathbb{R}^N)$ such that*

$$0 < u(y) \leq C e^{-\delta |y|} \qquad \text{for any } |y| \geq R ,$$

where $0 < \delta < m - \mu$ for $\mu \geq 0$, and $\delta = m$ for $\mu < 0$.

The proof of Theorem 3.1 is based on variational methods. By the Dirichlet-to-Neumann extension seen in Section 2, we only have to find nontrivial critical points of the functional $I \colon H^1(\mathbb{R}^{N+1}_+) \to \mathbb{R}$ defined by

$$I(v) = \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} \left(|\nabla v|^2 + m^2 |v|^2 \right) dx\, dy -$$
$$- \int_{\mathbb{R}^N} \left(\frac{\mu}{2} |\gamma(v)|^2 + \frac{\nu}{p} |\gamma(v)|^p + \frac{\sigma}{4} (W * \gamma(v)^2) \gamma(v)^2 \right) dy .$$

It follows from the following Proposition that $I \in C^1(H^1(\mathbb{R}^{N+1}_+))$.

Proposition 3.2. *Assume that W is as in Theorem 3.1. Then*

$$\int_{\mathbb{R}^N} \left(W * \gamma(v)^2 \right) \gamma(v)^2\, dy \leq C_W \|v\|_{H^1(\mathbb{R}^{N+1}_+)} \tag{3.2}$$

for some constant $C_W > 0$. Furthermore,

$$\left\| \left(W * \gamma(v)^2 \right) \gamma(v) \right\|_{L^m(\mathbb{R}^N)} \leq C \|W\|_{L^r((\mathbb{R}^N)} \|\gamma(v)\|^3_{L^\alpha(\mathbb{R}^N)} \tag{3.3}$$

for all $2 \leq \alpha \leq 2N/(N-1)$, $r > N/2$ and $2N/(N+4) < m < 2N/(N-3)$.

Proof. It follows from Young's inequality for convolutions that

$$\left\| \left(W * \gamma(v)^2 \right) \gamma(v)^2 \right\|_{L^1(\mathbb{R}^N)} \leq \|W\|_{L^r(\mathbb{R}^N)} \|\gamma(v)^2\|^2_{L^q(\mathbb{R}^N)}$$

provided that $1/r + 2/q = 2$. By the Sobolev embedding, $\gamma(v) \in L^{2q}(\mathbb{R}^N)$ whenever $2q \in [2, 2N/(N-1)]$. Under our assumption, $W = W_1 + W_2$ with $W_1 \in L^r(\mathbb{R}^N)$ and $W_2 \in L^\infty(\mathbb{R}^N)$. As a consequence,

$$\int_{\mathbb{R}^N} \left(W * \gamma(v)^2 \right) \gamma(v)^2\, dy =$$
$$= \int_{\mathbb{R}^N} \left(W_1 * \gamma(v)^2 \right) \gamma(v)^2\, dy + \int_{\mathbb{R}^N} \left(W_2 * \gamma(v)^2 \right) \gamma(v)^2\, dy \leq$$

$$\leq \|W_1\|_{L^r(\mathbb{R}^N)} \|\gamma(v)\|^4_{L^{4r/(2r-1)}(\mathbb{R}^N)} + \|W_2\|_L^\infty(\mathbb{R}^N) \|\gamma(v)\|^4_{L^2(\mathbb{R}^N)} \leq$$
$$\leq C_W \|v\|_{H^1(\mathbb{R}^{N+1}_+)}$$

for some suitable constant $C_W > 0$ depending on W, since $2 \leq 4r/(2r-1) < 2N/(N-1)$. This proves (3.2).

Similarly,

$$\int_{\mathbb{R}^N} \left|W * \gamma(v)^2 |\gamma(v)|\right|^m \, dy \leq \left(\int_{\mathbb{R}^N} \left|W * \gamma(v)^2\right|^{mq} \right)^{1/q} \left(\int_{\mathbb{R}^N} |\gamma(v)|^{mp} \right)^{1/p} \leq$$
$$\leq C \left(\int_{\mathbb{R}^N} |W|^r \right)^{m/r} \left(\int_{\mathbb{R}^N} |\gamma(v)|^{2s} \right)^{m/s} \left(\int_{\mathbb{R}^N} |\gamma(v)|^{mp} \right)^{1/p}$$

where $1/p + 1/q = 1$ and $1 + (1/mq) = 1/r + 1/s$. If we set $mp = \alpha = 2s$ we get $1 + 1/m = 1/r + 3/\alpha$ and therefore (3.3).

□

Moreover, every critical point v of I is a weak solution to the system

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ -\dfrac{\partial v}{\partial x} = \mu v + \nu |v|^{p-2} v + \sigma (W * v^2) v & \text{on } \mathbb{R}^N = \partial \mathbb{R}^{N+1}_+ \ . \end{cases}$$

The following space of symmetric function is introduced:

$$H^1_\# = \left\{ u \in H^1(\mathbb{R}^{N+1}_+) \mid u(x, Ry) = u(x,y) \text{ for all } R \in O(N) \right\} \ .$$

Here $O(N)$ stands for the set of rotations in $\mathbb{R}^N$, namely the set of orthogonal $N \times N$ matrices.

It is easily checked that I has the Mountain-Pass geometry ([9, Lemma 4.1]). Furthermore

Proposition 3.3 ([9, Lemma 4.2])**.** *The funcitonal I satisfies the Palais-Smale condition in $H^1_\#$: every sequence $\{v_n\}_n \subset H^1_\#$ such that $I(v_n) \to c$ and $I'(v_n) \to 0$ possesses a convergent subsequence.*

Now, standard results in Critical Point Theory yield that there exists a critical point $v_0 \in H^1_\#$ of I. Since I is invariant with respect to rotations in the y variable, the Symmetric Criticality Principle by Palais [26] implies that v_0 is also a critical point of I in $H^1(\mathbb{R}^{N+1}_+)$. The variational characterization of the Mountain-Pass Theorem says that

$$I(v_0) = c_\# = \inf_{g \in \Gamma_\#} \sup_{0 \leq t \leq 1} I(g(t)) \ ,$$

where $\Gamma_\# = \{g \in C([0,1], H^1_\#) \mid g(0) = 0, \ I(g(1)) < 0\}$. Now we remark that, for every critical point w of I $H^1(\mathbb{R}^{N+1}_+)$, the map $s \mapsto I(sw)$ has exactly one strict maximum point at $s = 1$. Moreover, $I(|v|) \leq I(v)$ for all $v \in H^1(\mathbb{R}^{N+1}_+)$. As a result, for all $s > 0$, $s \neq 1$, we have $I(s|v_0|) \leq I(sv_0) < I(v_0)$.

The path $s \mapsto s|v_0|$ belongs to $\Gamma_\#$ and hence

$$c_\# \leq \sup_{s>0} I(s|v_0|) \leq I(v_0) = c_\# \ .$$

If, by contradiction, $|v_0|$ is not a critical point of I, then we use the (pseudo)gradient flow to deform the path $s \mapsto s|v_0|$ into a new path $g \in \Gamma_\#$ such that $I(g(s)) < c_\#$ for all s, and this is clearly impossible due to the definition of $c_\#$. Hence $|v_0|$ is a weak solution of (3.1), so that we can always assume that there exists a nonnegative solution. The regularity of the solution is proved via a boostrap procedure, see [9, Section 3], while the last part of Theorem 3.1 is proved by a suitable application of the maximum principle in $\mathbb{R}^{N+1}_+$, see [9, Theorem 5.1].

In another recent paper, Mugnai (see [25]) studied the pseudorelativistic Hartree equation

$$\sqrt{-\Delta + m^2}\, u - \omega u - \lambda \left(W * u^2\right) + \frac{\partial F}{\partial s}(x, u) = 0 \quad \text{in } \mathbb{R}^N \tag{3.4}$$

where $F = F(x, s)$ can depend explicitly on the x variable.

Theorem 3.4 (Mugnai). *Assume that the following conditions hold:*

- $(F1)$ *$F : \mathbb{R}^N \times \mathbb{R} \to [0, +\infty)$ is such that the partial derivative $\partial F/\partial s$ is a Carathéodory function, $F(x, s) = F(|x|, s)$ for a.e. $x \in \mathbb{R}^N$ and every $s \in \mathbb{R}$, and $F(x, 0) = (\partial F/\partial s)(x, 0) = 0$ for a.e. $x \in \mathbb{R}^N$.*
- $(F2)$ *There exist constants $C_1 > 0$, $C_2 > 0$ and $2 < \ell < p < 2N/(N - 1)$ such that $|(\partial F/\partial s)(x, s)| \leq C_1 |s|^{\ell-1} + C_2 |s|^{p-1}$ for a.e. $x \in \mathbb{R}^N$ and every $s \in \mathbb{R}$.*
- $(F3)$ *There exists $2 \leq k \leq 4$ such that $0 \leq s(\partial F/\partial s)(x, s) \leq kF(x, s)$ for a.e. $x \in \mathbb{R}^N$ and every $s \in \mathbb{R}$.*
- (W) *$W : \mathbb{R}^N \to [0, +\infty)$ is such that $W(x) = W(|x|)$, $W = W_1 + W_2$ where $W_1 \in L^r(\mathbb{R}^N)$ for some $r > N/2$ and $W_2 \in L^\infty(\mathbb{R}^N)$.*

The for every $\lambda > 0$ and $\omega < m$ there exists a nontrivial solution $u \in H^{1/2}(\mathbb{R}^N)$ of equation (3.4) *which is radially symmetric. If, in addition, $F(x, s) \geq F(|x|, s)$ for a.e. $x \in \mathbb{R}^N$ and every $s \in \mathbb{R}$, then this solution is strictly positive.*

The proof of this theorem, which we omit, is again based on the Dirichlet-to-Neumann extension introduced in Section 2.

4. Existence of solutions with prescribed mass

When dealing with a pseudorelativistic Hartree equation, the most relevant variational problem from a physical point of view is the minimization of the action functional

$$\mathcal{A}(u) = \frac{1}{2} \int_{\mathbb{R}^N} u \left(\sqrt{-\Delta + m^2} - m\right) u\, dy + \\ + \frac{\nu}{p} \int_{\mathbb{R}^N} |u|^p\, dy - \frac{\sigma}{4} \int_{\mathbb{R}^N} \left(W * |u|^2\right) u\, dy$$

under the constraint

$$\int_{\mathbb{R}^N} |u|^2\, dy = M\,.$$

As we anticipated in the Introduction, the case $\nu = 0$ was studied in [22]. A very recent generalization appears in [10]. We first recall a useful function space.

Definition 4.1. Let $q \geq N$. A (measurable) function f belongs to the *weak L^q space*, written $f \in L^q_w(\mathbb{R}^N)$, if

$$\sup_{\alpha > 0} \alpha\, \mathcal{L}^N \left(\left\{x \in \mathbb{R}^N \mid |f(x)| > \alpha\right\}\right) < +\infty\,.$$

Here $\mathcal{L}^N$ is the Lebesgue measure in $\mathbb{R}^N$. For $f \in L^q_w(\mathbb{R}^N)$ we define

$$\|f\|_{q,w} = \sup_A \mathcal{L}^N(A)^{1/r} \int_A |f|\,,$$

where $1/q + 1/r = 1$ and A denotes any set with finite Lebesgue measure.

The main result proved in [10] is the following.

Theorem 4.2 (Coti Zelati and Nolasco, 2013)**.** *Let $W \in L^q_w(\mathbb{R}^N)$, where $q \geq N \geq 2$, and $W(y) \geq 0$ for all $y \in \mathbb{R}^N$. Suppose that for some $\alpha > 0$: $W(\lambda^{-1}y) \geq \lambda^\alpha W(y)$ for all $0 < \lambda < 1$ and all $y \in \mathbb{R}^N$. Finally, assume that W is rotationally symmetric and that $\lim_{|x|\to+\infty} W(|x|) = 0$.*

Take $\nu \geq 0$, $\sigma > 0$ and $2 + 2/q < p < 2N/(N-1)$. Then:

- *if $\nu > 0$ or $\eta = 0$ and $q > N$, then for all $M > 0$ there exists a strictly positive minimizer $u \in H^{1/2}(\mathbb{R}^N)$ of $\mathcal{A}$ such that $\int_{\mathbb{R}^N} |u|^2\, dy = M$.*
- *If $\nu = 0$ and $q = N$, there is a critical value $M_c > 0$ such that for all $0 < M < M_c$ there is a strictly positive minimizer $u \in H^{1/2}(\mathbb{R}^N)$ of $\mathcal{A}$ such that $\int_{\mathbb{R}^N} |u|^2\, dy = M$.*

Moreover there exists $\mu > 0$ such that u is a smooth, exponentially decaying at infinity, solution of

$$\left(\sqrt{-\Delta + m^2} - m\right) u = -\mu u - \nu |u|^{p-2} u + \sigma (W * |u|^2) u \qquad \text{in } \mathbb{R}^N$$

and u is radial if $W = W(r)$ is a decreasing function of $r > 0$.

Remark 4.3. The existence of *solitons* in the range $2 < p \leq 2 + 2/q$ seems to be an open problem. Furthermore, a singular convolution kernel like $W(x) = |x|^{-\alpha}$ is allowed in Theorem 4.2 only if $\alpha = N/q$.

To prove Theorem 4.2 a few technical facts are needed.

Proposition 4.4 ([20])**.** *The following weak Young inequality holds: for every $g \in L^q_w(\mathbb{R}^N)$, $f \in L^p(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$ where p, q, $r \in (1, +\infty)$ and $1/q + 1/p + 2/r = 2$, there results*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(y) g(y-z) h(y)\, dy\, dz \leq C(p,q,r) \|g\|_{q,w} \|f\|_{L^p} \|h\|_{L^r} .$$

Lemma 4.5. *Let $W \in L^q_w(\mathbb{R}^N)$ with $q \geq N$. If $4q/(2q-1) < p \leq 2N/(N-1)$ then*

$$\int_{\mathbb{R}^N} (W * |w|^2) w\, dy \leq C \|W\|_{q,w} \|w\|_{L^2}^{4-2p/q(p-2)} \|w\|_{L^p}^{2p/q(p-2)} .$$

In particular, if $q = N$, then

$$\int_{\mathbb{R}^N} (W * |w|^2) w\, dy \leq C \|W\|_{N,w} \|w\|_{L^2}^2 \|w\|_{L^{2N/(N-1)}}^2 .$$

The proof is a straightforward consequence of Proposition 4.4 and the Hölder inequality. Putting together these estimates, it follows that the functional

$$J(v) = \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} \left(|\nabla v|^2 + m^2 |v|^2 \right) dx\, dy -$$

$$- \int_{\mathbb{R}^N} \left(m |\gamma(v)|^2 + \frac{\nu}{p} |\gamma(v)|^p + \frac{\sigma}{4} (W * \gamma(v)^2) \gamma(v)^2 \right) dy .$$

is of class C^1 on $H^1(\mathbb{R}^{N+1}_+)$. The following Lemma connects the functionals J and $\mathcal{A}$.

