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Results on standing waves for the nonlinear Chern-Simons-Schrödinger equations

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Abstract. We review recent results on standing waves for a nonlinear Schrödinger equations with the Chern-Simons gauge field. The existence, nonexistence, regularity and multiplicity issues are discussed. Assuming some suitable solution forms, we will see the nonlinear Chern-Simons-Schrödinger equations are reduced to single nonlocal equations, which are of variational. Then the critical point theory is applicable to investigate the existence of solutions.

1. INTRODUCTION

The aim of this paper is to review recent results on the so-called nonlinear Chern-Simons-Schrödinger equations, which is a nonlinear Schrödinger equations augmented the Chern-Simons gauge as follows:

$$\begin{cases} iD_0\phi + (D_1D_1 + D_2D_2)\phi = -\lambda|\phi|^{p-2}\phi \ , \\ \partial_0 A_1 - \partial_1 A_0 = -\mathrm{Im}(\bar{\phi}D_2\phi) \ , \\ \partial_0 A_2 - \partial_2 A_0 = \mathrm{Im}(\bar{\phi}D_1\phi) \ , \\ \partial_1 A_2 - \partial_2 A_1 = -1/2|\phi|^2 \ , \\ \lim\limits_{|x|\to\infty} |\phi(x)| = 0 \ , \end{cases} \tag{1.1}$$

where i denotes the imaginary unit, $\partial_0 := \partial/\partial t$, $\partial_1 := \partial/\partial x_1$, $\partial_2 := \partial/\partial x_2$ for $(t, x_1, x_2) \in \mathbb{R}^{1+2}$, $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is a complex scalar field, $A_\mu : \mathbb{R}^{1+2} \to \mathbb{R}$ is a gauge field, $D_\mu = \partial_\mu + iA_\mu$ is a covariant derivative for $\mu = 0, 1, 2$, $p > 2$ and $\lambda > 0$ is a parameter representing the strength of interaction potential. The Chern-Simons gauge model was suggested in the 1980's to explain electromagnetic phenomena of anyon physics such as the high temperature superconductivity or the fractional quantum Hall effect. The exact model (1.1) was proposed by Jackiw and Pi [10, 11].

The system of equations (1.1) enjoys a gauge symmetry. Observe that if (ϕ, A_0, A_1, A_2) is a solution of (1.1) and $\chi(x, t)$ is a smooth function, then a transform

$$\phi \to e^{i\chi}\phi \quad , \quad A_\mu \to A_\mu - \partial_\mu\chi \quad , \quad \mu = 0, 1, 2 \ , \tag{1.2}$$

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also gives a solution of (1.1). In other words, for any solution (ϕ, A_μ) to (1.1) and any C^∞ function χ, a 4-tuple $(e^{i\chi}\phi,\, A_\mu - \partial_\mu\chi)$ is also a solution to (1.1). We say two solutions $(\phi,\, A_\mu)$ and $(\phi',\, A'_\mu)$ to (1.1) are gauge equivalent each other if there is a χ satisfying

$$(e^{i\chi}\phi,\, A_\mu - \partial_\mu\chi) = (\phi',\, A'_\mu)\ .$$

Throughout the paper, we mean by a solution, a 4-tuple (u, A_0, A_1, A_2) which solves (1.1) in classical sense. Now, we focus on standing wave solutions to (1.1). If a solution (ψ, A_0, A_1, A_2) to (1.1) is of the form

$$\phi(t,x) = \varphi(x)e^{i\omega t} \quad , \quad A_\mu(t,x) = A_\mu(x) \quad , \quad \mu = 0,1,2 \tag{1.3}$$

for some function $\varphi : \mathbb{R}^2 \to \mathbb{C}$ and $\omega \in \mathbb{R}$, we call it a standing wave of frequency ω. A standing wave with $\omega = 0$ is called a static solution. We always assume that every standing wave considered in this paper has *finite energy*, i.e., $(\varphi,\, A_\mu) \in H^1(\mathbb{R}^2;\mathbb{C}) \times (L^\infty(\mathbb{R}^2;\mathbb{R}))^3$. Here, $H^1(\mathbb{R}^2;\mathbb{C})$ denotes the set of functions $\varphi : \mathbb{R}^2 \to \mathbb{C}$ such that both of φ and $\nabla\varphi$ are square integrable.

2. Equivalence and nonexistence results

With a special choice $p = 4$, $\lambda = 1$, static solutions to (1.1) are studied in [12] by reducing (1.1) to a first order system, which is called the self-dual system:

$$(D_1 + iD_2)\varphi = 0 \quad , \quad \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}\,|\varphi|^2 \quad , \quad A_0 = \frac{1}{2}\,|\varphi|^2\ . \tag{2.1}$$

Applying the following decomposition identity,

$$(D_1D_1 + D_2D_2)\varphi = (D_1 - iD_2)(D_1 + iD_2)\varphi + (\partial_1 A_2 - \partial_2 A_1)\varphi\ , \tag{2.2}$$

it is easy to see that every solution to (2.1) is a time independent solution to (1.1). The self-dual system (2.1) can be transformed, by the transformation $u = \log|\varphi|^2$, to the so-called Liouville equation

$$\Delta u + e^u = \sum_{i=1}^{k} \delta_{x_i}\ ,$$

where $\{x_1, \cdots, x_k\}$ is the zero set of φ and δ_{x_i} is a dirac mass at x_i. It is well known that the Liouville equation is an integrable equation whose solutions are explicitly known.

One can ask whether the self-dual system is actually equivalent to the static case of (1.1) when $p = 4$, $\lambda = 1$, i.e., is every static solution of (1.1) also a solution of self-dual system? This equivalence is first discussed by Jackiw and Pi in [12] but their arguments consist of formal calculations although it is enough in physical context. A rigorous mathematical proof is given in [8]. Note that from a gauge transform (1.2) with $\chi = -\omega t$, one may regard a standing wave of frequency ω as a static solution with different A_0. This means that we eventually know all the standing waves of (1.1) at least when $p = 4$, $\lambda = 1$.

On the other hand, for the other choice $p = 4$, $\lambda \in (0,\, 1)$, it is proved in [4] that (1.1) never admits nontrivial standing waves. In fact, this nonexistence result and the equivalence result mentioned above can be obtained from one essential integral identity.