Lemma 4.6 ([10]). *For $u \in H^1(\mathbb{R}^{N+1}_+)$, let $w = \gamma(u) \in H^{1/2}(\mathbb{R}^N)$, $\hat{w} = \mathcal{F}(w)$ and*

$$v(x,y) = \mathcal{F}^{-1}\left(e^{-x\sqrt{m^2+|\cdot|^2}}\;\hat{w}\right) .$$

Then $v \in H^1(\mathbb{R}^{N+1}_+)$, $\|v\|_{H^1} = \|u\|_{H^{1/2}}$, $J(v) \leq J(u)$ and $J(v) = \mathcal{A}(w)$.

In [10] the following minimization problem is considered:

$$I(M) = \inf\left\{J(v) \mid v \in \mathcal{M}_M\right\} ,$$

where

$$\mathcal{M}_M = \left\{v \in H^1(\mathbb{R}^{N+1}_+) \mid \int_{\mathbb{R}^N} |\gamma(v)|^2 = M\right\} .$$

By a direct application of P.-L. Lions' *principle of concentration-compactness* (see [23]), the authors can prove that $I(M)$ is always attained.

Proposition 4.7 ([10, Proposition 3.5]). *For every $M > 0$ there exists a function $v \in H^1(\mathbb{R}^{N+1}_+)$ such that*

$$J(v) = I(M) \quad , \quad \int_{\mathbb{R}^N} |\gamma(v)|^2 = M ,$$

i.e. v is a minimizer for J in $\mathcal{M}_M$.

The proof of Theorem 4.2 can be now accomplished as follows. By the previous Proposition, there exists a function $u \in H^1(\mathbb{R}^{N+1}_+)$ that minimizes J on $\mathcal{M}_M$. Therefore u can be assumed to be non-negative and of the form

$$u(x,y) = \mathcal{F}^{-1}\left(e^{-x\sqrt{m^2+|\cdot|^2}}\;\hat{w}\right)$$

where $w = \gamma(u) \in H^{1/2}(\mathbb{R}^N)$. If W is a nonincreasing radial function, then w can be assumed to be a radial nonincreasing function. Indeed let w^* be the spherically symmetric decreasing rearrangement of w and define

$$u^*(x,y) = \mathcal{F}^{-1}\left(e^{-x\sqrt{m^2+|\cdot|^2}}\;\widehat{w^*}\right) .$$

Then $J(u^*) = \mathcal{A}(w^*)$. The properties of the symmetric drecreasing rearrangement yield that $J(u^*) = \mathcal{A}(w^*) \leq \mathcal{A}(w) = J(u) = I(M)$. Moreover there is a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$\begin{cases} -\Delta u + m^2 u = 0 & \text{in } \mathbb{R}^{N+1}_+ , \\ -\dfrac{\partial u}{\partial x} + \mu u = mu - \nu|u|^{p-2}u + \sigma(W * |u|^2)u & \text{on } \mathbb{R}^N . \end{cases}$$

It can be proved that $\mu > 0$ by testing against u and remarking that $J(u) < 0$. We refer to [10] for more details about the regularity properties of the weak solution.

5. Eigenvalues

In [12] Coti Zelati and Nolasco offered a variational characterization of the eigenvalues and eigenvectors of the operator

$$H = H_0 + V = \sqrt{-c^2\Delta + m^2c^4} + V ,$$

where H_0 is the relativistic (free) Hamiltonian operator and V ia real-valued potential function. They assumed that (h1) $V \in L^3_w(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, $V \in L^\infty(\mathbb{R}^3 \setminus B_{R_0})$ for some $R_0 > 0$ and

$$\lim_{R\to+\infty} \|V\|_{L^\infty(\mathbb{R}^N\setminus B(0,R))} = 0 \quad ; \quad \lim_{R\to+\infty} \operatorname{ess\,sup}_{|x|>R} V(x)|x|^2 = -\infty$$

(h2) V is H_0–form bounded with bound less than one, i.e. there exists $a \in (0,1)$ such that $|\langle\phi \mid V\phi\rangle_{L^2}| \le a\langle\phi \mid H_0\phi\rangle_{L^2}$ for all $\phi \in H^{1/2}(\mathbb{R}^3,\mathbb{C})$.

As a particularly important case, the Coulomb potential of a nucleus with Z protons

$$V(x) = -\frac{Ze^2}{|x|}$$

is allowed in assumption (h2) by virtue of two inequalities, the first proved by Hardy and the second proved by Kato and Herbst (see the references in [12]):

$$\text{for all } \psi \in H^1(\mathbb{R}^3): \qquad \left\||x|^{-1}\psi\right\|_{L^2} \le 2\|\nabla\psi\|_{L^2} \le \frac{2}{c\hbar}\left\|\sqrt{-c^2\hbar^2\Delta + m^2c^4}\;\psi\right\|_{L^2}$$

$$\text{for all } \psi \in H^{1/2}(\mathbb{R}^3): \quad \langle\psi \mid |x|^{-1}\psi\rangle_{L^2} \le \frac{\pi}{2}\langle\psi \mid \sqrt{-\Delta}\;\psi\rangle_{L^2} \le$$
$$\le \frac{\pi}{2c\hbar}\langle\psi \mid \sqrt{-c^2\Delta + m^2c^4}\;\psi\rangle_{L^2} .$$

As a consequence, (h2) is satisfied for all $0 < Z < 68$ by Hardy and for all $0 < Z < 87$ by Kato.

By considering the functional $J : H^1(\mathbb{R}^4_+,\mathbb{C})$ defined by

$$J(\phi) = \int_{\mathbb{R}^4_+} \left(|\partial_x\phi|^2 + c^2|\nabla_y\phi|^2 + m^2c^4|\phi|^2\right) dx\,dy + \int_{\mathbb{R}^3} \langle\gamma(\phi) \mid V\gamma(\phi)\rangle\,dy$$

Coti Zelati and Nolasco proved the following results.

Theorem 5.1 ([12, Theorem 1 and Theorem 2])**.** *Let* $m > 0$ *and* (h1)–(h2) *hold. Then there exist* $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots$ *and* $\phi_1,\ \phi_2, \cdots, \phi_k, \cdots \in H^1(\mathbb{R}^4_+,\mathbb{C})$ *such that, for all* $k \in \mathbb{N}$,

$$\lambda_k = J(\phi_k) = \inf_{X_k} J$$

where

$$X_1 = \left\{\phi \in H^1(\mathbb{R}^4_+,\mathbb{C}) \mid \|\gamma(\phi)\|_{L^2} = 1\right\}$$

and, for $1 < k \in \mathbb{N}$,

$$X_k = \left\{\phi \in H^1(\mathbb{R}^4_+,\mathbb{C}) \mid \|\gamma(\phi)\|_{L^2} = 1,\ \langle\gamma(\phi) \mid \gamma(\phi_i)\rangle_{L^2} = 0,\ i = 1, \cdots, k-1\right\} .$$

Moreover $\{\lambda_k\}_k \subset \sigma_{\rm disc}(H_0 + V)$ *and*

$$0 < \lambda_1 \le \cdots \le \lambda_k \le \lambda_{k+1} \to \inf\ \sigma_{\rm ess}(H_0 + V) = mc^2 \qquad \text{as } k \to +\infty .$$

The functions $\varphi_k = \gamma(\phi_k) \in H^{1/2}(\mathbb{R}^3,\mathbb{C})$ *are the eigenfunctions of the operator* $H_0 + V$ *, and the functions* $\phi_k \in H^1(\mathbb{R}^4_+,\mathbb{C})$ *are weak solution of the Neumann problem*

$$\begin{cases} -\partial_x^2\phi_k + \Delta_y\phi_k + m^2c^4\phi_k = 0 & \text{in } \mathbb{R}^4_+ \\[2ex] \dfrac{\partial\phi_k}{\partial x} + V\varphi_k = \lambda_k\varphi_k & \text{on } \mathbb{R}^3 . \end{cases}$$

Finally, for all $0 \le \beta < \sqrt{m^2c^4 - \lambda_k^2}$ *there exists* $R > 0$ *such that* $e^{(\beta/c)|y|}\varphi_k \in L^2(\mathbb{R}^3 \setminus B_R)$.

Here the symbol σ_{disc} and σ_{ess} denote, respectively, the discrete and the essential spectrum of an operator. In the same paper, the following regularity result was proved.

Theorem 5.2 ([12, Theorem 3]). *Let $\phi_k \in H^1(\mathbb{R}^4_+, \mathbb{C})$ and $\varphi_k = \gamma(\phi_k)$ be the functions given by Theorem 5.1 and R_0 be given by assumption* (h1). *Then we have*

1. *$\phi_k \in W^{1,q}([0,r) \times (\mathbb{R}^3 \setminus B_{R_0}))$ for any $q \in [2,+\infty)$, $r > 0$;*
2. *$\phi_k \in C^{0,\alpha}([0,+\infty) \times (\mathbb{R}^3 \setminus B_{R_0}))$ for any $\alpha \in [0,1]$ and $\varphi_k \in C^{0,\alpha}(\mathbb{R}^3 \setminus B_{R_0})$;*
3. *if in addition $V \in L^3_{\text{loc}}(\mathcal{U})$ for some $\mathcal{U} \subset \mathbb{R}^3$ then for every $\mathcal{V} \Subset \mathcal{U}$ (i.e. such that the closure of $\mathcal{V}$ is a compact subset of $\mathcal{U}$) $\phi_k \in W^{1,p}([0,r) \times \mathcal{V})$ for every $p \in [2,+\infty)$ and $r > 0$, and $\varphi_k \in C^{0,\alpha}(\mathcal{V})$ for every $\alpha \in [0,1)$.*

6. Ground states with potentials

In [6] the problem

$$\sqrt{-\Delta + m^2}\, u + Vu = \left(W * |u|^\theta\right) |u|^{\theta-2} u \qquad \text{in } \mathbb{R}^N \tag{6.1}$$

was considered. Here $N \geq 3$, $m > 0$, V is an external potential and $W \geq 0$ is a radially symmetric convolution kernel such that $\lim_{|x|\to+\infty} W(|x|) = 0$; moreover

(V1) $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous and bounded function, and $V(y) + V_0 \geq 0$ for every $y \in \mathbb{R}^N$ and for some $V_0 \in (0, m)$.

(V2) There exist $R > 0$ and $k \in (0, 2m)$ such that

$$V(x) \leq V_\infty - e^{-k|x|} \qquad \text{for all } |x| \geq R$$

where $V_\infty = \liminf_{|x|\to+\infty} V(x) > 0$.

(W) $W \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ for some $r > \max\left\{1, \dfrac{N}{N(2-\theta)+\theta}\right\}$ and $2 \leq \theta < \dfrac{2N}{N-1}$.

Theorem 6.1. *Retain assumptions* (V1), (V2) *and* (W). *Then equation* (6.1) *has at least a positive solution $u \in H^{1/2}(\mathbb{R}^N)$.*

Remark 6.2. Theorem 6.1 applies for a large class of bounded electric potentials without symmetric constraints and covers the physically relevant cases of Newton or Yukawa type two body interaction, i.e. $W(x) = 1/|x|^\lambda$ with $0 < \lambda < 2$, $W(x) = e^{-|x|}/|x|$.

The main idea exploits the same Dirichlet-to-Neumann extension that we introduced in Section 2. We start by collecting some useful inequalities.

Writing $W = W_1 + W_2 \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ we can estimate the convolution term as follows:

$$\int_{\mathbb{R}^N} (W * |\gamma(v)|^\theta)|\gamma(v)|^\theta = \int_{\mathbb{R}^N} (W_1 * |\gamma(v)|^\theta)|\gamma(v)|^\theta + \int_{\mathbb{R}^N} (W_2 * |\gamma(v)|^\theta)|\gamma(v)|^\theta \leq$$

$$\leq |W_1|_r |\gamma(v)|^{2\theta}_{2r\theta/(2r-1)} + |W_2|_\infty |\gamma(v)|^{2\theta}_\theta \leq |W_1|_r \|v\|^{2\theta} + |W_2|_\infty \|v\|^{2\theta}\,. \tag{6.2}$$

Since

$$r > \frac{N}{N(2-\theta)+\theta} \quad \text{and} \quad 2 \le \theta < \frac{2N}{N-1} \ ,$$

there results

$$\frac{2r-1}{2\theta r} = \frac{1}{\theta} - \frac{1}{2\theta r} > \frac{1}{\theta} - \frac{N(2-\theta)+\theta}{2\theta N} = \frac{N-1}{2N} \ ,$$

and thus

$$\frac{2\theta r}{2r-1} < \frac{2N}{N-1} \ .$$

6.1. **The limit problem.** Let us consider the space of the symmetric functions

$$H^\sharp = \{u \in H^1(\mathbb{R}^{N+1}_+) \mid u(x,Ry) = u(x,y) \text{ for all } R \in O(N)\} \ .$$

Let us consider the functional $J_\alpha : H^\sharp \to \mathbb{R}$ defined by setting

$$J_\alpha(v) = \frac{1}{2} \iint_{\mathbb{R}^{N+1}_+} \left(|\nabla v|^2 + m^2 v^2\right) dx\, dy + \frac{1}{2} \int_{\mathbb{R}^N} \alpha \gamma(v)^2\, dy -$$
$$-\frac{1}{2\theta} \int_{\mathbb{R}^N} (W * |\gamma(v)|^\theta)|\gamma(v)|^\theta\, dy \ .$$

where $W \ge 0$ is radially symmetric, $\lim_{|x|\to+\infty} W(|x|) = 0$ and assumption (W) holds. If $\alpha > -m$, we can extend the arguments in Theorem 4.3 [9], for the case $\theta = 2$, and prove that the functional J_α has a Mountain Pass critical point $v_\alpha \in H^\sharp$, namely

$$J_\alpha(v_\alpha) = E_\alpha = \inf_{g\in\Gamma_\sharp} \max_{t\in[0,1]} J_\alpha(g(t))$$

where $\Gamma_\sharp = \{g \in C([0,1]; H^\sharp) \mid g(0) = 0,\ J_\alpha(g(1)) < 0\}$. The critical point v_α corresponds to a weak solution of

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ -\dfrac{\partial v}{\partial x} = -\alpha v + (W * |v|^\theta)|v|^{\theta-2} v & \text{in } \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+ \ . \end{cases}$$

In the sequel, we need a standard characterization of the mountain-pass level E_α. Let us define the *Nehari manifold* $\mathcal{N}_\alpha$ associated to the functional J_α:

$$\mathcal{N}_\alpha = \Big\{ v \in H^\sharp \mid \iint_{\mathbb{R}^N\times\mathbb{R}^N} |\nabla v|^2 + m^2 v^2\, dx\, dy =$$
$$= -\alpha \int_{\mathbb{R}^N} \gamma(v)^2\, dy + \int_{\mathbb{R}^N} \left(W * |\gamma(v)|^\theta\right) |\gamma(v)|^\theta\, dy \Big\} \ .$$

Lemma 6.3. *There results*

$$\inf_{v\in\mathcal{N}_\alpha} J_\alpha(v) = \inf_{v\in H^\sharp} \max_{t>0} J_\alpha(tv) = E_\alpha \ . \tag{6.3}$$

Proof. The proof is straightforward, since J_α is the sum of homogeneous terms; we follow [29]. First of all, for $v \in H^\sharp$ we compute

$$J_\alpha(tv) = \frac{t^2}{2} \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} |\nabla v|^2 + m^2 v^2\, dx\, dy + \alpha \int_{\mathbb{R}^N} \gamma(v)^2\, dy \right) -$$
$$-\frac{t^{2\theta}}{2\theta} \int_{\mathbb{R}^N} \left(W * |\gamma(v)|^\theta\right) |\gamma(v)|^\theta\, dy \ .$$

Since $\theta \geq 2$ and by using (7.12), it is easy to check that $t \in (0, +\infty) \mapsto J(tv)$ possesses a unique critical point $t = t(v) > 0$ such that $t(v)v \in \mathcal{N}_\alpha$. Moreover, since J_α has the mountain-pass geometry, $t = t(v)$ is a maximum point. It follows that

$$\inf_{v \in \mathcal{N}_\alpha} J_\alpha(v) = \inf_{v \in H^\sharp} \max_{t>0} J_\alpha(tv) .$$

The manifold $\mathcal{N}_\alpha$ splits $H^\sharp$ into two connected components, and the component containing 0 is open. In addition, J_α is non-negative on this component, because $\langle J'_\alpha(tv), v \rangle \geq 0$ when $0 < t \leq t(v)$. It follows immediately that any path $\gamma : [0,1] \to H^\sharp$ with $\gamma(0) = 0$ and $J_\alpha(\gamma(1)) < 0$ must cross $\mathcal{N}_\alpha$, so that

$$E_\alpha \geq \inf_{v \in \mathcal{N}_\alpha} J_\alpha(v) .$$

The proof of (6.3) is complete.

□

Following completely analogous argument in Theorem 3.14 and Theorem 5.1 in [9], we can state the following result.

Theorem 6.4. *Let $\alpha + m > 0$ and (W) holds. Then $v_\alpha \in C^\infty([0,+\infty) \times \mathbb{R}^N)$, $v_\alpha(x,y) > 0$ in $[0,\infty) \times \mathbb{R}^N$ and for any $0 \leq \sigma \in (-\alpha, m)$ there exists $C > 0$ such that*

$$0 < v_\alpha(x,y) \leq C e^{-(m-\sigma)\sqrt{x^2+|y|^2}} e^{-\sigma x}$$

for all $(x,y) \in [0,+\infty) \times \mathbb{R}^N$. In particular,

$$0 < v_\alpha(0,y) \leq C e^{-\delta |y|} \qquad \text{for every } y \in \mathbb{R}^N$$

where $0 < \delta < m + \alpha$ if $\alpha \leq 0$, and $\delta = m$ if $\alpha > 0$.

6.2. **The Palais-Smale condition.** For any $v \in H^1(\mathbb{R}^{N+1}_+)$ we denote

$$\mathbb{D}(v) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y) |\gamma(v)(x)|^\theta |\gamma(v)(y)|^\theta dxdy .$$

Inequality (6.2) yields immediately

$$\mathbb{D}(v) \leq K \|v\|^{2\theta}$$

for every $v \in H^1(\mathbb{R}^{N+1}_+)$.

Lemma 6.5. *Let $\{v_n\}_n$ be a sequence in $H^1(\mathbb{R}^{N+1}_+)$ such that $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^{N+1}_+)$*

$$I(v_n) \to c < E_{V_\infty} \quad \text{and} \quad I'(v_n) \to 0$$

where $V_\infty := \liminf_{|x| \to \infty} V(x) > 0$. Then a subsequence of $\{v_n\}_n$ converges strongly to 0 in $H^1(\mathbb{R}^{N+1}_+)$.

Proof. First of all, we recall (7.12), and we rewrite $I(v)$ as

$$I(v) = \frac{1}{2} \int_{\mathbb{R}^{N+1}_+} |\nabla v|^2 + m^2 |v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} (V + V_0) \gamma(v)^2 - \frac{V_0}{2} \int_{\mathbb{R}^N} \gamma(v)^2 - \frac{1}{2\theta} \mathbb{D}(v) ,$$

so that $V + V_0 \geq 0$ everywhere. Now,

$$c + 1 + \|v_n\| \geq I(v_n) - \frac{1}{2} \langle I'(v_n), v_n \rangle = \left(\frac{1}{2} - \frac{1}{2\theta} \right) \mathbb{D}(v_n) ,$$

which implies that, for some constants C_1 and C_2,

$$\frac{1}{2\theta}\,\mathbb{D}(v_n) \le C_1\|v_n\| + C_2\ .$$

But then, using (7.12),

$$\begin{aligned}
c+1 \ge I(v_n) &\ge \\
&\ge \frac{1}{2}\int_{\mathbb{R}^{N+1}_+}|\nabla v_n|^2 + \frac{m^2}{2}\int_{\mathbb{R}^{N+1}_+}|v_n|^2 - \\
&\quad -\frac{V_0}{2}\left(m\int_{\mathbb{R}^{N+1}_+}|v_n|^2 + \frac{1}{m}\int_{\mathbb{R}^{N+1}_+}|\nabla v_n|^2\right) - C_1\|v_n\| - C_2 = \\
&= \frac{1}{2}\left(1-\frac{V_0}{m}\right)\int_{\mathbb{R}^{N+1}_+}|\nabla v_n|^2 + \frac{m(m-V_0)}{2}\int_{\mathbb{R}^{N+1}_+}|v_n|^2 - C_1\|v_n\| - C_2
\end{aligned}$$

and since $m - V_0 > 0$ we deduce that $\{v_n\}$ is a bounded sequence in $H^1(\mathbb{R}^{N+1}_+)$.

A standard argument shows that $\|v_n\|$ is bounded in $H^1(\mathbb{R}^{N+1}_+)$ and

$$\frac{\theta-1}{2\theta}\left(\|v_n\|^2 + \int_{\mathbb{R}^N} V(y)\gamma(v_n)^2\,dy\right) \to c \quad \text{and} \quad \frac{\theta-1}{2\theta}\,\mathbb{D}(v_n) \to c\ .$$

Therefore $c \ge 0$. If $c = 0$, then

$$\begin{aligned}
o(1) &= \left(\|v_n\|^2 + \int_{\mathbb{R}^N} V(y)\gamma(v_n)^2\,dy\right) \ge \\
&\ge \left(1-\frac{V_0}{m}\right)\int_{\mathbb{R}^{N+1}_+}|\nabla v_n|^2 + m(m-V_0)\int_{\mathbb{R}^{N+1}_+}|v_n|^2\ ,
\end{aligned}$$

and $m - V_0 > 0$ yields that $v_n \to 0$ strongly in $H^1(\mathbb{R}^{N+1}_+)$.

Assume therefore that $c > 0$. Fix $\alpha < V_\infty$ such that $c < E_\alpha$, and $R_0 > 0$ such that $V(x) \ge \alpha$ if $|x| \ge R_0$. Let $\varepsilon \in (0,1)$. Since $\{v_n\}_n$ is bounded in $H^1(\mathbb{R}^{N+1}_+)$ there exists $R_\varepsilon > R_0$ such that $R_\varepsilon \to +\infty$ as $\varepsilon \to 0$ and, after passing to a subsequence,

$$\iint_{S_{R_\varepsilon}} \left(|\nabla v_n|^2 + m^2 v_n^2\right) dx\,dy + \int_{A_{R_\varepsilon}} V(y)\gamma(v_n)^2\,dy < \varepsilon \qquad \text{for all } n \in \mathbb{N}\ .$$

where

$$\begin{aligned}
S_{R_\varepsilon} &= \left\{z = (x,y) \in \mathbb{R}^{N+1}_+ \ \middle|\ R_\varepsilon < |z| < R_\varepsilon + 1\right\} \\
A_{R_\varepsilon} &= \left\{y \in \mathbb{R}^N \ \middle|\ R_\varepsilon < |y| < R_\varepsilon + 1\right\}\ .
\end{aligned}$$

If this is not the case, for any $m \in \mathbb{N}$, $m \ge R_0$ there exists $\nu(m) \in \mathbb{N}$ such that

$$\iint_{S_m} \left(|\nabla v_n|^2 + m^2 v_n^2\right)\,dx\,dy + \int_{A_m} V(y)\gamma(v_n)^2\,dy \ge \varepsilon$$

for any $n \in \mathbb{N}$, $n \ge \nu(m)$. We can assume that $\nu(m)$ is nondecreasing. Therefore for any integer $m \ge R_0$ there exists an integer $\nu(m)$ such that

$$\|v_n\|^2 + \int_{\mathbb{R}^N} V(y)\gamma(v_n)^2\,dy \ge$$

$$\geq \iint_{T_m} \left(|\nabla v_n|^2 + m^2 v_n^2\right) \, dx \, dy + \int_{B_m} V(y)\gamma(v_n)^2 \, dy \geq$$
$$\geq (m - R_0)\varepsilon$$

for any $n \geq \nu(m)$, where $T_m = \left\{z = (x,y) \in \mathbb{R}^{N+1}_+ \mid R_0 < |z| < m\right\}$ and $B_m = \{y \in \mathbb{R}^N \mid R_0 < |y| < m\}$, which contradicts the fact that $\|v_n\|$ is bounded.

We may assume that $|v_n| \to 0$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ with $p < 2N/(N-1)$ and thus $|\gamma(v_n)| \to 0$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$.

Let $\xi_\varepsilon \in C^\infty(\mathbb{R}^{N+1}_+)$ be a symmetric function, namely $\xi_\varepsilon(x, gy) = \xi_\varepsilon(x,y)$ for all $g \in O(N)$, $x > 0$, $y \in \mathbb{R}^N$. Moreover assume that $\xi_\varepsilon(z) = 0$ if $|z| \leq R_\varepsilon$ and $\xi_\varepsilon(z) = 1$ if $|z| \geq R_\varepsilon + 1$ and $\xi(z) \in [0,1]$ for all $z \in \mathbb{R}^{N+1}_+$. Set $w_n = \xi_\varepsilon v_n$. We apply now Young's inequality with $p = q = 2r/(2r-1)$ and $h = W$, $f = |\gamma(v_n)|^\theta$, $g = |\gamma(v_n)|^\theta - |\gamma(w_n)|^\theta$:

$$|\mathbb{D}(v_n) - \mathbb{D}(w_n)| \leq$$
$$\leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y) \left| |\gamma(v_n)(x)|^\theta |\gamma(v_n)(y)|^\theta - |\gamma(w_n)(x)|^\theta |\gamma(w_n)(y)|^\theta \right| \, dx \, dy =$$
$$= \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y) \left| |\gamma(v_n)(x)|^\theta |\gamma(v_n)(y)|^\theta - |\gamma(v_n)(x)|^\theta |\gamma(w_n)(y)|^\theta + \right.$$
$$\left. + |\gamma(v_n)(x)|^\theta |\gamma(w_n)(y)|^\theta - |\gamma(w_n)(x)|^\theta |\gamma(w_n)(y)|^\theta \right| \, dx \, dy \leq$$
$$\leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y) |\gamma(v_n)(x)|^\theta \left| |\gamma(v_n)(y)|^\theta - |\gamma(w_n)(y)|^\theta \right| \, dx \, dy +$$
$$+ \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y) |\gamma(w_n)(y)|^\theta \left| |\gamma(v_n)(x)|^\theta - |\gamma(w_n)(x)|^\theta \right| \, dx \, dy \leq$$
$$\leq 2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} W(x-y) |\gamma(v_n)(x)|^\theta \left| |\gamma(v_n)(y)|^\theta - |\gamma(w_n)(y)|^\theta \right| \, dx \, dy \leq$$
$$\leq 2C |W|_r \, |\gamma(v_n)|^\theta_{2r\theta/(2r-1)} \left| |\gamma(v_n)|^\theta - |\gamma(w_n)|^\theta \right|_{2r/(2r-1)} = o(1) \, ,$$

since $|\gamma(v_n)|^\theta - |\gamma(w_n)|^\theta \to 0$ strongly in $L^{2r/(2r-1)}_{\mathrm{loc}}(\mathbb{R}^N)$. Here and in the following C denotes some positive constant independent of n, not necessarily the same one. Similarly

$$\left| \int_{\mathbb{R}^N} \left(W * |\gamma(v_n)|^\theta \right) |\gamma(v_n)|^{\theta - 2} \gamma(v_n) \gamma(w_n) - \right.$$
$$\left. - \int_{\mathbb{R}^N} \left(W * |\gamma(w_n)|^\theta \right) |\gamma(w_n)|^{\theta-2} \gamma(w_n) \gamma(w_n) \right| \leq$$
$$\leq 2C |W|_r \, |\gamma(v_n)|^\theta_{2r\theta/(2r-1)} \left| |\gamma(v_n)|^\theta - |\gamma(w_n)|^\theta \right|_{2r/(2r-1)} = o(1) \, .$$

Therefore,

$$|I'(v_n)w_n - I'(w_n)w_n| \leq C \iint_{S_\varepsilon} \left(|\nabla v_n|^2 + m^2 v_n^2\right) \, dx \, dy + \int_{A_\varepsilon} V(y)\gamma(v_n)^2 \, dy + o(1) \, .$$

Set $u_n = (1 - \xi) v_n$. Analogously we have

$$|I'(v_n)u_n - I'(u_n)u_n| \leq C \iint_{S_\varepsilon} \left(|\nabla u_n|^2 + m^2 u_n^2\right) \, dx \, dy + \int_{A_\varepsilon} V(y)\gamma(u_n)^2 \, dy + o(1) \, .$$

Therefore

$$I'(u_n)u_n = O(\varepsilon) + o(1) \tag{6.4}$$

and

$$I'(w_n)w_n = O(\varepsilon) + o(1) \; . \tag{6.5}$$

From (6.4)), we derive that $I(u_n) = [(\theta-1)/2\theta]\mathbb{D}(u_n) + O(\varepsilon) + o(1) \geq O(\varepsilon) + o(1)$.

Consider $t_n > 0$ such that $I'(t_n w_n)(t_n w_n) = 0$ for any n, namely

$$t_n^{2(\theta-1)} = \frac{\|w_n\|^2 + \int_{\mathbb{R}^N} V(y)\gamma(w_n)^2\, dy}{\mathbb{D}(w_n)} \; .$$

From (6.5)), we have that $t_n = 1 + O(\varepsilon) + o(1)$. Therefore from the characterization of E_α we have

$$\begin{aligned} c + o(1) = I(v_n) = I(u_n) + I(w_n) + O(\varepsilon) \geq I(w_n) + O(\varepsilon) + o(1) \geq \\ \geq I(t_n w_n) + O(\varepsilon) + o(1) \geq E_\alpha + O(\varepsilon) + o(1) \; . \end{aligned}$$

As $n \to +\infty$, $\varepsilon \to 0$, we derive that $c \geq E_\alpha$ which is a contradiction. Hence, $c = 0$ and $v_n \to 0$ strongly in $H^1(\mathbb{R}^{N+1}_+)$.

□

Lemma 6.6. *Let $\{v_n\}_n$ be a sequence in $H^1(\mathbb{R}^{N+1}_+)$ such that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^{N+1}_+)$. The following hold:*

(i) $\mathbb{D}'(v_n)u \to \mathbb{D}'(v)u$ for all $u \in H^1(\mathbb{R}^{N+1}_+)$.

(ii) After passing to a subsequence, there exists a sequence $\{\widetilde{v}_n\}_n$ in $H^1(\mathbb{R}^{N+1}_+)$ such that $\widetilde{v}_n \to v$ strongly in $H^1(\mathbb{R}^{N+1}_+)$,

$$\begin{aligned} &\mathbb{D}(v_n) - \mathbb{D}(v_n - \widetilde{v}_n) \to \mathbb{D}(v) && \text{in } \mathbb{R} \; , \\ &\mathbb{D}'(v_n) - \mathbb{D}'(v_n - \widetilde{v}_n) \to \mathbb{D}'(v) && \text{in } H^{-1}(\mathbb{R}^{N+1}_+) \; . \end{aligned}$$

Proof. We omit the proof.