2.1. **Integral identity.** Now, we introduce an integral identity which holds for every standing wave solutions to (1.1). First, we consider the following system of equations:

$$(\omega + A_0)\varphi - (D_1D_1 + D_2D_2)\varphi - V'(|\varphi|^2)\varphi = 0 \ , \tag{2.3}$$

$$\partial_1 A_0 = \mathrm{Im}(\bar\varphi D_2\varphi) \ , \tag{2.4}$$

$$\partial_2 A_0 = -\mathrm{Im}(\bar\varphi D_1\varphi) \ , \tag{2.5}$$

$$\partial_1 A_2 - \partial_2 A_1 = -1/2|\varphi|^2 \ . \tag{2.6}$$

Observe that the equations (2.3)–(2.6) with $V(s) = (\lambda/2)s^2$ are exactly the one obtained when we insert the standing wave ansatz (1.3) into equations (1.1), i.e., if $(\varphi e^{i\omega t},\, A_\mu)$ is a standing wave solution to (1.1), (φ, A_μ) solves (2.3)–(2.6).

Proposition 2.1 ([4])**.** *Suppose* $V \in C^1(\mathbb{R};\mathbb{R})$ *such that* $V(0) = V'(0) = 0$*. Let* $(\varphi,\, A_\mu) \in (C^2(\mathbb{R}^2;\mathbb{C}) \cap H^1(\mathbb{R}^2;\mathbb{C})) \times (C^1(\mathbb{R}^2;\mathbb{R}) \cap L^\infty(\mathbb{R}^2;\mathbb{R}))^3$ *be a solution to equations* (2.3)–(2.6) *satisfying*

$$V(|\varphi|^2),\ V'(|\varphi|^2)|\varphi|^2 \in L^1(\mathbb{R}^2;\mathbb{R}) \ .$$

Then one has

$$\int_{\mathbb{R}^2} |(D_1 + iD_2)\varphi|^2 + \frac{1}{2}\,|\varphi|^4 + V(|\varphi|^2) - V'(|\varphi|^2)|\varphi|^2\, dx = 0 \ . \tag{2.7}$$

The identity (2.7) is essentially established in [8]. The main idea of the proof is to integrate (2.3) after multiplying it by $x_k\overline{D_k\varphi}$ and φ to derive the Derrick-Pohozaev identity and the Nehari identity, respectively. Then the Bogomol'nyi trick (2.8)

$$|D_j\varphi|^2 = |(D_1 + iD_2)\varphi|^2 + \nabla \times J - F_{12}|\varphi|^2 \tag{2.8}$$

where

$$J \equiv (\mathrm{Im}(\bar\varphi D_1\varphi),\ \mathrm{Im}(\bar\varphi D_2\varphi)) \ ,$$

is used to obtain (2.7). We give here a sketch of proof adopted from [4].

Proof. For the simplicity, we just assume that the integration by parts on whole $\mathbb{R}^2$ is applicable without taking care of boundary terms. We refer to [8] for a rigorous argument.

From now on, we adopt the summation convention for repeated indices. Suppose that $(\varphi,\, A_\mu)$ is a solution of (2.3)–(2.6). Multiplying (2.3) by $x_k\overline{D_k\varphi}$ and integrating over $\mathbb{R}^2$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^2} (\omega + A_0)\varphi\, x_k\overline{D_k\varphi}\, dx - \int_{\mathbb{R}^2} D_jD_j\varphi\, x_k\overline{D_k\varphi}\, dx - \\ &- \int_{\mathbb{R}^2} V'(|\varphi|^2)\varphi\, x_k\overline{D_k\varphi}\, dx = 0 \ . \end{aligned} \tag{2.9}$$

We set

$$\mathrm{I} := \int_{\mathbb{R}^2} (\omega + A_0)\varphi\, x_k\overline{D_k\varphi}\, dx \qquad , \quad \mathrm{II} := \int_{\mathbb{R}^2} D_jD_j\varphi\, x_k\overline{D_k\varphi}\, dx \ ,$$

$$\mathrm{III} := \int_{\mathbb{R}^2} V'(|\varphi|^2)\varphi\, x_k\overline{D_k\varphi}\, dx \ .$$

Integrating by parts and taking real parts, we have

$$\begin{aligned} \mathrm{Re}\{\mathrm{I}\} &= -\int_{\mathbb{R}^2} (\omega + A_0)|\varphi|^2 + \frac{1}{2}\,|\varphi|^2\, x_j \partial_j A_0 \, dx \ , \\ \mathrm{Re}\{\mathrm{III}\} &= -\int_{\mathbb{R}^2} V(|\varphi|^2)\, dx \ . \end{aligned} \tag{2.10}$$

For II, we define $F_{jk} := \partial_j A_k - \partial_k A_j$. Then we see from integrating by parts that

$$\mathrm{II} = -\int_{\mathbb{R}^2} |D_j\varphi|^2 + x_k D_j\varphi \overline{D_j D_k \varphi}\, dx =$$

$$= -\int_{\mathbb{R}^2} |D_j\varphi|^2 + x_k D_j\varphi \, \overline{(D_k D_j \varphi + iF_{jk}\varphi)}\, dx \ ,$$

where we used the following identity

$$D_j D_k \varphi = D_k D_j \varphi + iF_{jk}\varphi \ .$$

Taking the real part, we obtain

$$\mathrm{Re}\{\mathrm{II}\} = -\int_{\mathbb{R}^2} |D_j\varphi|^2 + \frac{1}{2}\, x_k \partial_k(|D_j\varphi|^2) + x_k F_{jk}\mathrm{Im}(\bar{\varphi} D_j\varphi)\, dx =$$

$$= -\int_{\mathbb{R}^2} x_k F_{jk} \mathrm{Im}(\bar{\varphi} D_j \varphi)\, dx \ . \tag{2.11}$$

We put (2.10) and (2.11) into the identity (2.9). Then equations (2.4)–(2.6) imply that

$$\int_{\mathbb{R}^2} V(|\varphi|^2) - (\omega + A_0)|\varphi|^2 \, dx = 0 \ . \tag{2.12}$$

On the other hand, multiplying (2.3) by $\bar{\varphi}$ and integrating it, we derive

$$\int_{\mathbb{R}^2} (\omega + A_0)|\varphi|^2 + |D_j\varphi|^2 - V'(|\varphi^2|)|\varphi|^2 \, dx = 0 \ . \tag{2.13}$$

Combining (2.12) and (2.13), we conclude that

$$\int_{\mathbb{R}^2} |D_j\varphi|^2 + V(|\varphi|^2) - V'(|\varphi|^2)|\varphi|^2 \, dx = 0 \ . \tag{2.14}$$

Now we apply the Bogomol'nyi trick. Observe that the following identity holds

$$|D_j\varphi|^2 = |(D_1 + iD_2)\varphi|^2 + \nabla \times J - F_{12}|\varphi|^2 \tag{2.15}$$

where

$$J \equiv (\mathrm{Im}(\bar{\varphi} D_1 \varphi),\ \mathrm{Im}(\bar{\varphi} D_2\varphi)) \ ,$$

and we use the notation $\nabla \times J = \partial_1 J_2 - \partial_2 J_1$ for a vector $J = (J_1,\, J_2)$. Then the equation (2.14), considering the equation (2.6) and the divergence theorem, gives

$$\int_{\mathbb{R}^2} |(D_1 + iD_2)\varphi|^2 + \frac{1}{2}\,|\varphi|^4 + V(|\varphi|^2) - V'(|\varphi|^2)|\varphi|^2 dx = 0 \ .$$

This completes the proof.

□

2.2. **Equivalence and nonexistence results depending on $\lambda > 0$.** The equivalence and nonexistence results are easily obtained by applying (2.7).

Theorem 2.1 ([4]). *Let* $(\varphi e^{i\omega t},\, A_0, A_1, A_2)$ *be a standing wave solution to* (1.1). *Then one has that for all* $\omega \in \mathbb{R}$,

(*i*) *if* $0 < \lambda < 1$, *then* $\varphi \equiv 0$;
(*ii*) *if* $\lambda = 1$ *and* $\varphi \not\equiv 0$, *then it follows that*

$$(D_1 + iD_2)\varphi = 0 \quad , \quad \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}\,|\varphi|^2 \quad , \quad A_0 + \omega = \frac{1}{2}\,|\varphi|^2 \; .$$

Proof. Let $V(s) = (\lambda/2)s^2$ in (2.3) and $(\varphi e^{i\omega t},\, A_\mu)$ be a standing wave solution to (1.1) so that $(\varphi,\, A_\mu)$ is a solution to (2.3)-(2.6) and $(\varphi, A_\mu) \in (C^2(\mathbb{R}^2;\mathbb{C}) \cap H^1(\mathbb{R}^2;\mathbb{C})) \times (C^1(\mathbb{R}^2;\mathbb{R}) \cap L^\infty(\mathbb{R}^2;\mathbb{R}))^3$. It is immediate from the Sobolev inequality that

$$V(|\varphi|^2) = \frac{\lambda}{2}\,|\varphi|^4 \in L^1(\mathbb{R}^2;\mathbb{R}) \quad , \quad V'(|\varphi|^2)|\varphi|^2 = \lambda|\varphi|^4 \in L^1(\mathbb{R}^2;\mathbb{R}) \; .$$

Then Proposition 2.1 implies that we have an identity,

$$\int_{\mathbb{R}^2} |(D_1 + iD_2)\varphi|^2 + \frac{1}{2}(1-\lambda)|\varphi|^4 dx = 0 \; ,$$

from which we easily deduce that if $\lambda \in (0,1)$, then $\varphi \equiv 0$, and if $\lambda = 1$, then $(D_1 + iD_2)\varphi = 0$. To complete the proof of Theorem 2.1, it remains to show that if $\lambda = 1$ and $\varphi \not\equiv 0$, then $A_0 + \omega = (1/2)|\varphi|^2$. Applying the identity (2.2) to the equation (2.3), we see that

$$(\omega + A_0 - \frac{1}{2}\,|\varphi|^2)\varphi = 0 \; .$$

Also, multiplying both side of the equation $(D_1 + iD_2)\varphi = 0$ by $\bar{\varphi}$ and taking real and imaginary part, we get

$$\mathrm{Im}(\bar{\varphi} D_2 \varphi) = \frac{1}{2}\,\partial_1|\varphi|^2 \quad , \quad -\mathrm{Im}(\bar{\varphi} D_1 \varphi) = \frac{1}{2}\,\partial_2|\varphi|^2 \; .$$

Then, the equations (2.4) and (2.5) imply

$$\nabla A_0 = \nabla \frac{1}{2}\,|\varphi|^2 \; ,$$

from which we deduce $A_0 = (1/2)|\varphi|^2 + c$ for some constant $c \in \mathbb{R}$. Since $\varphi \not\equiv 0$, we finally get $c = -\omega$.

□

2.3. **Further results.** One can extend our nonexistence result to the equations (2.3)–(2.6) with general potentials V as follows.

Theorem 2.2 ([4]). *Let* $V \in C^1(\mathbb{R};\mathbb{R})$ *and* $V(0) = V'(0) = 0$. *Let also* $(\varphi,\, A_\mu) \in (C^2(\mathbb{R}^2;\mathbb{C}) \cap H^1(\mathbb{R}^2;\mathbb{C})) \times (C^1(\mathbb{R}^2;\mathbb{R}) \cap L^\infty(\mathbb{R}^2;\mathbb{R}))^3$ *be a solution to equations* (2.3)–(2.6) *satisfying*

$$V(|\varphi|^2),\; V'(|\varphi|^2)|\varphi|^2 \in L^1(\mathbb{R}^2;\mathbb{R}) \; .$$

(*i*) *If* $(1/2)s^2 + V(s) - V'(s)s > 0$ *for* $s > 0$, *then* $\varphi \equiv 0$.
(*ii*) *If* $(1/2)s^2 + V(s) - V'(s)s \geq 0$ *for* $s \geq 0$ *and there is a positive sequence* $\{s_n\} \to 0$ *such that* $(1/2)s_n^2 \neq V(s_n)$, *then* $\varphi \equiv 0$.