□

Proposition 6.7. *The functional $I : H^1(\mathbb{R}^{N+1}_+) \to \mathbb{R}$ satisfies the Palais-Smale condition $(PS)_c$ at each $c < E_{V_\infty}$, where $V_\infty := \liminf_{|x|\to\infty} V(x)$.*

Proof. Let $v_n \in H^1(\mathbb{R}^{N+1}_+)$ satisfy

$$I(v_n) \to c < E_{V_\infty} \quad \text{and} \quad I'(v_n) \to 0$$

strongly in the dual space $H^{-1}(\mathbb{R}^{N+1}_+)$. Since $\{v_n\}_n$ is bounded in $H^1(\mathbb{R}^{N+1}_+)$ it contains a subsequence such that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^{N+1}_+)$ and $\gamma(v_n) \rightharpoonup \gamma(v)$ in $L^p(\mathbb{R}^N)$ for any $p \in [2, 2N/(N-1)]$.

By Lemma 6.6, v solves (6.1) and, after passing to a subsequence, there exists a sequence $\{\widetilde{v}_n\}_n$ in $H^1(\mathbb{R}^{N+1}_+)$ such that $u_n := v_n - \widetilde{v}_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^{N+1}_+)$,

$$\begin{aligned} &I(v_n) - I(u_n) \to I(v) && \text{in } \mathbb{R} \; , \\ &I'(v_n) - I'(u_n) \to 0 && \text{strongly in } H^{-1}(\mathbb{R}^{N+1}_+) \; . \end{aligned}$$

Hence, $I(v) = [(\theta-2)/2\theta]\mathbb{D}(v) \geq 0$,

$$I(u_n) \to c - I(v) \leq c \quad , \quad \text{and} \quad I'(u_n) \to 0$$

strongly in $H^{-1}(\mathbb{R}^{N+1}_+)$. By Lemma 6.5 a subsequence of $\{u_n\}_n$ converges strongly to 0 in $H^1(\mathbb{R}^{N+1}_+)$. This implies that a subsequence of $\{v_n\}_n$ converges strongly to v in $H^1(\mathbb{R}^{N+1}_+)$.

□

6.3. **Mountain Pass Geometry.** Let us consider the limit problem

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ -\dfrac{\partial v}{\partial x} = -V_\infty v + (W * |v|^\theta)|v|^{\theta-2}v & \text{in } \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+ \end{cases} \tag{6.6}$$

where $V_\infty := \liminf_{|x|\to\infty} V(x) > 0$. By Theorem 6.4, the first mountain pass value E_{V_∞} of the functional J_{V_∞} associated to problem (6.6) is attained at a positive function $\omega_\infty \in H^1(\mathbb{R}^{N+1}_+)$, which is symmetric $\omega_\infty(x, gy) = \omega_\infty(x, y)$ for all $g \in O(N)$, $x > 0$, $y \in \mathbb{R}^N$. Moreover, since $V_\infty > 0$, we are allowed to choose $\sigma = 0$, and there exists $C > 0$ such that

$$0 < \omega_\infty(x, y) \leq Ce^{-m\sqrt{x^2+|y|^2}} \tag{6.7}$$

for all $(x, y) \in [0, +\infty) \times \mathbb{R}^N$. In particular, $\gamma(\omega_\infty)$ is radially symmetric in $\mathbb{R}^N$ and

$$0 < \gamma(\omega_\infty)(y) \leq Ce^{-m|y|}$$

for any $y \in \mathbb{R}^N$. As in Theorem 6.4, a bootstrap procedure shows that $\omega_\infty \in C^\infty([0, +\infty) \times \mathbb{R}^N)$.

Lemma 6.8. *We have*

$$|\nabla\omega_\infty(z)| = O(e^{-m|z|}) \qquad \textit{as} \qquad |z| \to \infty \, . \tag{6.8}$$

Proof. We consider the equation

$$\sqrt{-\Delta + m^2}\, u + V_\infty u = \left(W * |u|^\theta\right)|u|^{\theta-2}u \qquad \text{in } \mathbb{R}^N$$

satisfied by ω_∞. For any index $i = 1, 2, \cdots, N$ we write $v_i = \partial\omega_\infty/\partial y_i$ and observe that v_i satisfies

$$\sqrt{-\Delta + m^2}\, v_i + V_\infty v_i = \theta(W * \omega_\infty^{\theta-1} v_i)\omega_\infty^{\theta-1} + (\theta - 1)(W * \omega_\infty^\theta)\omega_\infty^{\theta-2} v_i$$

or, equivalently,

$$-\Delta v_i + m^2 v_i = 0 \qquad \text{in } \mathbb{R}^{N+1}_+$$

$$-\frac{\partial v_i}{\partial x} = -V_\infty v_i + \theta(W * \omega_\infty^{\theta-1} v_i)\omega_\infty^{\theta-1} + (\theta - 1)(W * \omega_\infty^\theta)\omega_\infty^{\theta-2} v_i \quad \text{in } \mathbb{R}^N \, .$$

The differentiation of the equation is allowed by the regularity of the solution ω_∞, see [9, Theorem 3.14]. Moreover, $\omega_\infty \in L^p(\mathbb{R}^{N+1}_+)$ for any $p > 1$, because it is bounded and decays exponentially fast at infinity. By elliptic regularity, $\omega_\infty \in W^{2,p}(\mathbb{R}^{N+1}_+)$ for any $p > 1$, and in particular $v_i \in L^p(\mathbb{R}^{N+1}_+)$ for any $p > 1$. An interpolation estimate shows that $\omega_\infty^{\theta-1} v_i \in L^p(\mathbb{R}^{N+1}_+)$ for any $p > 1$. Then the convolution $W * (\omega_\infty^{\theta-1} v_i) \in L^\infty(\mathbb{R}^{N+1}_+)$, and the term $(W * (\omega_\infty^{\theta-1} v_i))\omega_\infty^{\theta-1} \in L^2(\mathbb{R}^{N+1}_+)$ by the summability properties of ω_∞. The term $(W * \omega_\infty^\theta)\omega_\infty^{\theta-2} v_i \in L^2(\mathbb{R}^{N+1}_+)$ trivially.

Now the proof of [9, Theorem 3.14] shows that $v_i(x, y) \to 0$ as $x + |y| \to +\infty$. A comparison with the function $e^{-m\sqrt{x^2+|y|^2}}$ as in [9, Theorem 5.1] shows the validity of (6.8).

□

Fix $\varepsilon \in (0, (2m - k)/(2m + k))$. For $R > 0$, we consider a symmetric cut off function $\xi_R \in C^\infty(\mathbb{R}^{N+1}_+)$, namely $\xi_R(x, gy) = \xi_R(x, y)$ for all $g \in O(N)$, $x > 0$,

$y \in \mathbb{R}^N$ such that $\xi_R(z) = 0$ if $|z| \geq R$ and $\xi_R(z) = 1$ if $|z| \leq R(1-\varepsilon)$ and $\xi_R(z) \in [0,1]$ for all $z \in \mathbb{R}^{N+1}_+$.

Let us define $\omega^R(z) := \omega_\infty(z)\xi_R(z)$, for any $z \in \mathbb{R}^{N+1}_+$.

Lemma 6.9. *As $R \to \infty$,*

$$\left| \iint_{\mathbb{R}^{N+1}_+} |\nabla \omega_\infty|^2 - \left|\nabla \omega^R\right|^2 \right| = O(R^{N-1}e^{-2m(1-\varepsilon)R}) \,, \tag{6.9}$$

$$\left|\mathbb{D}(\omega_\infty) - \mathbb{D}(\omega^R)\right| = O(R^{N-1}e^{-\theta m(1-\varepsilon)R}) \,. \tag{6.10}$$

Proof. The proof of (6.9) is standard. Indeed, using (6.8) and cylindrical coordinates in $\mathbb{R}^{N+1}_+$,

$$\begin{aligned}\left| \iint_{\mathbb{R}^{N+1}_+} |\nabla \omega^R|^2 - |\nabla \omega_\infty|^2 \right| &\leq C \iint_{\{z\in\mathbb{R}^{N+1}_+ \,|\, (1-\varepsilon)R<|z|\}} |\nabla \omega_\infty|^2 \leq \\ &\leq C_1 \iint_{\{z\in\mathbb{R}^{N+1}_+ \,|\, (1-\varepsilon)R<|z|\}} e^{-2m|z|}\,dz \leq \\ &\leq C_1 R^{N-1} e^{-2m(1-\varepsilon)R} \,.\end{aligned}$$

To prove (6.10), we recall that $W = W_1 + W_2 \in L^r(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$. The difference $\mathbb{D}(\gamma(\omega_\infty)) - \mathbb{D}(\gamma(\omega^R))$ can be split in two parts, the one with W_1 and the one with W_2. The former can be estimated as follows:

$$\begin{aligned}&\left|\mathbb{D}(\gamma(\omega_\infty)) - \mathbb{D}(\gamma(\omega^R))\right| \leq \\ &\leq \int_{\mathbb{R}^N\times\mathbb{R}^N} \left||\gamma(\omega_\infty)(x)|^\theta |\gamma(\omega_\infty)(y)|^\theta - |\gamma(\omega^R)(x)|^\theta |\gamma(\omega^R)(y)|^\theta\right| W_1(x-y)\,dx\,dy \leq \\ &\leq 2\int_{\mathbb{R}^N\times\mathbb{R}^N} W_1(x-y)|\gamma(\omega_\infty)(x)|^\theta \left||\gamma(\omega_\infty)(y)|^\theta - |\gamma(\omega^R)(y)|^\theta\right|\,dx\,dy \leq \\ &\leq 2\left\||\gamma(\omega_\infty)|^\theta - |\gamma(\omega^R)|^\theta\right\|_{2r/(2r-1)} \|\gamma(\omega_\infty)\|^\theta_{2r\theta/(2r-1)} \|W_1\|_r \leq \\ &\leq C\left(\int_{(1-\varepsilon)R}^\infty t^{N-1}e^{-m[2r\theta/(2r-1)]t}\,dt\right)^{(2r-1)/2r} = C_2 R^{N-1}e^{-\theta m(1-\varepsilon)R} \,.\end{aligned}$$

The latter is simpler, since we use directly the L^∞-norm of W_2.

□

For $s \in \mathbb{R}^N$, set $R_s := [(k+2m)/4m]|s|$. Since $k \in (0, 2m)$, it results that $R_s \in (0, |s|)$. Hence $|s| - R_s \to +\infty$, as $|s| \to +\infty$. With this notation, we define the function

$$\omega_s^{R_s}(z) := \omega_\infty(x, y-s)\xi_{R_s}(x, y-s)$$

where $z = (x,y) \in \mathbb{R}^{N+1}$.

Lemma 6.10. *There exist ϱ_0, $d_0 \in (0,\infty)$ such that*

$$I(t(\omega_s^{R_s})) \leq E_{V_\infty} - d_0 e^{-k|y|} \qquad \textit{for all } t \geq 0 \,,$$

provided that $|s| \geq \varrho_0$

Proof. For $u \in H^1(\mathbb{R}^{N+1}_+)$ we have by (7.12) that $\max_{t\geq 0} I(tu) = I(t_u u)$ if and only if

$$t_u = \left(\frac{\|u\|^2 + \int_{\mathbb{R}^N} V(y)\gamma(u)^2\,dy}{\mathbb{D}(u)} \right)^{1/(2\theta-2)} .$$

Indeed

$$\|u\|^2 + \int_{\mathbb{R}^N} V(y)\gamma(u)^2\,dy \geq$$

$$\geq \left(1-\frac{V_0}{m}\right)\int_{\mathbb{R}^{N+1}_+} |\nabla u|^2 + m(m-V_0)\int_{\mathbb{R}^{N+1}_+}|u|^2 > 0\,. \tag{6.11}$$

So, since $\omega_\infty^{R_s} \to \omega_\infty$ in $H^1(\mathbb{R}^{N+1}_+)$ as $|s|\to\infty$, and taking into account that $I_{V_\infty}(\omega_\infty) = \max_{t\geq 0} I_{V_\infty}(t(\omega_\infty))$ there exist $0<t_1<t_2<+\infty$ such that

$$\max_{t\geq 0} I(t(\omega_s^{R_s})) = \max_{t_1\leq t\leq t_2} I(t(\omega_s^{R_s}))$$

for all large enough $|s|$.

Let $t\in[t_1,t_2]$. Write $V=V^+-V^-$, where $V^+(x)=\max\{V(x),0\}$ and $V^-(x)=\max\{-V(x),0\}$, and remark that the assumption $V_\infty>0$ implies $V(x)=V^+(x)$ whenever $|x|$ is sufficiently large. Assumption (V2) yields therefore

$$\int_{\mathbb{R}^N} V(y)(t\gamma(\omega_s^{R_s}))^2(y)\,dy \leq t^2\int_{|y|\leq R_s} V^+(y+s)(\gamma(\omega^{R_s}))^2(y)\,dy \leq$$

$$\leq t^2 \int_{|y|\leq R_s} (V_\infty - c_0\, e^{-k|y+s|})(\gamma(\omega_\infty))^2(y)\,dx \leq$$

$$\leq \int_{\mathbb{R}^N} V_\infty\,(t\gamma(\omega_\infty))^2 -$$

$$-\left(c_0\, t_1^2 \int_{|y|\leq 1} e^{-k|y|}(\gamma(\omega_\infty))^2(y)\,dy\right) e^{-k|s|}$$

for $|s|$ large enough.

Therefore, using Lemma 6.9, we get

$$I(t(\omega^{R_s})_s) = \frac{1}{2}\left\|t(\omega^{R_s})_s\right\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(y)\,(t\gamma(\omega_\infty))^2\;dy - \frac{1}{2\theta}\,\mathbb{D}(t\omega_s^{R_s}) \leq$$

$$\leq \frac{1}{2}\left\|t\omega_\infty\right\|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty\,(t\gamma(\omega_\infty))^2\;dy - \frac{1}{2\theta}\,\mathbb{D}(t\omega_\infty) -$$

$$-Ce^{-k|s|} + O(R_s^{N-1}e^{-2m(1-\varepsilon)R_s}) \leq$$

$$\leq \max_{t\geq 0} I_{V_\infty}(t\omega_\infty) - d_0\, e^{-\kappa|s|} = E_{V_\infty} - d_0\, e^{-k|s|}$$

for sufficiently large $|s|$, because our choices of ε and R_s guarantee that $2m(1-\varepsilon)R_s > k|s|$.

□

6.4. **Proof of theorem 6.1.** The proof of Theorem 6.1 is now immediate. The Euler functional I satisfies the geometric assumptions of the Mountain Pass Theorem on $H^1(\mathbb{R}^{N+1}_+)$. Since it also satisfies the Palais-Smale condition, as we showed in the previous sections, we conclude that I possesses at least a critical point $v \in H^1(\mathbb{R}^{N+1}_+)$. In addition,

$$I(v) = c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I(\gamma(t)) ,$$

where $\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^{N+1}_+)) \mid \gamma(0) < 0,\ I(\gamma(1)) < 0\}$.