Proof. If $(1/2)s^2 + V(s) - V'(s)s \geq 0$ for $s \geq 0$ we get from the identity (2.7) that

$$\frac{1}{2}\,|\varphi|^4 + V(|\varphi|^2) - V'(|\varphi|^2)|\varphi|^2 \equiv 0\ .$$

This implies that

$$\frac{1}{2}\,s^2 + V(s) - V'(s)s = 0 \qquad \text{for all } s \in [0, M]\ ,$$

where $M = \sup_{x\in\mathbb{R}^2}\,|\varphi(x)|^2$. Then, if $(1/2)s^2 + V(s) - V'(s)s > 0$ for $s > 0$, then $M = 0$, which proves the first claim (i).

If $M > 0$, solving the above ODE, we get

$$V(s) = \frac{1}{2}\,s^2 \qquad \text{for all } s \in [0, M]\ .$$

This implies the second claim (ii).

□

In [9], the following nonlinear coupled Schrödinger system equipped with Chern-Simons gauge fields is considered:

$$\text{(2.16)} \qquad \begin{cases} iD_0\phi + (D_1D_1 + D_2D_2)\phi + 2\lambda_1|\phi|^2\phi + \lambda_2|\psi|^2\phi = 0\ , \\ iD_0\psi + (D_1D_1 + D_2D_2)\psi + \lambda_2|\phi|^2\psi - (\partial_1A_2 - \partial_2A_1)\psi = 0\ , \\ \partial_0A_1 - \partial_1A_0 = -\mathrm{Im}(\bar{\phi}D_2\phi) - \mathrm{Im}(\bar{\psi}D_2\psi) + \dfrac{1}{2}\,\partial_1|\psi|^2\ , \\ \partial_0A_2 - \partial_2A_0 = \mathrm{Im}(\bar{\phi}D_1\phi) + \mathrm{Im}(\bar{\psi}D_1\psi) + \dfrac{1}{2}\,\partial_2|\psi|^2\ , \\ \partial_1A_2 - \partial_2A_1 = -\dfrac{1}{2}(|\phi|^2 + |\psi|^2)\ . \end{cases}$$

By making use of an integral identity corresponding to (2.16), one can get similar equivalence and nonexistence results.

Theorem 2.3 ([9]). *Suppose that* $(\phi,\ \psi,\ A_0,\ A_1,\ A_2)$ *is a static solution of* (2.16) *with* $\lambda_1 = \lambda_2 = 1/2$ *such that*

- *(i)* $\phi,\ \psi \in H^1(\mathbb{R}^2;\mathbb{C})$ *and* $\phi \not\equiv 0$,
- *(ii)* $A_0 \in L^p(\mathbb{R}^2)$ *for some* $1 < p \leq \infty$,
- *(iii)* $A_1,\ A_2 \in L^q(\mathbb{R}^2)$ *for some* $2 < q \leq \infty$.

Then $(\phi,\ \psi,\ A_0,\ A_1,\ A_2)$ *also solves the self dual system:*

$$D_1u + iD_2u = 0 \quad , \quad D_1v + iD_2v = 0\ ,$$
$$\partial_1A_2 - \partial_2A_1 = -\frac{1}{2}(|u|^2 + |v|^2) \quad , \quad A_0 - \frac{1}{2}\,|u|^2 = 0\ .$$

Theorem 2.4 ([9]). *Suppose that* $(\phi,\ \psi,\ A_0,\ A_1,\ A_2)$ *is a standing wave solution of* (2.16) *satisfying (i)–(iii) of Theorem* 2.3 *and* $0 < \lambda_1 < 1/2$, $0 < \lambda_2 \leq 1/2$. *Then, one has* $\phi \equiv 0$. *In addition, one also has* $\psi \equiv 0$ *if* $A_0 \neq -\omega_2$.

Theorem 2.5 ([9]). *Suppose that* $(\phi,\ \psi,\ A_0,\ A_1,\ A_2)$ *is a standing wave solution of* (2.16) *satisfying (i)–(iii) of Theorem* 2.3 *and* $\lambda_1 = 1/2$, $0 < \lambda_2 < 1/2$. *Then, one has either* $\phi \equiv 0$ *or* $\psi \equiv 0$. *In addition, one has* $\psi \equiv 0$ *if* $A_0 \neq -\omega_2$.

The coupled Chern-Simons-Schrödinger system with a general potential is also dealt with in [9].

3. EXISTENCE OF BARE SOLUTIONS

In this section, we discuss about the existence of standing wave solution to (1.1) for $p > 2$. In other words, we look for a 4-tuple of functions $(\varphi, A_\mu) \in H^1(\mathbb{R}^2;\mathbb{C}) \times (L^\infty(\mathbb{R}^2;\mathbb{R}))^3$, which solves

$$\begin{aligned} &(\omega + A_0)\varphi - (D_1D_1 + D_2D_2)\varphi - |\varphi|^{p-2}\varphi = 0 \ , \\ &\partial_1 A_0 = \mathrm{Im}(\bar{\varphi} D_2\varphi) \ , \\ &\partial_2 A_0 = -\mathrm{Im}(\bar{\varphi} D_1\varphi) \ , \\ &\partial_1 A_2 - \partial_2 A_1 = -1/2|\varphi|^2 \ . \end{aligned} \tag{3.1}$$

Since directly solving (3.1) is not an easy task at all, it would be helpful to introduce more specific ansatz than (1.3) as follows:

$$\begin{aligned} &\varphi(x) = u(|x|) \quad &, \quad &A_0(x) = k(|x|) \ , \\ &A_1(x) = \frac{x_2}{|x|^2}\, h(|x|) \quad &, \quad &A_2(x) = -\frac{x_1}{|x|^2}\, h(|x|) \ , \end{aligned} \tag{3.2}$$

where u, k, h are real valued functions on $[0,\infty)$ such that $h(0) = 0$. We plug our ansatz (3.2) into (3.1) for obtaining that

$$\Delta u - \omega u - ku - h^2 u + \lambda |u|^{p-2}u = 0 \ , \tag{3.3}$$

$$\frac{1}{s}\, h'(s) = \frac{1}{2}\, u^2(s) \ , \tag{3.4}$$

$$k'(s) = -\frac{h(s)}{s}\, u^2(s) \ . \tag{3.5}$$

We choose a function space $H^1_r(\mathbb{R}^2)$, the set of radially symmetric functions in Sobolev space $H^1(\mathbb{R}^2)$, for u. Due to the radially symmetric property of H^1_r, one has the following inequality called the Strauss inequality for $u \in H^1_r(\mathbb{R}^2)$:

$$|u(x)| \le C\frac{\|u\|}{|x|^{1/2}} \quad , \quad |x| > 0 \tag{3.6}$$

where $\|u\|^2 = \int_{\mathbb{R}^2} |\nabla u|^2 + |u|^2\, dx$ denotes the Sobolev norm on $H^1(\mathbb{R}^2)$.

Integrating (3.4) and using the condition $h(0) = 0$, we see that

$$h(r) = \int_0^r \frac{s}{2}\, u^2(s)\, ds \ ,$$

thus we obtain

$$A_1(x) = \frac{x_2}{|x|^2} \int_0^{|x|} \frac{s}{2}\, u^2(s)\, ds \quad , \quad A_2(x) = -\frac{x_1}{|x|^2} \int_0^{|x|} \frac{s}{2}\, u^2(s)\, ds \ . \tag{3.7}$$

Also, by integrating both sides of (3.5) from r to ∞, we get

$$A_0(x) = k(|x|) = \xi + \int_{|x|}^\infty \frac{h(s)}{s}\, u^2(s)\, ds \ , \tag{3.8}$$

where ξ is an integral constant which represents the value of A_0 at infinity. The following regularity result is proved in [3] by using the Strauss inequality.

Proposition 3.1 ([3])**.** *If* $u \in H^1_r(\mathbb{R}^2)$*, then* A_1 *and* A_2 *defined by* (3.7) *are in* $L^\infty(\mathbb{R}^2)$*. If* u *is in* $H^1_r(\mathbb{R}^2) \cap L^\infty_{loc}(\mathbb{R}^2)$*, then* A_0 *defined by* (3.8) *is in* $L^\infty(\mathbb{R}^2)$*. Furthermore, if* u *is in* $H^1_r(\mathbb{R}^2) \cap C(\mathbb{R}^2)$*, then* A_0, A_1 *and* A_2 *defined by* (3.7) *and* (3.8) *are in* $L^\infty(\mathbb{R}^2) \cap C^1(\mathbb{R}^2)$*.*

Now, the equations (3.3)-(3.5) can be rewritten by

$$\begin{aligned}\Delta u - \omega u - \left(\xi + \int_{|x|}^{\infty} \frac{h(s)}{s}\, u^2(s)\, ds\right) u- \\ - \frac{h^2(|x|)}{|x|^2}\, u + \lambda |u|^{p-2} u = 0 \quad , \quad u \in H^1_r(\mathbb{R}^2)\end{aligned} \tag{3.9}$$

where $h(s) = \int_0^s (l/2) u^2(l)\, dl$. It turns out that the equation (3.9) is the Euler-Lagrange equation of the following functional

$$\begin{aligned}J(u) \equiv \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + (\omega + \xi) u^2 + \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2}\, u^2(s)\, ds\right)^2 dx \\ - \frac{\lambda}{p} \int_{\mathbb{R}^2} |u|^p\, dx \qquad \text{for } u \in H^1_r \ .\end{aligned} \tag{3.10}$$

The following proposition is proved in [3]. The proof consist of combining Proposition 3.1 with well known Moser iteration technique, Strauss inequality and Schauder estimate (see [6]).

Proposition 3.2 ([3])**.** *The functional* J *is continuously differentiable on* H^1_r*. If* u *is a critical point of* J*, it belongs* $C^2(\mathbb{R}^2)$ *and is a classical solution of* (3.9)*.*

Therefore, our problem of finding a standing wave solution of (1.1) is reduced to the problem of finding a nontrivial critical point of J. The following inequality plays a crucial role for studying the geometry of J.

Proposition 3.3 ([3])**.** *For* $u \in H^1_r(\mathbb{R}^2)$*, the following inequality holds*

$$\begin{aligned}\int_{\mathbb{R}^2} |u|^4\, dx \le \\ \le 4 \left(\int_{\mathbb{R}^2} |\nabla u|^2\, dx\right)^{1/2} \left(\int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2}\, u^2(s)\, ds\right)^2 dx\right)^{1/2} .\end{aligned} \tag{3.11}$$

Proof. Since $C_0^\infty(\mathbb{R}^2)$ is dense in H_r^1, it suffices to prove the inequality for $u \in C_0^\infty$. From the Fubini theorem and Hölder's inequality, we see that

$$
\begin{aligned}
\int_{\mathbb{R}^2} |u|^4\,dx &= \int_0^\infty |u|^4(r) 2\pi r\,dr = \\
&= \int_0^\infty 2\pi r u^2(r) \left(\int_r^\infty -(u^2(s))'\,ds \right) dr \leq \\
&\leq \int_0^\infty \int_0^\infty 4\pi r u^2(r) |u(s)||u'(s)| \chi_{\{s>r\}}\,ds\,dr = \\
&= \int_0^\infty 8\pi |u(s)||u'(s)| \left(\int_0^s \frac{r}{2}\,u^2(r)\,dr \right) ds = \\
&= 4 \int_{\mathbb{R}^2} \frac{|u||\nabla u|}{|x|} \left(\int_0^{|x|} \frac{s}{2}\,u^2(s)\,ds \right) dx \leq \\
&\leq 4 \left(\int_{\mathbb{R}^2} |\nabla u|^2\,dx \right)^{1/2} \left(\int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left(\int_0^{|x|} \frac{s}{2}\,u^2(s)\,ds \right)^2 dx \right)^{1/2}.
\end{aligned}
$$

This shows that the inequality holds.

□

This inequality tells us that the H^1 norm and nonlocal term $\int_{\mathbb{R}^2} (u^2/|x|^2) \left(\int_0^{|x|} (s/2) u^2(s)\,ds \right)^2 dx$ control the local nonlinear term $(1/p) \int_{\mathbb{R}^2} |u|^p\,dx$ for $2 < p < 4$ and $(\lambda/4) \int_{\mathbb{R}^2} |u|^4\,dx$ for $0 < \lambda < 1$. Consequently, the behavior of J when $p > 4$ differs from when $2 < p \leq 4$. One may also say that the effect of the local nonlinear term is stronger than the effect of the nonlocal term when $p > 4$, the opposite is true when $2 < p < 4$ and some competition takes places when $p = 4$, $\lambda > 1$.