To prove that $v \geq 0$, we notice that, reasoning as in (6.11), the map $t \mapsto I(tw)$ has one and only one strict maximum point at $t = 1$ whenever $w \in H^1(\mathbb{R}^{N+1}_+)$ is a critical point of I. Since $I(|w|) \leq I(w)$ for all $w \in H^1(\mathbb{R}^{N+1}_+)$, and

$$I(t|w|) \leq I(tw) < I(w) \qquad \text{for every } t > 0,\ t \neq 1 ,$$

we conclude that

$$c \leq \sup_{t\geq 0} I(t|v|) \leq I(v) = c .$$

We claim that $|v|$ is also a critical point of I. Indeed, otherwise, we could deform the path $t \mapsto t|v|$ into a path $\gamma \in \Gamma$ such that $I(\gamma(t)) < c$ for every $t \geq 0$, a contradiction with the definition of c.

6.5. **Further properties of the solution.** We collect in the next statement some additional features of the weak solution found above.

Theorem 6.11. *Let u be the solution to equation* (6.1) *provided by Theorem 6.1. Then $u \in C^\infty(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for every $q \geq 2$. Moreover,*

$$0 < u(y) \leq Ce^{-m|y|} .$$

Proof. The regularity of u can be established by mimicking the proofs in Section 3 of [9].

The potential function V is harmless, being bounded from above and below.

To prove the exponential decay at infinity, we introduce a comparison function

$$W_R(x,y) = C_R e^{-m\sqrt{x^2+|y|^2}} \quad , \quad \text{for every } (x,y) \in \mathbb{R}^{N+1}_+ ,$$

and we will fix $R > 0$ and $C_R > 0$ in a suitable manner. We also introduce the notation

$$\begin{aligned} B_R^+ &= \left\{(x,y) \in \mathbb{R}^{N+1}_+ \mid \sqrt{x^2+|y|^2} < R\right\} \\ \Omega_R^+ &= \left\{(x,y) \in \mathbb{R}^{N+1}_+ \mid \sqrt{x^2+|y|^2} > R\right\} \\ \Gamma_R &= \left\{(0,y) \in \partial\mathbb{R}^{N+1}_+ \mid |y| \geq R\right\} . \end{aligned}$$

It is easily seen that

$$\begin{cases} -\Delta W_R + m^2 W_R \geq 0 & \text{in } \Omega_R^+ \\ -\dfrac{\partial W_R}{\partial x} = 0 & \text{on } \Gamma_R^+ . \end{cases}$$

Call $w(x,y) = W_R(x,y) - v(x,y)$, and remark that $-\Delta w + m^2 w \geq 0$ in Ω_R^+. If $C_R = e^{mR} \max_{\partial B_R^+} v$, then $w \geq 0$ on ∂B_R^+ and $\lim_{x+|y| \to +\infty} w(x,y) = 0$. We claim that $w \geq 0$ in the closure $\overline{\Omega_R^+}$.

If not, $\inf_{\overline{\Omega_R^+}} w < 0$, and the strong maximum principle provides a point $(0,y_0) \in \Gamma_R$ such that

$$w(0,y_0) = \inf_{\overline{\Omega_R^+}} w < w(x,y) \qquad \text{for every } (x,y) \in \Omega_R^+ \ .$$

For some $0 < \lambda < m$, we introduce $z(x,y) = w(x,y) e^{\lambda x}$.
As before, $\lim_{x+|y| \to +\infty} z(x,y) = 0$ and $z \geq 0$ on ∂B_R^+. Since

$$0 \leq -\Delta w + m^2 w = e^{-\lambda x} \left(-\Delta z + 2\lambda \frac{\partial z}{\partial x} + (m^2 - \lambda^2) z \right)$$

the strong maximum principle applies and yields that $\inf_{\Gamma_R} z = \inf_{\overline{\Omega_R^+}} z < z(x,y)$ for every $(x,y) \in \Omega_R^+$. Therefore $z(0,y_0) = \inf_{\Gamma_R} z = \inf_{\Gamma_R} w < 0$. Hopf's lemma now gives

$$-\frac{\partial w}{\partial x}(0,y_0) - \lambda w(0,y_0) < 0 \ .$$

But this is impossible. Indeed

$$-\frac{\partial w}{\partial x}(0,y_0) = -V(y_0) v(0,y_0) - (W * |v|^\theta) |v(0,y_0)|^{\theta-2} v(0,y_0) \ ,$$

and hence

$$-\frac{\partial w}{\partial x}(0,y_0) - \lambda v(0,y_0) = -\lambda v(0,y_0) - V(y_0) v(0,y_0) - (W * |v|^\theta) |v(0,y_0)|^{\theta-2} v(0,y_0) \ .$$

Recall that $v(0,y_0) < 0$ and $\lambda > 0$; if we can show that

$$-V(y_0) v(0,y_0) - (W * |v|^\theta) |v(0,y_0)|^{\theta-2} v(0,y_0) \geq 0 \ ,$$

we will be done. First of all, we recall that (see [9, p. 70]

$$\lim_{|y| \to +\infty} (W * |v|^\theta) |v(0,y)|^{\theta-2} v(0,y) = 0 \ ,$$

since $\lim_{|y| \to +\infty} W(y) = 0$. So we pick $R > 0$ so large that $\left| (W * |v|^\theta) |v(0,y_0)|^{\theta-2} v(0,y_0) \right|$ is very small. Choosing R even larger, we can also assume that $V(y_0) > 0$, since $V_\infty > 0$. Hence $-V(y_0) v(0,y_0) - (W * |v|^\theta) |v(0,y_0)|^{\theta-2} v(0,y_0) \geq 0$, and the proof is finished.

To summarize, we have proved that, whenever $x + |y|$ is sufficiently large, then

$$v(x,y) \leq W_R(x,y) \ ,$$

and hence the validity of (7.10).

□

7. Semiclassical analysis

The semiclassical limit ($\varepsilon \to 0^+$) for the pseudo-relativistic Hartree equation

$$i\varepsilon \frac{\partial \psi}{\partial t} = \left(\sqrt{-\varepsilon^2 \Delta + m^2} - m \right) \psi + V\psi - \left(\frac{1}{|x|} * |\psi|^2 \right) \psi \quad , \quad x \in \mathbb{R}^3 \tag{7.1}$$

was studied in the very recent paper [7]. Here $\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is the wave field, $m > 0$ is a physical constant, ε is the semiclassical parameter $0 < \varepsilon \ll 1$, a dimensionless scaled Planck constant (all other physical constant are rescaled to be 1), V is bounded external potential in $\mathbb{R}^3$. The pseudo-differential operator $\sqrt{-\varepsilon^2\Delta + m^2}$ is simply defined in Fourier variables by the symbol $\sqrt{\varepsilon^2|\xi|^2 + m^2}$.

Remark 7.1. Equation (7.1) corresponds to (1.1) with $c = 1$ and $\varepsilon = \hbar$.

Equation (7.1) has interesting applications in the quantum theory for large systems of self-interacting, relativistic bosons with mass $m > 0$. As recently shown by Elgart and Schlein [14], equation (7.1) emerges as the correct evolution equation for the mean-field dynamics of many-body quantum systems modelling pseudo-relativistic boson stars in astrophysics. The external potential, $V = V(x)$, accounts for gravitational fields from other stars. In what follows, we will assume that V is a smooth, bounded function. The pseudo-relativistic Hartree equation can be also derived coupling together a pseudo-relativistic Schrödinger equation with a Poisson equation, i.e.

$$\begin{cases} i\varepsilon \dfrac{\partial \psi}{\partial t} = \left(\sqrt{-\varepsilon^2\Delta + m^2} - m\right)\psi + V\psi - U\psi \ , \\ -\Delta U = |\psi|^2 \end{cases}$$

More generally, we will focus on the generalized pseudo-relativistic Hartree equation

$$\sqrt{-\varepsilon^2\Delta + m^2}\, u + Vu = (I_\alpha * |u|^p)\, |u|^{p-2}u \quad , \quad \text{in } \mathbb{R}^N \ , \tag{7.2}$$

where $m > 0$, $2 \le p < \frac{2N}{N-1}$, $V : \mathbb{R}^N \to \mathbb{R}$ is an external scalar potential,

$$I_\alpha(y) = \frac{c_{N,\alpha}}{|y|^{N-\alpha}} \quad (yx \neq 0) \quad , \quad \alpha \in (0, N)$$

is a convolution kernel and $c_{N,\alpha}$ is a positive constant; for our purposes we can choose $c_{N,\alpha} = 1$. For $N = 3$, $\alpha = p = 2$, equation (7.2) becomes the pseudo-relativistic Hartree equation with Coulomb kernel

$$\sqrt{-\varepsilon^2\Delta + m^2}\, u + Vu = \left(\frac{1}{|y|} * |u|^2\right) u \quad , \quad \text{in } \mathbb{R}^3$$

Replacing $u(y)$ by $\varepsilon^{\alpha/2(1-p)}u(\varepsilon y)$, equation (7.2) becomes equivalent to following Hartree equation

$$\sqrt{-\Delta + m^2}\, u + V_\varepsilon(y)u = (I_\alpha * |u|^p)\, |u|^{p-2}u \quad , \quad \text{in } \mathbb{R}^N \ . \tag{7.3}$$

where $V_\varepsilon(y) = V(\varepsilon y)$. In what follows we will assume that

(V) $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous and bounded function such that $V_{\min} = \inf_{\mathbb{R}^N} V > -m$ and there exists a bounded open set $O \subset \mathbb{R}^N$ with the property that

$$V_0 = \inf_O V < \min_{\partial O} V \ .$$

Let us define

$$\mathcal{M} = \{y \in O \mid V(y) = V_0\} \ .$$

We will establish the existence of a single-spike solution concentrating around a point close to $\mathcal{M}$. Precisely, our main result is the following.

Theorem 7.2. *Retain assumption* (V) *and assume that* $2 \le p < 2N/(N-1)$ *and* $(N-1)p - N < \alpha < N$. *Then, for every sufficiently small* $\varepsilon > 0$, *there exists a solution* $u_\varepsilon \in H^{1/2}(\mathbb{R}^N)$ *of equation* (7.3) *such that* u_ε *has a local maximum point* y_ε *satisfying*

$$\lim_{\varepsilon \to 0} \operatorname{dist}(\varepsilon y_\varepsilon, \mathcal{M}) = 0 \ ,$$

and for which

$$u_\varepsilon(y) \le C_1 \exp\left(-C_2|y-y_\varepsilon|\right)$$

for suitable constants $C_1>0$ *and* $C_2>0$*. Moreover, for any sequence* $\{\varepsilon_n\}_n$ *with* $\varepsilon_n \to 0$*, there exists a subsequence, still denoted by the same symbol, such that there exist a point* $y_0 \in \mathcal{M}$ *with* $\varepsilon_n y_{\varepsilon_n} \to y_0$*, and a positive least-energy solution* $U \in H^{1/2}(\mathbb{R}^N)$ *of the equation*

$$\sqrt{-\Delta+m^2}\, U + V_0 U = \left(I_\alpha * U^p\right) U^{p-1}$$

for which we have

$$u_{\varepsilon_n}(y) = U\left(y-y_{\varepsilon_n}\right) + \mathcal{R}_n(y)$$

where $\lim_{n\to+\infty} \|\mathcal{R}_n\|_{H^{1/2}} = 0$.

To prove the main result, the *nonlocal* problem (7.3) in $\mathbb{R}^N$ is replaced by a local Neumann problem in the half space $\mathbb{R}^{N+1}_+$ as above. Critical points of the Euler functional associated to the local Neumann problem are then found by means of a variational approach introduced in [1, 2] (see also [5]) for nonlinear Schrödinger equations and extended in [8] to deal with non-relativistic Hartree equations.

More precisely, for any $\varepsilon>0$, given $u \in \mathcal{S}(\mathbb{R}^N)$, there exists one and only one function $v \in \mathcal{S}(\mathbb{R}^{N+1}_+)$ such that

$$\begin{cases} -\varepsilon^2 \Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ v(0,y) = u(y) & \text{for } y \in \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+ \,. \end{cases}$$

Setting

$$T_\varepsilon u(y) = -\varepsilon\, \frac{\partial v}{\partial x}(0,y)\,,$$

we easily see that the problem

$$\begin{cases} -\varepsilon^2 \Delta w + m^2 w = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ w(0,y) = T_\varepsilon u(y) & \text{for } y \in \partial\mathbb{R}^{N+1}_+ = \{0\}\times\mathbb{R}^N \simeq \mathbb{R}^N \end{cases}$$

is solved by $w(x,y) = -\varepsilon(\partial v/\partial x)(x,y)$. From this we deduce that

$$T_\varepsilon(T_\varepsilon u)(y) = -\varepsilon\, \frac{\partial w}{\partial x}(0,y) = \varepsilon^2\, \frac{\partial^2 v}{\partial x^2}(0,y) = \left(-\varepsilon^2 \Delta_y v + m^2 v\right)(0,y)\,,$$

and hence $T_\varepsilon \circ T_\varepsilon = (-\varepsilon^2\Delta_y + m^2)$, namely T_ε is a square root of the Schrödinger operator $-\varepsilon^2\Delta_y + m^2$ on $\mathbb{R}^N = \partial\mathbb{R}^{N+1}_+$.

From the previous construction, we can replace the *nonlocal* problem (7.3) in $\mathbb{R}^N$ with the local Neumann problem in the half space $\mathbb{R}^{N+1}_+$

$$\begin{cases} -\varepsilon^2 \Delta v(x,y) + m^2 v(x,y) = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ -\varepsilon\, \dfrac{\partial v}{\partial x}(0,y) = -V(y)v(0,y) + \left(I_\alpha * |v(0,\cdot)|^p\right)|v(0,y)|^{p-2} v(0,y) & \text{for } y \in \mathbb{R}^N \,. \end{cases}$$

Setting $v_\varepsilon(x,y) = \varepsilon^{\alpha/2(1-p)} v(\varepsilon x, \varepsilon y)$ and $V_\varepsilon(y) = V(\varepsilon y)$, we are led to the *local* boundary-value problem

$$\begin{cases} -\Delta v_\varepsilon + m^2 v_\varepsilon = 0 & \text{in } \mathbb{R}^{N+1}_+ \\ -\dfrac{\partial v_\varepsilon}{\partial x}(0,y) = -V_\varepsilon(y)v_\varepsilon(0,y) + \left(I_\alpha * |v_\varepsilon(0,\cdot)|^p\right)|v_\varepsilon(0,y)|^{p-2} v_\varepsilon(0,y) & \text{for } y \in \mathbb{R}^N \,. \end{cases}$$

It follows easily that the functional $\mathcal{E}_\varepsilon : H \to \mathbb{R}$ defined by

$$\mathcal{E}_\varepsilon(v) = \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} |\nabla v|^2\,dx\,dy + \frac{m^2}{2}\int_{\mathbb{R}^{N+1}_+} v^2\,dx\,dy + \\ +\frac{1}{2}\int_{\mathbb{R}^N} V_\varepsilon(x)\gamma(v)^2\,dy - \frac{1}{2p}\int_{\mathbb{R}^N} \left(I_\alpha * |\gamma(v)|^p\right)|\gamma(v)|^p\,dy$$

is of class C^1, and its critical points are (weak) solutions to problem (7.3).