With aid of (3.11), the existence of critical point of J is studied in [3]. Hereafter, we normalize the value ξ by 0, that is, we only consider solutions (ϕ, A_μ) such that $\lim_{x\to\infty} A_0(x) = 0$.

Theorem 3.1 ([3])**.** *Set $\xi = 0$. We denote J by $J_{\omega,\lambda}$ to clarify the dependence of ω and λ. Then one has the following:*

- *(i) if $p > 4$, then for any ω, $\lambda > 0$, $J_{\omega,\lambda}$ admits a critical point;*
- *(ii) if $p = 4$, then for any $\lambda > 1$, a dichotomy takes place that either $J_{0,\lambda}$ admits a nontrivial critical point or there exists some $\omega_0 > 0$ such that $J_{\omega_0,\lambda}$ admits a critical point.*
- *(iii) if $2 < p < 4$, then for any sufficiently small $\alpha > 0$, there exists ω_α such that for any $\lambda > 0$, $J_{\omega_\alpha,\lambda}$ admits a nontrivial critical point u_α satisfying $\|u_\alpha\|_{L^2(\mathbb{R}^2)} = \alpha$.*

Every critical point above is positive everywhere. In other words, solutions are bare.

The proof consists of devising three minimization problems corresponding to the range of p. We refer to [3] for the whole proof.

Later, Pomponio and Ruiz [13] improve the case (*iii*) of Theorem 3.1. By carrying out more refined analysis, they succeed to classify the geometry of J to obtain sharp existence results as follows.

Theorem 3.2 ([13])**.** *Fix* $2 < p < 4$. *Then, there exist* $\bar{\omega} > \tilde{\omega} > \omega_0$ *such that*

(*i*) *if* $\omega > \bar{\omega}$, *then* $J_{\omega,\lambda}$ *admits no nontrivial critical point;*

(*ii*) *if* $\omega \in (\omega_0, \tilde{\omega})$, *then* $J_{\omega,\lambda}$ *admits at least two nontrivial critical points: one of them is a global minimizer for* $J_{\omega,\lambda}$ *and the other is of mountain pass type* (*see* [15, 19])*;*

(*iii*) *for almost every* $\omega \in (0, \omega_0)$, $J_{\omega,\lambda}$ *admits a nontrivial critical point;*

Every critical point above is positive everywhere.

Here, the value ω_0 is explicitly given by the formula:

$$\omega_0 = \frac{4-p}{2+p}\, 3^{(p-2)/2(4-p)}\, 2^{2/(4-p)} \left(\frac{m^2(2+p)}{p-2}\right)^{-(p-2)/2(4-p)}, \tag{3.12}$$

where

$$m = \int_{-\infty}^{\infty} \left(\frac{2}{p}\cosh^2\left(\frac{p-2}{2}\, r\right)\right)^{2/(2-p)} dr\,.$$

We note that when $p = 4$, $J_{0,\lambda}(u)$ is invariant under a scaling $tu(t\cdot)$. Combining this fact with Theorem 3.1 and Theorem 3.2, we finally deduce the following.

Corollary 3.1. (*i*) *If* $p > 4$, *then for any* ω, $\lambda > 0$, *the equation* (1.1) *admits a nontrivial standing wave of the form* (3.2) *with frequency* ω.

(*ii*) *If* $p = 4$ *then for any* $\lambda > 1$, *the equation* (1.1) *admits one parameter family of nontrivial standing waves of the form* (3.2). *In this case, the frequencies are not specified.*

(*iii*) *If* $2 < p < 4$, *then for* $\lambda > 0$, *the equation* (1.1) *admits no nontrivial standing wave of the form* (3.2) *for frequency* $\omega > \bar{\omega}$, *admits at least two standing waves of the form* (3.2) *for frequency* $\omega \in (\omega_0, \tilde{\omega})$ *and admits a standing wave of the form* (3.2) *for almost every frequency* $\omega \in (0, \omega_0)$.

Some remarks are in order.

Remark 3.1. Recall that when $p = 4$ and $\lambda = 1$, any standing wave of (1.1) satisfies the self dual system (2.1). If the standing wave is of the form (3.2), it is given by the following explicit formula:

$$(\phi, A_0, A_1, A_2) = \left(\frac{\sqrt{8}\, l\, e^{i\omega t}}{1+|lx|^2}\,,\ \left(\frac{2l}{1+|lx|^2}\right)^2 - \omega,\ \frac{2l^2 x_2}{1+|lx|^2}\,,\ \frac{-2l^2 x_1}{1+|lx|^2}\right),$$

where $l > 0$ is an arbitrary real constant.

Remark 3.2. If we don't normalize the value of A_0 at infinity as 0, then for any frequency $\omega > 0$, we can prove the existence of a nontrivial standing wave for $p \le 4$ by using the gauge transform (1.2). Moreover, they are not gauge equivalent for different frequencies. See [3] for the precise way of construction.

3.1. **Further results.** Huh [7] proved the existence of infinitely many critical points of J for $p > 6$ by applying the symmetric mountain pass lemma (see [15, 19]). For $4 < p \leq 6$, one may also see that J still enjoys the symmetric mountain pass geometry but it is not possible to check whether J satisfies the (PS) condition. In [16], infinitely many critical points of J for $p > 4$ are found. The proof follows arguments in [17], based on so-called Struwe's monotonicity trick [18]. In fact, a general class of nonlinearities for existence is proposed in [16]. Under the following structure condition for V:

- (V1) $V \in C^1(\mathbb{R}, \mathbb{R})$ such that $V(0) = V'(0) = 0$;
- (V2) $\limsup_{s\to\infty} |V'(s)/s^{p-1}| < \infty$ and $\limsup_{s\to\infty} |V(s)/s^p| < \infty$ for some $p > 1$;
- (V3) there exists some $\alpha > 1$ such that $\alpha V'(s)/s^{2-1/\alpha} - V(s)/s^{3-1/\alpha}$ is monotonically increasing to ∞ as $s \to \infty$,

there exists a nontrivial solution of the equation

$$\Delta u - \omega u - \left(\int_{|x|}^{\infty} \frac{h(s)}{s} \, u^2(s) \, ds \right) u - \frac{h^2(|x|)}{|x|^2} \, u + V(u^2)u = 0 \, .$$

For other existence results dealing with general potentials different from (V1)–(V3), we refer to [5] and [20]. The former paper consider Berestycki-Gallouet-Kavian type nonlinearity[1], which is a planar version of Berestycki-Lions type nonlinearity[2] and The latter studies asymptotically linear nonlinearities.

Finally, Ruiz and Pomponio studied a problem that the equation (3.9) is restricted to $B(0, R)$, the ball with radius R centered at origin, under the zero Dirichlet condition. They constructed a family of solutions u_R which concentrates near the boundary $\partial B(0, R)$ as $R \to \infty$ by using a singular perturbation method based on a Lyapunov-Schmidt reduction. See [14].

4. Existence of vortex solutions

The standing waves constructed in [3] and [13] admit no vortices. In other words, every standing wave (ϕ, A_μ) considered satisfies $|\phi| > 0$. In [4], the authors construct standing waves with a vortex point of order N for arbitrarily given $N \in \mathbb{N}$. Precisely speaking, this means solutions (ϕ, A_μ) of (1.1) satisfying $\phi(0) = 0$ and $\lim_{|x|\to 0} |\phi(x)|/|x|^N > 0$. To do this, they propose an ansatz of the form

$$\begin{aligned} \phi(t, x) &= e^{i(N\theta + \omega t)} u(|x|) \quad , \quad A_0(t, x) = k(|x|) \, , \\ (A_1(t, x), A_2(t, x)) &= (x_2, \, -x_1) \, \frac{h(|x|)}{|x|^2} \, , \end{aligned} \tag{4.1}$$

where $\tan\theta = x_2/x_1$ and u, k, h are real valued such that $h(0) = 0$. We note that if (ϕ, A_μ) is a solution of (1.1) of the form (4.1), then $u(0)$ should be 0 due to the singularity of $e^{i(N\theta+\omega t)}$ at 0. It is proved in [4] that the vortex point 0 of solutions of (1.1) which has the form (4.1) must be of order N under some suitable assumptions.

Proposition 4.1 ([4]). *Let N be a positive integer and (ϕ, A_0, A_1, A_2) a solution of* (1.1)*, which has the form* (4.1) *such that the origin is a unique zero of $u = |\phi|$, i.e.,*

$$u(0) = 0 \quad \textit{and} \quad u(x) > 0 \ \textit{if} \ |x| > 0 \, .$$

Then the origin is a zero of order N, i.e., there is a positive function $\eta \in C^2(\mathbb{R}^2)$ such that $u(x) = |x|^N \eta(x)$.

For further purpose, we replace the nonlinearity $\lambda|\phi|^2\phi$ of (1.1) with $V'(|\phi|^2)\phi$. Then, inserting (4.1) into (1.1) with the nonlinearity $V'(|\phi|^2)\phi$, we obtain

$$\Delta u - \omega u - \frac{(h(r) - N)^2}{r^2}\, u - k(r)u + V'(u^2)u = 0\ , \tag{4.2}$$

$$k'(r) = \frac{1}{r}(N - h(r))u(r)^2\ , \tag{4.3}$$

$$\frac{h'(r)}{r} = \frac{1}{2}\, u(r)^2\ . \tag{4.4}$$

We assume that $V \in C^1(\mathbb{R};\mathbb{R})$ and $V(0) = V'(0) = 0$. We can deduce from (4.3) and (4.4) that h and k are formally expressed by u as

$$h_u(r) = \int_0^r \frac{s}{2}\, u^2(s)\, ds \quad , \quad k_u(r) = \xi + \int_r^\infty \frac{h_u(s) - N}{s}\, u^2(s)\, ds\ , \tag{4.5}$$

where $\xi \in \mathbb{R}$ is an integral constant. As the previous section we set $\xi = 0$. We put (4.5) into (4.2) to get a reduced single equation:

$$\Delta u - \omega u - \frac{(h_u - N)^2}{|x|^2}\, u - \left(\int_{|x|}^\infty \frac{h_u(s) - N}{s}\, u^2(s)\, ds\right) u + V'(u^2)u = 0\ , \tag{4.6}$$

where $h_u(s) = \int_0^s (l/2)u^2(l)\, dl$. Thus, if we find a solution $u \in C^2(\mathbb{R}^2 \setminus \{0\};\mathbb{R})$ of (4.6) such that every integrals in (4.6) are finite, then a 4-tuple (ϕ, A_0, A_1, A_2) defined by

$$(\phi, A_0, A_1, A_2) := \left(e^{iN\theta} u e^{i\omega t},\ k_u(|x|),\ \frac{x_2}{|x|^2}\, h_u(|x|),\ -\frac{x_1}{|x|^2}\, h_u(|x|)\right) \tag{4.7}$$

solves (1.1) on $\mathbb{R} \times \{\mathbb{R}^2 \setminus \{0\}\}$. To show that (ϕ, A_0, A_1, A_2) also satisfies (1.1) at the origin, it is sufficient to prove that $(\phi, A_0, A_1, A_2) \in C^2(\mathbb{R}^{1+2};\mathbb{C}) \times (C^1(\mathbb{R}^2;\mathbb{R}))^3$.

It is worth to mention here that a suitable function space for solutions of (4.6) is not $H^1(\mathbb{R}^2)$ because $k_u(0)$ is not finite for $u \in H^1(\mathbb{R}^2)$. An appropriate function space turns out to be H defined by the completion of a set $D := \{\, u \in C_c^\infty(\mathbb{R}^2 \setminus \{0\};\mathbb{R}) \mid u(x) = u(|x|)\,\}$ with respect to the norm

$$\|u\|_H^2 := \int_{\mathbb{R}^2} |\nabla u|^2 + \left(1 + \frac{N^2}{|x|^2}\right) u^2\, dx\ .$$

The first observation is that every $u \in H$ is continuous and bounded such that $u(0) = 0$.