7.1. **Compactness properties for the limiting problem.** For $a > -m$, the equation

$$\sqrt{-\Delta + m^2}\,u + au = \left(I_\alpha * |u|^p\right)|u|^{p-2}u \tag{7.4}$$

plays the rôle of a limiting problem for (7.3). Its Euler functional $L_a : H \to \mathbb{R}$ is defined (via the local realization of Section 2) by

$$L_a(v) = \frac{1}{2}\int_{\mathbb{R}^{N+1}_+} \left(|\nabla v|^2 + m^2|v|^2\right)\,dx\,dy + \\ +\frac{a}{2}\int_{\mathbb{R}^N} |\gamma(v)|^2\,dy - \frac{1}{2p}\int_{\mathbb{R}^N} \left(I_\alpha * |\gamma(v)|^p\right)|\gamma(v)|^p\,dy\,.$$

We define the ground-state level

$$m_a = \inf\left\{L_a(v) \mid L_a'(v) = 0,\ v \in H \setminus \{0\}\right\}$$

and the set S_a of elements $v \in H \setminus \{0\}$ such that $v > 0$, $L_a(v) = m_a$, and for every $x \geq 0$:

$$\max_{y \in \mathbb{R}^N} v(x,y) = v(x,0)\,. \tag{7.5}$$

Proposition 7.3. *The set S_a is non-empty for any $a > -m$.*

Proof. The proof is indeed standard, and we will be sketchy. First of all, we invoke [10, Lemma 2.1] to deduce that ground states of L_a correspond to ground states of the functional $\mathcal{L}_a : H^{1/2}(\mathbb{R}^N) \to \mathbb{R}$ defined as

$$\mathcal{L}_a(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(\left|\sqrt{(m^2-\Delta)^{1/2} - m}\;u\right|^2 + (a+m)|u|^2\right) - \\ -\frac{1}{2p}\int_{\mathbb{R}^N} \left(I_\alpha * |u|^p\right)|u|^p\,. \tag{7.6}$$

We claim that $\mathcal{L}_a$ possesses a ground state. We fix $a > -m$ and consider the minimization problem associated to (7.6)

$$\widetilde{m}_a = \inf_{u \in H^{1/2}(\mathbb{R}^N)\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^N} \left|\sqrt{(m^2-\Delta)^{1/2} - m}\;u\right|^2 + (a+m)\,|u|^2}{\left(\displaystyle\int_{\mathbb{R}^N} \left(I_\alpha * |u|^p\right)|u|^p\right)^{1/p}}\,. \tag{7.7}$$

Since $\sqrt{m^2-\Delta} - m > 0$ in the sense of functional calculus and $a + m > 0$, it follows easily that $\widetilde{m}_a > 0$. As in [24, Proof of Proposition 2.2] we can show that $\widetilde{m}_a$ is attained. Since the quotient in (7.7) is homogeneous of degree zero, as in the local case we see that any minimizer of $\widetilde{m}_a$ is, up to a rescaling and a translation, a ground state for (7.6). Therefore the claim is proved, and in particular $S_a \neq \emptyset$. It is easy to check that ground states are non-negative, and, as in [9, Theorem 5.1], actually strictly positive.

□

Remark 7.4. By [22, Formula (A.3)], the quotient to be minimized in (7.7) decreases under polarization. This implies, reasoning as in [24, Section 5] that ground states are radially symmetric around a point of $\mathbb{R}^N$.

For $U \in S_a$, we write $E_a = L_a(U)$. By an immediate extension of [27, Lemma 3.17], the map $a \mapsto E_a$ is strictly increasing and continuous. The following is the main result of this section.

Proposition 7.5. *The set S_a is compact in H, and for some $C > 0$ and any $\sigma \in (-V_{\min}, m) \cap [0, +\infty)$ we have*

$$v(x,y) \leq C e^{-(m-\sigma)\sqrt{x^2+|y|^2}}\, e^{-\sigma x} \tag{7.8}$$

for every $v \in S_a$.

Proof. If $v \in S_a$, it follows easily from [9, Theorem 5.1] or [6, Theorem 7.1] that v decays exponentially fast at infinity and (7.8) holds. Moreover, since

$$m_a = L_a(v) = \left(\frac{1}{2} - \frac{1}{2p} \right) \left(|\nabla v|_2^2 + m^2 |v|_2^2 \right) ,$$

S_a is bounded in H. We claim that S_a is also bounded in $L^\infty(\mathbb{R}^{N+1}_+)$.

Indeed, by [9, Theorem 3.2] it follows that $\gamma(v) \in L^q(\mathbb{R}^N)$ for any $q \in [2,\infty]$, then also $g(\cdot) = -a\gamma(v) + (I_\alpha * |\gamma(v)|^p)\, |\gamma(v)|^{p-2}\gamma(v) \in L^q(\mathbb{R}^N)$ for $q \in [2,\infty]$. Following [3], we let $u(x,y) = \int_0^x v(t,y)\, dt$. It follows that $u \in H^1((0,R) \times \mathbb{R}^N)$ for all $R > 0$. Arguing as in [9, Proposition 3.9], we can deduce that u is a weak solution of the Dirichlet problem

$$\begin{cases} -\Delta u + m^2 u = g & \text{in } \mathbb{R}^{N+1}_+ \\ u = 0 & \text{for } y \in \mathbb{R}^N \ . \end{cases} \tag{7.9}$$

where $g(x,y) = g(y)$ for every $x > 0$ and $y \in \mathbb{R}^N$. We sketch the proof for the sake of completeness. Pick an arbitrary function $\eta \in C_0^\infty(\mathbb{R}^{N+1}_+)$ and write $\omega_t(x,y) = \eta(x+t,y)$ for any $t \geq 0$. Then

$$\begin{aligned} &\int_0^{+\infty} \int_0^{+\infty} \int_{\mathbb{R}^N} \nabla v(x,y) \cdot \nabla \eta(x+t,y)\, dy\, dx\, dt = \\ &= \int_0^{+\infty} \int_x^{+\infty} \int_{\mathbb{R}^N} \nabla v(x,y) \cdot \nabla \eta(s,y)\, dy\, ds\, dx = \\ &= \int_0^{+\infty} \int_0^{s} \int_{\mathbb{R}^N} \nabla v(x,y) \cdot \nabla \eta(s,y)\, dy\, dx\, ds = \\ &= \int_0^{+\infty} \int_{\mathbb{R}^N} \nabla \left(\int_0^s v(x,y)\, dx \right) \cdot \nabla \eta(s,y)\, dy\, ds \end{aligned}$$

and this readily implies that

$$\int_{\mathbb{R}^{N+1}_+} \left(\nabla v \cdot \nabla w_t + m^2 v w_t \right)\, dx\, dy = \int_{\mathbb{R}^N} g w_t\, dy \ .$$

An integration with respect to t from 0 to $+\infty$ gives

$$\int_{\mathbb{R}^{N+1}_+} \left(\nabla u \cdot \nabla \eta + m^2 u \eta - g\eta\right) \, dx \, dy = 0 \, ,$$

and hence the validity of (7.9) is proved.

Moreover for any given $R > 0$ we can define $u_{\text{odd}} \in H^1((-R,R) \times \mathbb{R}^N)$ and $g_{\text{odd}} \in \bigcap_{q \geq 2} L^q((-R,R) \times \mathbb{R}^N)$ by

$$u_{\text{odd}} = \begin{cases} u(x,y) & \text{if } x \geq 0 \\ -u(-x,y) & \text{if } x < 0 \, , \end{cases} \qquad g_{\text{odd}}(x,y) = \begin{cases} g(y) & \text{if } x \geq 0 \\ -g(y) & \text{if } x < 0 \, . \end{cases}$$

It is easy to check as before that

$$-\Delta u_{\text{odd}} + m^2 u_{\text{odd}} = g_{\text{odd}} \qquad \text{in } \mathbb{R}^{N+1} \, .$$

Since $g_{\text{odd}} \in L^q((-R,R) \times \mathbb{R}^N)$ for any $q \in [2,+\infty[$, $R > 0$, we can invoke standard regularity results to conclude that

$$u_{\text{odd}} \in W^{2,q}((-R,R) \times \mathbb{R}^N)$$

for every $q \geq 2$ and every $R > 0$, and hence $u_{\text{odd}} \in C^{1,\beta}(\mathbb{R}^{N+1})$, $u \in C^{1,\beta}(\mathbb{R}^{N+1}_+)$ and $v = \partial u / \partial x \in C^{0,\beta}(\mathbb{R}^{N+1}_+)$ by Sobolev's Embedding Theorem. Therefore $g \in C^{0,\beta/(p-1)}(\mathbb{R}^N)$, and Schauder estimates yield $u \in C^{2,\beta/(p-1)}(\mathbb{R}^{N+1}_+)$ and $v \in C^{1,\beta/(p-1)}(\mathbb{R}^{N+1}_+)$. Moreover, the $C^{1,\beta}$-norm of v can be estimated by the L^q-norm of g, which immediately implies that S_a is a bounded subset of $L^\infty(\mathbb{R}^{N+1}_+)$.

Next, we claim that $\lim_{|(x,y)| \to +\infty} v(x,y) = 0$ uniformly with respect to $v \in S_a$. We assume by contradiction that this is false: there exist a number $\delta > 0$, a sequence of points $(x_n, y_n) \in \mathbb{R}^{N+1}_+$ and a sequence of elements $v_n \in S_a$ such that $x_n + |y_n| \to +\infty$ but $v_n(x_n,y_n) \geq \delta$ for every n. Let us write $z_n = (x_n,y_n)$, and call $\tilde{v}_n(z) = v_n(z + z_n)$ for $z = (x,y) \in \mathbb{R}^{N+1}_+$. By the previous arguments, $\{\tilde{v}_n\}_n$ is a bounded sequence in $H \cap L^\infty(\mathbb{R}^{N+1}_+)$. Moreover, up to a subsequence, we can assume that $v_n \rightharpoonup v$, $\tilde{v}_n \rightharpoonup \tilde{v}$ in H and locally uniformly in $\mathbb{R}^{N+1}_+$. As in [8, page 989], both v and $\tilde{v}$ weakly solve (7.4). We now show that they are non-trivial weak solutions. The conclusion is obvious for $\tilde{v}$, since $\tilde{v}_n(0) = v_n(z_n) \geq \delta$, so that $\tilde{v}(0) \geq \delta$. We consider instead v, and remark that [9, Equation 3.16] implies

$$\sup_{y \in \mathbb{R}^N} |v_n(x,y)| \leq C |\gamma(v_n)|_2 \; e^{-mx}$$

for some universal constant $C > 0$. Hence $\delta \leq v_n(z_n) \leq |\gamma(v_n)|_2 \; e^{-m x_n}$, and the boundedness of $\gamma(v_n)$ in L^2 yields the boundedness of $\{x_n\}_n$ in $\mathbb{R}$. Without loss of generality, we can assume that $x_n \to \bar{x} \in [0,+\infty)$. Therefore, by (7.5),

$$v_n(\bar{x},0) \geq v_n(\bar{x},y_n) \geq v_n(x_n,y_n) + o(1) \geq \frac{\delta}{2}$$

by locally uniform convergence, and we conclude that v is also nontrivial.

Now, for every $n \in \mathbb{N}$,

$$L_a(v_n) = \left(\frac{1}{2} - \frac{1}{2p}\right) \left(\int_{\mathbb{R}^{N+1}_+} \left(|\nabla v_n|^2 + m^2 v_n^2\right) \, dx \, dy + a \int_{\mathbb{R}^N} \gamma(v_n)^2 \, dy \right) = m_a \, ,$$

and

$$L_a(v) \geq m_a \quad , \quad L_a(\tilde{v}) \geq m_a \, .$$

If $R > 0$ satisfies $2R \leq x_n + |y_n|$, then

$$
\begin{aligned}
m_a &= L_a(v_n) \geq \\
&\geq \left(\frac{1}{2} - \frac{1}{2p} \right) \liminf_{n \to +\infty} \int_{B(0,R)} \left(|\nabla v_n|^2 + m^2 v_n^2 \right) \, dx \, dy + \\
&\quad + a \int_{B(0,R) \cap (\{0\} \times \mathbb{R}^N)} \gamma(v_n)^2 \, dy + \\
&\quad + \left(\frac{1}{2} - \frac{1}{2p} \right) \liminf_{n \to +\infty} \int_{B(0,R)} \left(|\nabla \tilde{v}_n|^2 + m^2 \tilde{v}_n^2 \right) \, dx \, dy + \\
&\quad + a \int_{B(0,R) \cap (\{0\} \times \mathbb{R}^N)} \gamma(\tilde{v}_n)^2 \, dy \geq \\
&\geq \left(\frac{1}{2} - \frac{1}{2p} \right) \left(\int_{B(0,R)} \left(|\nabla \tilde{v}|^2 + m^2 \tilde{v}^2 \right) \, dx \, dy + \right. \\
&\quad \left. + a \int_{B(0,R) \cap (\{0\} \times \mathbb{R}^N)} \gamma(\tilde{v})^2 \, dy \right) = \\
&= L_a(v) + L_a(\tilde{v}) + o(1) = 2m_a + o(1)
\end{aligned}
$$

as $R \to +\infty$. This contradiction proves that

$$
\lim_{|(x,y)| \to +\infty} v(x,y) = 0 \qquad \text{uniformly with respect to } v \in S_a \,. \tag{7.10}
$$

From [9, page 70] it follows immediately that

$$
\lim_{|y| \to +\infty} I_\alpha * |\gamma(v)|^p(y) = 0 \quad , \quad \text{uniformly w.r.t. } v \in S_a \,.
$$

Pick $R_a > 0$, independent of $v \in S_a$, such that $|y| \geq R_a$ implies

$$
|I_\alpha * |\gamma(v)|^p(y)| \, |\gamma(v)(y)|^{p-2} \leq \frac{a}{2} \,.
$$

As a consequence,

$$
\begin{cases}
-\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+ \\
-\dfrac{\partial v}{\partial x} \leq -\dfrac{a}{2} \, v & \text{in } \{0\} \times \{|y| \geq R_a\}
\end{cases}
$$

As in [9, Theorem 5.1] or [6, Theorem 7.1], and recalling the uniform decay at infinity of (7.10), it follows that v decays exponentially fast at infinity, with constants that are uniform with respect to $v \in S_a$.

We are ready to conclude: let $\{v_n\}_n$ be a sequence from S_a. Our previous arguments show that $\{v_n\}_n$ converges — up to a subsequence — weakly to some $v \in H$, and this limit v is also a solution to equation (7.4). Fix

$$
r > \max \left\{ 1 \,, \, \frac{N}{N(2-p)+p} \right\}
$$

and split I_α as $I_\alpha^1 + I_\alpha^2$, where $I_\alpha^1 \in L^r(\mathbb{R}^N)$ and $I_\alpha^2 \in L^\infty(\mathbb{R}^N)$. This induces a decomposition of the non-local term $\mathcal{N}(v) = \mathcal{N}^1(v) + \mathcal{N}^2(v)$ as

$$\begin{aligned}\mathcal{N}(v) &= \frac{1}{2p}\int_{\mathbb{R}^N} (I_\alpha * |\gamma(v)|^p)\, |\gamma(v)|^p\, dy \\ \mathcal{N}^1(v) &= \frac{1}{2p}\int_{\mathbb{R}^N} \left(I_\alpha^1 * |\gamma(v)|^p\right) |\gamma(v)|^p\, dy \\ \mathcal{N}^2(v) &= \frac{1}{2p}\int_{\mathbb{R}^N} \left(I_\alpha^2 * |\gamma(v)|^p\right) |\gamma(v)|^p\, dy\ .\end{aligned}$$

We obtain immediately that

$$0 = \lim_{n\to+\infty}\left(\int_{\mathbb{R}^{N+1}_+} \left(|\nabla v_n|^2 + m^2 v_n^2\right) dx\, dy - \mathcal{N}(v_n)\right) =$$

$$= \int_{\mathbb{R}^{N+1}_+} \left(|\nabla v|^2 + m^2 v^2\right) dx\, dy - \mathcal{N}(v)\ . \tag{7.11}$$

We complete the proof by showing that $\lim_{n\to+\infty} \mathcal{N}(v_n) = \mathcal{N}(v)$. Now, by the Hardy-Littlewood-Sobolev inequality (see [20, Theorem 4.3])

$$\left|\mathcal{N}^1(v_n) - \mathcal{N}^1(v)\right| \le$$

$$\le \int_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha^1(x-y)\, \big||\gamma(v_n)(x)|^p|\gamma(v_n)(y)|^p - |\gamma(v)(x)|^p|\gamma(v)(y)|^p\big|\, dx\, dy =$$

$$= \int_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha^1(x-y)\, \big||\gamma(v_n)(x)|^p|\gamma(v_n)(y)|^p - |\gamma(v_n)(x)|^p|\gamma(v)(y)|^p +$$

$$+ |\gamma(v_n)(x)|^p|\gamma(v)(y)|^p - |\gamma(v)(x)|^p|\gamma(v)(y)|^p\big|\, dx\, dy \le$$

$$\le \int_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha^1(x-y)|\gamma(v_n)(x)|^p\, \big||\gamma(v_n)(y)|^p - |\gamma(v)(y)|^p\big|\, dx\, dy +$$

$$+ \int_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha^1(x-y)|\gamma(v)(y)|^p\, \big||\gamma(v_n)(x)|^p - |\gamma(v)(x)|^p\big|\, dx\, dy =$$

$$= 2\int_{\mathbb{R}^N\times\mathbb{R}^N} I_\alpha^1(x-y)|\gamma(v_n)(x)|^p\, \big||\gamma(v_n)(y)|^p - |\gamma(v)(y)|^p\big|\, dx\, dy \le$$

$$\le 2C|I_\alpha^1|_r\, |\gamma(v_n)|^p_{2rp/(2r-1)}\, \big||\gamma(v_n)|^p - |\gamma(v)|^p\big|_{2r/(2r-1)} = o(1)\ ,$$

since $|\gamma(v_n)|^p - |\gamma(v)|^p \to 0$ strongly in $L^{2r/(2r-1)}_{\mathrm{loc}}(\mathbb{R}^N)$ by the choice of r. On the other hand,

$$\left|\mathcal{N}^2(v_n) - \mathcal{N}^2(v)\right| \le$$

$$\le \|I_\alpha^2\|_\infty \int_{\mathbb{R}^N\times\mathbb{R}^N} \big||\gamma(v_n)(x)|^p|\gamma(v_n)(y)|^p - |\gamma(v)(x)|^p|\gamma(v)(y)|^p\big|\, dx\, dy$$

and the conclusion follows as before. Since $\lim_{n\to+\infty} \mathcal{N}(v_n) = \mathcal{N}(v)$, equation (7.11) yields $\lim_{n\to+\infty} \|v_n\|^2 = \|v\|^2$, and the proof is complete.

□

7.2. **The penalization scheme.** For

$$\delta = \frac{1}{10}\operatorname{dist}(\mathcal{M}, \mathbb{R}^N \setminus O) \quad \text{and} \quad \beta \in (0,\delta)$$

we fix a cut-off $\varphi \in C_0^\infty(\mathbb{R}^{N+1}_+)$ such that $0 \le \varphi \le 1$ everywhere, $\varphi(x,y) = 1$ if $x + |y| \le \beta$, and $\varphi(x,y) = 0$ if $x + |y| \ge 2\beta$. Setting $\varphi_\varepsilon(x,y) = \varphi(\varepsilon x, \varepsilon y)$, for any $U \in S_{V_0}$ and any point $y_0 \in \mathcal{M}^\beta$ we define

$$U_\varepsilon^{y_0}(x,y) = \varphi_\varepsilon\left(x, y - \frac{y_0}{\varepsilon}\right) U\left(x, y - \frac{y_0}{\varepsilon}\right)$$

We also define, for all $\varepsilon > 0$,

$$\chi_\varepsilon(y) = \begin{cases} 0 & \text{if } y \in O_\varepsilon \\ \varepsilon^{-6/\mu} & \text{if } y \notin O_\varepsilon \end{cases}$$

and

$$Q_\varepsilon(v) = \left(\int_{\mathbb{R}^N} \chi_\varepsilon \gamma(v)^2 \, dy - 1 \right)_+^{(2p+1)/2}$$

for $v \in H$. Finally, let

$$\Gamma_\varepsilon(v) = \mathcal{E}_\varepsilon(v) + Q_\varepsilon(v) \quad , \quad v \in H \ .$$

We want to find a solution, for $\varepsilon > 0$ sufficiently small, near the set

$$X_\varepsilon = \left\{ U_\varepsilon^{y_0} \mid y_0 \in \mathcal{M}^\beta \text{ and } U \in S_{V_0} \right\} \ .$$

We define the (trivial) path $\psi_\varepsilon(s) = sU_\varepsilon^{y_0}$ for every $s \in [0,1]$.

Lemma 7.6. *There exists $T > 0$ such that $\Gamma_\varepsilon(\psi_\varepsilon(T)) < -2$ for all ε sufficiently small. Moreover,*

$$\lim_{\varepsilon\to 0} \max_{s\in[0,T]} \Gamma_\varepsilon(\psi_\varepsilon(s)) = E_{V_0}$$

where we recall that $E_{V_0} = L_{V_0}(U)$ for $U \in S_{V_0}$.

Proof. Indeed, by our definition of the penalization term Q_ε, by a simple change of variables and by the exponential decay of U at infinity,

$$\begin{aligned} \Gamma_\varepsilon(\psi_\varepsilon(s)) &= \mathcal{E}_\varepsilon(\psi_\varepsilon(s)) = \\ &= \frac{s^2}{2} \int_{\mathbb{R}^{N+1}_+} |\nabla \psi_\varepsilon(s)|^2 + \frac{m^2 s^2}{2} \int_{\mathbb{R}^{N+1}_+} \psi_\varepsilon(s)^2 + \frac{s^2}{2} \int_{\mathbb{R}^N} V_\varepsilon \gamma(\psi_\varepsilon(s))^2 - \\ &\quad - \frac{s^{2p}}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\psi_\varepsilon(s))|^p) \, |\gamma(\psi_\varepsilon(s))|^p = \\ &= \left(\frac{1}{2} \int_{\mathbb{R}^{N+1}_+} |\nabla U|^2 + \frac{m^2}{2} \int_{\mathbb{R}^{N+1}_+} U^2 + \frac{1}{2} \int_{\mathbb{R}^N} V_0 \gamma(U)^2 + o(1) \right) s^2 - \\ &\quad - \left(\int_{\mathbb{R}^N} (I_\alpha * |U|^p) \, |U|^p + o(1) \right) \frac{s^{2p}}{2p} \end{aligned}$$

where $o(1) \to 0$ as $\varepsilon \to 0$ uniformly with respect to s. The conclusion follows easily. □

We are ready to introduce our mini-max scheme. For $\varepsilon > 0$ sufficiently small, we define the set of paths

$$\Phi_\varepsilon = \{\psi \in C([0,T],H) \mid \psi(0) = 0,\ \psi(T) = \psi_\varepsilon(T) = TU_\varepsilon^{y_0}\} ,$$

where $T > 0$ is the number we found in Lemma 7.6. To this set we associate the min-max level

$$C_\varepsilon = \inf_{\psi\in\Phi_\varepsilon} \max_{s\in[0,T]} \Gamma_\varepsilon(\psi(s)) .$$

By well-known arguments (see for instance [5, Proposition 3.2] for a proof in a local setting that extends smoothly to our case) it is possible to prove that

$$\lim_{\varepsilon\to 0} C_\varepsilon = E_{V_0} .$$

For $\alpha \in \mathbb{R}$ define the sublevel

$$\Gamma_\varepsilon^\alpha = \{v \in H \mid \Gamma_\varepsilon(v) \le \alpha\} .$$

Proposition 7.7. *Let $d > 0$ be small enough, and let $\{\varepsilon_j\}_j$ be such that $\lim_{j\to+\infty} \varepsilon_j = 0$ and let $\{v_{\varepsilon_j}\} \subset X_{\varepsilon_j}^d$ be such that*

$$\lim_{j\to+\infty} \Gamma_{\varepsilon_j}(v_{\varepsilon_j}) \le E_{V_0} \quad , \quad \lim_{j\to+\infty} \Gamma'_{\varepsilon_j}(v_{\varepsilon_j}) = 0 .$$

Then there exist - up to a subsequence - $\{\tilde{y}_j\}_j \subset \mathbb{R}^N$, a point $\bar{y} \in \mathcal{M}$ and $U \in S_{V_0}$ such that

$$\lim_{j\to+\infty} |\varepsilon_j\tilde{y}_j - \bar{y}| = 0$$
$$\lim_{j\to+\infty} \left\| v_{\varepsilon_j} - \varphi_{\varepsilon_j}(\cdot,\cdot - \tilde{y}_j)U(\cdot,\cdot - \tilde{y}_j)\right\| = 0 .$$

Proof. We omit the very long proof. □

7.3. **Critical points of the penalized functional.** We are now ready to show that the penalized functional Γ_ε possesses a critical point for every $\varepsilon > 0$ sufficiently small.

Lemma 7.8. *For $d > 0$ sufficiently small, there exist positive constants ε_0 and ω such that $|\Gamma'_\varepsilon(v)| \ge \omega$ for every $v \in \Gamma_\varepsilon^{D_\varepsilon} \cap \left(X_\varepsilon^d \setminus X_\varepsilon^{d/2}\right)$ and $\varepsilon \in (0, \varepsilon_0)$.*

Proof. If not, for $d > 0$ so small that Proposition 7.7 applies, there exist sequences $\{\varepsilon_j\}_j$ with $\lim_j \varepsilon_j = 0$ and $\{v_{\varepsilon_j}\}_j$ with $v_{\varepsilon_j} \in X_{\varepsilon_j}^d \setminus X_{\varepsilon_j}^{d/2}$ satisfying

$$\lim_{j\to+\infty} \Gamma_{\varepsilon_j}(v_{\varepsilon_j}) \le E_{V_0} \quad \text{and} \quad \lim_{j\to+\infty} \Gamma'_{\varepsilon_j}(v_{\varepsilon_j}) = 0 .$$

Hence Proposition 7.7 applies and provides points $y_{\varepsilon_j} \in \mathbb{R}^N$, $\bar{y} \in \mathcal{M}$ and a ground state $U \in S_{V_0}$ such that

$$\lim_{j\to+\infty} |\varepsilon_j y_j - \bar{y}| = 0$$
$$\lim_{j\to+\infty} \left\| v_{\varepsilon_j} - \varphi_{\varepsilon_j}(\cdot,\cdot - y_j)U(\cdot,\cdot - y_j)\right\| = 0 .$$

The definition of X_{ε_j} implies $\lim_{j\to+\infty} \operatorname{dist}(v_{\varepsilon_j}, X_{\varepsilon_j}) = 0$, and this contradicts the assumption $v_{\varepsilon_j} \notin X_{\varepsilon_j}^{d/2}$. □

Let now $d > 0$ be chosen so that Lemma 7.8 applies.

Proposition 7.9. *For $\varepsilon > 0$ sufficiently small, the functional Γ_ε has a critical point $v_\varepsilon \in X_\varepsilon^d \cap \Gamma_\varepsilon^D$.*

Proof. Pick $R_0 > 0$ so large that $O \subset \left(\{0\} \times \mathbb{R}^N\right) \cap B(0, R_0)$ and $\psi_\varepsilon(s) \in H_0^1(B(0, R/\varepsilon))$ for any $s \in [0, T]$, $R > R_0$ and $\varepsilon > 0$ sufficiently small. We write $D_\varepsilon = \max_{0 \leq s \leq T} \Gamma_\varepsilon(\psi_\varepsilon(s))$. By Lemma 7.6, there exists $\mathfrak{a} \in (0, E_{V_0})$ such that, for sufficiently small $\varepsilon > 0$,

$$\Gamma_\varepsilon(\psi_\varepsilon(s)) \geq D_\varepsilon - \mathfrak{a} \quad \text{implies} \quad \psi_\varepsilon(s) \in X_\varepsilon^{d/2} \cap H_0^1(B(0, R/\varepsilon)) \ .$$

We claim that, for sufficiently small $\varepsilon > 0$ and $R > R_0$, there is a sequence $\{v_n^R\}_n \subset X_\varepsilon^{d/2} \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0, R/\varepsilon))$ such that $\Gamma_\varepsilon'(v_n^R) \to 0$ is $H_0^1(B(0, R/\varepsilon))$ as $n \to +\infty$.

Arguing by contradiction, we assume that for sufficiently small $\varepsilon > 0$ there exists a number $a_R(\varepsilon) > 0$ such that

$$|\Gamma_\varepsilon'(v)| \geq a_R(\varepsilon)$$

on $X_\varepsilon^{d/2} \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0, R/\varepsilon))$. With a slight abuse of notation, we will identify any $v \in H_0^1(B(0, R/\varepsilon))$ with its extension to H as the null function outside $B(0, R/\varepsilon)$. Applying Lemma 7.8, we find a number $\omega > 0$, independent of $\varepsilon > 0$, such that $|\Gamma_\varepsilon'(v)| \geq \omega$ for $v \in \Gamma_\varepsilon^{D_\varepsilon} \cap (X_\varepsilon^d \setminus X_\varepsilon^{d/2})$. By a classical deformation argument that starts from ψ_ε, there exist some $\mu \in (0, \mathfrak{a})$ and a path $\psi \in C([0, T], H)$ satisfying

$$\psi(s) = \psi_\varepsilon(s) \quad \text{for } \psi_\varepsilon(s) \in \Gamma_\varepsilon^{D_\varepsilon - \mathfrak{a}} \quad , \quad \psi(s) \in X_\varepsilon^d \quad \text{for } \psi_\varepsilon(s) \notin \Gamma_\varepsilon^{D_\varepsilon - \mathfrak{a}}$$

and

$$\Gamma_\varepsilon(\psi(s)) < D_\varepsilon - \mu \qquad \text{for every } s \in [0, T] \ . \tag{7.12}$$

Let $\zeta \in C_0^\infty(\mathbb{R}_+^{N+1})$ be a cut-off function such that $\zeta(x, y) = 1$ for $0 < x < \delta$ and $y \in O^\delta$, $\zeta(x, y) = 0$ for $x \geq 2\delta$ and $y \notin O^{2\delta}$, $\zeta(\cdot, \cdot) \in [0, 1]$, and $|\nabla \zeta| \leq 2/\delta$. For $\psi(s) \in X_\varepsilon^d$ we denote $\psi_1(s) = \zeta_\varepsilon \psi(s)$ and $\psi_2(s) = (1 - \zeta_\varepsilon)\psi(s)$, where $\zeta_\varepsilon(x, y) = \zeta(\varepsilon x, \varepsilon y)$. We remark that we understand the dependency on ε in the notation of ψ_1 and ψ_2. Observe that

$$\begin{aligned}
\Gamma_\varepsilon(\psi(s)) = \Gamma_\varepsilon(\psi_1(s)) + \Gamma_\varepsilon(\psi_2(s)) + Q_\varepsilon(\psi(s)) - Q_\varepsilon(\psi_1(s)) - Q_\varepsilon(\psi_2(s)) - \\
- \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\psi(s))|^p) |\gamma(\psi(s))|^p + \\
+ \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\psi_1(s))|^p) |\gamma(\psi_1(s))|^p + \\
+ \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\psi_2(s))|^p) |\gamma(\psi_2(s))|^p \ .
\end{aligned}$$

The elementary inequality $(h + k - 1)_+ \geq (h - 1)_+ + (k - 1)_+$ valid for $h \geq 0$ and $k \geq 0$ immediately implies that

$$Q_\varepsilon(\psi(s)) \geq Q_\varepsilon(\psi_1(s)) + Q_\varepsilon(\psi_2(s))$$

and, similarly to (6.7), we find that

$$\int_{\mathbb{R}^N \setminus O_\varepsilon} |\gamma(\psi(s))|^2 \, dy \leq C \varepsilon^{6/\mu} \ . \tag{7.13}$$

On the other hand, writing $\kappa = (I_\alpha * |\gamma(\psi(s))|^p)|\gamma(\psi(s))|^p - (I_\alpha * |\gamma(\psi_1(s))|^p)|\gamma(\psi_1(s))|^p - (I_\alpha * |\gamma(\psi_2(s))|^p)|\gamma(\psi_2(s))|^p$ for simplicity,

$$\int_{\mathbb{R}^N} \kappa = 2 \int_{O_\varepsilon^{2\delta} \times (\mathbb{R}^N \setminus O_\varepsilon^{2\delta})} (I_\alpha * |\gamma(\psi(s))|^p)|\gamma(\psi(s))|^p -$$

$$-2 \int_{(O_\varepsilon^{2\delta} \setminus O_\varepsilon^{\delta}) \times (\mathbb{R}^N \setminus O^\delta)} (I_\alpha * |\gamma(\psi(s))|^p)|\gamma(\psi(s))|^p$$

and from (7.13) via interpolation we deduce that

$$\lim_{\varepsilon \to 0} \int_{O_\varepsilon^{2\delta} \times (\mathbb{R}^N \setminus O_\varepsilon^{2\delta})} (I_\alpha * |\gamma(\psi(s))|^p)|\gamma(\psi(s))|^p = 0 \tag{7.14}$$

$$\lim_{\varepsilon \to 0} \int_{(O_\varepsilon^{2\delta} \setminus O_\varepsilon^{\delta}) \times (\mathbb{R}^N \setminus O_\varepsilon^\delta)} (I_\alpha * |\gamma(\psi(s))|^p)|\gamma(\psi(s))|^p = 0 \; . \tag{7.15}$$

Equations (7.14) and (7.15) yield

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\psi(s))|^p)|\gamma(\psi(s))|^p - \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\psi_1(s))|^p)|\gamma(\psi_1(s))|^p -$$

$$- \int_{\mathbb{R}^N} (I_\alpha * |\gamma(\psi_2(s))|^p)|\gamma(\psi_2(s))|^p = 0 \; ,$$

and hence, as $\varepsilon \to 0$,

$$\Gamma_\varepsilon(\psi(s)) \geq \Gamma_\varepsilon(\psi_1(s)) + \Gamma_\varepsilon(\psi_2(s)) + o(1) \; .$$

By similar arguments,

$$\Gamma_\varepsilon(\psi_2(s)) \geq$$

$$-\frac{1}{2p} \int_{(\mathbb{R}^N \setminus O_\varepsilon) \times (\mathbb{R}^N \setminus O_\varepsilon)} I_\alpha(x-y)|\gamma(\psi_2(s)(x)|^p|\gamma(\psi_2(s)(y)|^p \, dx \, dy \geq o(1) \; ,$$

and we finally conclude that

$$\Gamma_\varepsilon(\psi(s)) \geq \Gamma_\varepsilon(\psi_1(s)) + o(1)$$

as $\varepsilon \to 0$. If we define

$$\psi_1^1(s)(x,y) = \begin{cases} \psi_1(s)(x,y) & \text{if } x > 0 \text{ and } y \in O^{2\delta} \\ 0 & \text{if } x > 0 \text{ and } y \notin O^{2\delta} \; , \end{cases}$$

we immediately see that $\Gamma_\varepsilon(\psi_1(s)) \geq \Gamma_\varepsilon(\psi_1^1(s))$, and $\psi_1^1 \in \Phi_\varepsilon$ because $0 < \mathfrak{a} < E_{V_0}$. Now [13, Proposition 3.4] implies that, as $\varepsilon \to 0$,

$$\max_{0 \leq s \leq T} \Gamma_\varepsilon(\psi(s)) \geq E_{V_0} + o(1) \; ,$$

and this contradicts (7.12).

For a fixed ε sufficiently small and for $R \gg 1$, we consider a sequence $\{v_n^R\}_n \subset X_\varepsilon^{d/2} \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0,R/\varepsilon))$ such that $\Gamma_\varepsilon'(v_n^R) \to 0$ is $H_0^1(B(0,R/\varepsilon))$ as $n \to +\infty$. The boundedness of $\{v_n^R\}_n$ in $H_0^1(B(0,R/\varepsilon))$ and the Sobolev embedding theorem imply that $v_n^R \to v^R$ strongly in $L^q(B(0,R/\varepsilon))$ for any $q < 2N/(N-1)$. Since $\{v_n^R\}_n$

is a Palais-Smale sequence, a standard argument shows that $v_n^R \to v^R$ strongly in $H_0^1(B(0,R/\varepsilon))$. Hence the limit v^R is a weak solution to the problem

$$-\Delta v^R + m^2 v^R = 0 \qquad \text{in } B(0, \frac{R}{\varepsilon})$$

with

$$-\frac{\partial v^R}{\partial x}(0,y) = -V_\varepsilon(y) v^R(0,y) + \left(I_\alpha * |v^R(0,\cdot)|^p\right) |v^R(0,y)|^{p-2} v^R(0,y) +$$

$$+(2p+1)\left(\int_{\mathbb{R}^N} \chi_\varepsilon \gamma(v^R)^2 \, dy - 1\right)_+^{(2p-1)/2} \chi_\varepsilon v^R(0,y)$$

for $y \in \mathbb{R}^N$ with $|y| = R/\varepsilon$.

Since $v^R \in X_\varepsilon^d \cap \Gamma_\varepsilon^{D_\varepsilon} \cap H_0^1(B(0,R/\varepsilon))$, we deduce that both $\{\|v^R\|\}_R$ and $\{\Gamma_\varepsilon(v^R)\}_R$ are uniformly bounded for $\varepsilon > 0$ sufficiently small. Hence also $\{Q_\varepsilon(v^R)\}_R$ is uniformly bounded for $\varepsilon > 0$ sufficiently small. Now a Moser iteration scheme like [9, Theorem 3.2] yields that $\{v^R\}_R$ is bounded in L^∞ uniformly for $\varepsilon > 0$ sufficiently small. Taking into account that $\{Q_\varepsilon(v^R)\}_R$ is uniformly bounded in L^∞ and

$$\left(I_\alpha * |v^R(0,\cdot)|^p\right) |v^R(0,y)|^{p-1} \leq \frac{1}{2}(V_\varepsilon + m)|v^R(0,y)|$$

when $|y| \geq 2R/\varepsilon$, we can perform a comparison argument as in [9, Theorem 5.1] and derive

$$|v^R(x,y)| \leq C e^{-m(\sqrt{x^2+|y|^2} - 2R_0)} .$$

We assume, without loss of generality, that $\{v^R\}_R$ weakly converges to some v_ε in H as $R \to +\infty$ that solves

$$-\Delta v_\varepsilon + m^2 v_\varepsilon = 0 \qquad \text{in } \mathbb{R}_+^{N+1}$$

with

$$-\frac{\partial v_\varepsilon}{\partial x}(0,y) = -V_\varepsilon(y) v_\varepsilon(0,y) + \left(I_\alpha * |v_\varepsilon(0,\cdot)|^p\right) |v_\varepsilon(0,y)|^{p-2} v_\varepsilon(0,y) +$$

$$+(2p+1)\left(\int_{\mathbb{R}^N} \chi_\varepsilon \gamma(v_\varepsilon)^2 \, dy - 1\right)_+^{(2p-1)/2} \chi_\varepsilon v_\varepsilon(0,y)$$

for $y \in \mathbb{R}^N$.

□

8. Some perspectives

A major generalization of the pseudorelativistic Hartree equation consists in, roughly speaking, putting a potential *inside* the square root. For example we can replace the operator $\sqrt{-\Delta + m^2}$ with $\sqrt{-\Delta + V(\cdot)}$, where the potential function V is such that $-\Delta + V$ is a positive operator.

Quite similarly, we may also replace the standard Laplacian $-\Delta$ with a magnetic Laplacian $(-i\nabla - A)^2$ for some magnetic potential $A \colon \mathbb{R}^N \to \mathbb{R}^N$. Needless to say, it looks rather delicate to develop a good analytic framework for these x-dependent *nonlocal* differential operators, because it is unclear if they can be represented by a symbol in the theory of pseudodifferential operators. In the literature at least three different definitions of the fractional operator

$$\sqrt{(-i\nabla - A)^2 + m^2}$$

were proposed, see [17]. A possible starting point for a PDE analysis of this *magnetic pseudorelativistic* operator could be the case of a linear potential A, i.e. $A(x) = \hat{A}x$ for some constant matrix $\hat{A}$.

References

[1] J. Byeon & L. Jeanjean, *Standing waves for nonlinear Schrödinger equations with a general nonlinearity*, Archives for Rational Mechanics and Analysis, 185(2)(2007), 185–200.

[2] J. Byeon & L. Jeanjean, *Erratum: Standing waves for nonlinear Schrödinger equations with a general nonlinearity*, Archives for Rational Mechanics and Analysis, 190(3)(2008), 549–551.

[3] X. Cabré & S. Solà-Morales, *Layer solutions in a half-space for boundary reactions*, Comm. Pure Appl. Math., 58(12)(2005), 1678–1732.

[4] L. Caffarelli & L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations, 32(7-9)(2007), 1245–1260, doi 10.1080/03605300600987306.

[5] S. Cingolani, L. Jeanjean & S. Secchi, *Multi-peak solutions for magnetic NLS equations without non-degeneracy conditions*, ESAIM: Control Optim. Calc. Variations, 15(2009), 653–675.

[6] S. Cingolani & S. Secchi, *Ground states for the pseudo-relativistic Hartree equation with external potential*, Proc. Roy. Soc. Edinburgh Sect. A, 145(1)(2015), 73–90, doi 10.1017/S0308210513000450.

[7] S. Cingolani & S. Secchi, *Semiclassical analysis for pseudo-relativistic Hartree equations*, J. Differential Equations, 258(12)(2015), 4156–4179, doi 10.1016/j.jde.2015.01.029.

[8] S. Cingolani, S. Secchi & M. Squassina, *Semi-classical limit for Schrödinger equations with magnetic field and Hartree-type nonlinearities*, Proceedings of the Royal Society of Edinburgh, 140A(2010), 973–1009.

[9] V. Coti Zelati & M. Nolasco, *Existence of ground states for nonlinear, pseudo-relativistic Schrödinger equations*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 22(1)(2011), 51–72, doi 10.4171/RLM/587.

[10] V. Coti Zelati & M. Nolasco, *Ground states for pseudo-relativistic Hartree equations of critical type*, Rev. Mat. Iberoam., 29(4)(2013), doi 10.4171/RMI/763.

[11] V. Coti Zelati & M. Nolasco, *Ground states for pseudo-relativistic equations with combined power and Hartree-type nonlinearities*, in Recent trends in nonlinear partial differential equations. II. Stationary problems, Contemp. Math., 595, Amer. Math. Soc., Providence, RI, 2013, 151–167, doi 10.1090/conm/595/11810.

[12] V. Coti Zelati & M. Nolasco, *A variational approach to the eigenfunctions of the one particle relativistic Hamiltonian*, J. Elliptic Parabol. Equ., 1(2015), 89–108.

[13] V. Coti Zelati & P.H. Rabinowitz, *Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials*, J. Amer. Math. Soc., 4(1991), 693–727.

[14] A. Elgart & B. Schlein, *Mean field dynamics of boson stars*, Comm. Pure Appl. Math., 60(4)(2007), 500–545, doi 10.1002/cpa.20134.

[15] J. Fröhlich, B.L.G. Jonsson & E. Lenzmann, *Boson stars as solitary waves*, Comm. Math. Phys., 274(1)(2007), 1–30, doi 10.1007/s00220-007-0272-9.

[16] J. Fröhlich & E. Lenzmann, *Blowup for nonlinear wave equations describing boson stars*, Comm. Pure Appl. Math., 60(11)(2007), 1691–1705, doi 10.1002/cpa.20186.

[17] T. Ichinose, *Magnetic relativistic Schrödinger operators and imaginary-time path integrals*, in Mathematical physics, spectral theory and stochastic analysis, Oper. Theory Adv. Appl., 232, Birkhäuser/Springer Basel AG, Basel, 2013, 247–297, doi 10.1007/978-3-0348-0591-9_5.

[18] E. Lenzmann, *Uniqueness of ground states for pseudorelativistic Hartree equations*, Anal. PDE, 2(1)(2009), 1–27, doi 10.2140/apde.2009.2.1.

[19] E. Lenzmann, *Well-posedness for semi-relativistic Hartree equations of critical type*, Math. Phys. Anal. Geom., 10(1)(2007),43–64, doi 10.1007/s11040-007-9020-9.

[20] E.H. Lieb & M. Loss, *Analysis*, American Mathematical Society, Providence, RI, Graduate Studies in Mathematics, 14, 2001.

[21] E.H. Lieb & R. Seiringer, *The stability of matter in quantum mechanics*, Cambridge University Press, Cambridge, 2010.

[22] E.H. Lieb & H.-T. Yau, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Comm. Math. Phys., 112(1)(1987), 147–174.

[23] P–L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. I*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1(2)(1984), 109–145,

[24] V. Moroz & J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, J. Funct. Anal., 265(2)(2013), 153–184.

[25] D. Mugnai, *Pseudorelativistic Hartree equation with general nonlinearity: existence, non-existence and variational identities*, Adv. Nonlinear Stud., 13(4)(2013), 799–823.

[26] R.S. Palais, *The principle of symmetric criticality*, Comm. Math. Phys., 69(1)(1979), 19–30.

[27] P. H. Rabinowitz, *On a class of nonlinear Schrödinger equations*, Z. Angew. Math. Phys., 43(1992), 270–291.

[28] L. Tartar, *An introduction to Sobolev spaces and interpolation spaces*, 3, Springer Science & Business Media, 2007.

[29] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, 24, Birkhäuser, 1996.