Proposition 4.2 ([4]). *Define*

$$C_v(\mathbb{R}^2) := \left\{ u \in C(\mathbb{R}^2;\mathbb{R}) \ \mid\ u(0) = 0,\ \sup_{x\in\mathbb{R}^2} |u(x)| < \infty \right\}\ .$$

Then H is continuously embedded in $C_v(\mathbb{R}^2)$. More precisely, $H \subset C_v(\mathbb{R}^2)$ and the following pointwise inequality holds

$$|u(r)|^2 \le \frac{1}{2\pi} \int_{B(0,r)} |\nabla u|^2 + \frac{1}{|x|^2}\, u^2\, dx$$

In particular we have

$$\sup_{x\in\mathbb{R}^2} |u(x)| \leq \frac{1}{\sqrt{2\pi}} \|u\|_H \ .$$

The following proposition shows the validity of integrals in (4.5) and the regularity of gauge fields A_μ defined by (4.5) and (4.7) when $u \in H$.

Proposition 4.3 ([4])**.** *If* $u \in H$*, then* $(A_1, A_2) \in C^1(\mathbb{R}^2;\mathbb{R}) \cap L^\infty(\mathbb{R}^2;\mathbb{R})$ *and* $A_0 \in C(\mathbb{R}^2;\mathbb{R}) \cap L^\infty(\mathbb{R}^2;\mathbb{R})$ *where* (A_0, A_1, A_2) *is defined by* (4.5) *and* (4.7)*.*

Now, we define an action functional I on H by

$$I(u) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + \omega u^2 + \frac{(h_u - N)^2}{|x|^2}\, u^2 - V(u^2)\, dx \ ,$$

We recall that $V \in C^1(\mathbb{R};\mathbb{R})$ such that $V(0) = V'(0) = 0$. We additionally assume on V that there is $p > 1$ such that

$$\limsup_{s\to\infty} \left|\frac{V'(s)}{s^{p-1}}\right| < \infty \quad , \quad \limsup_{s\to\infty} \left|\frac{V(s)}{s^p}\right| < \infty \ . \tag{4.8}$$

Then one is able to show (4.6) is an Euler-Lagrange equation of I.

Proposition 4.4 ([4])**.** *I is well-defined, continuously differentiable on H and every critical points of I are weak solutions of* (4.6)*, i.e., if $u \in H$ is a critical point of I, we have*

$$\int_{\mathbb{R}^2} \nabla u \cdot \nabla\varphi + \omega u\varphi + \frac{(h_u - N)^2}{|x|^2}\, u\varphi +$$

$$+ \left(\int_{|x|}^{\infty} \frac{h_u(s) - N}{s}\, u^2(s)\, ds\right) u\varphi + V'(u^2)u\varphi\, dx = 0 \ ,$$

for every $\varphi \in H$, in particular, every $\varphi \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$.

The following proposition says that if $u \in H$ is a weak solution of (4.6), the 4-tuple of functions (4.7) defined by u is a (classical) solution of (1.1). Note also that Proposition 4.1 is a special case of it.

Proposition 4.5 ([4])**.** *Suppose that V satisfies* (4.8)*, $V \in C^{1,\alpha}(\mathbb{R};\mathbb{R})$ for some $\alpha \in (0, 1)$ and $V(0) = V'(0) = 0$. Consider the equation* (1.1) *with the nonlinearity $V'(|\phi|^2)\phi$ replacing $\lambda|\phi|^2\phi$.*

- *(i) If $u \in H$ is a weak solution of* (4.6) *and $u > 0$ on $\mathbb{R}^2 \setminus \{0\}$, then*
  - (1) *$u \in C^2(\mathbb{R}^2 \setminus \{0\};\mathbb{R})$ and there is $\eta \in C^2(\mathbb{R}^2;\mathbb{R})$ such that $u(x) = |x|^N\eta(x)$ and $\eta > 0$ on $\mathbb{R}^2$;*
  - (2) *$(\phi, A_0, A_1, A_2) \in C^2(\mathbb{R}^{2+1};\mathbb{C}) \times (C^1(\mathbb{R}^2;\mathbb{R}) \cap L^\infty(\mathbb{R}^2;\mathbb{R}))^3$ and is a solution of* (1.1) *for (ϕ, A_0, A_1, A_2) defined by* (4.7)*.*
- *(ii) If $(\phi = e^{i(N\theta+\omega t)}u, A_0, A_1, A_2)$ is a solution of finite energy to* (1.1) *such that $u > 0$ on $\mathbb{R}^2 \setminus \{0\}$, then u is in H.*

Once more, our problem of finding a standing wave solution to (1.1) carrying a vortex point of order N to is reduced to the problem of finding a nontrivial critical point of I. One also has an inequality analogous to (3.11).

Proposition 4.6 ([4])**.** *For $u \in H$, the following inequality holds:*

$$\int_{\mathbb{R}^2} |u|^4\,dx \le 4\left(\int_{\mathbb{R}^2} |\nabla u|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^2} \frac{(h_u - N)^2}{|x|^2}\,u^2\,dx\right)^{1/2}. \tag{4.9}$$

With the inequality (4.9) in hand, one can expect an existence result analogous to Corollary 3.1 holds.

Theorem 4.1 ([4])**.** (*i*) *If $p > 4$, then for any ω, $\lambda > 0$, the equation* (1.1) *admits a nontrivial standing wave of the form* (4.1) *with frequency ω.*

(*ii*) *If $p = 4$ then for any $\lambda > 1$, the equation* (1.1) *admits one parameter family of nontrivial standing waves of the form* (4.1)*. In this case, the frequencies are not specified.*

Moreover, every standing wave mentioned above satisfies $|\phi(0)| = 0$ and $|\phi| > 0$ in $\mathbb{R}^2 \setminus \{0\}$.

An existence result for the equations (1.1) with general potentials is also available. In fact, the statement (i) of theorem 4.1 is a special case of the following theorem.

Theorem 4.2 ([4])**.** *Let $V \in C^{1,\alpha}(\mathbb{R},\mathbb{R})$ be such that $V(0) = V'(0) = 0$. Suppose that V satisfies the following conditions:*

($B1$) $\limsup_{s\to\infty} |V'(s)/s^{p-1}| < \infty$, $\limsup_{s\to\infty} |V(s)/s^{p}| < \infty$ *for some $p > 1$;*

($B2$) *there is $\alpha \in (0,1)$ such that $0 < (1+\alpha)V(s) \le \alpha V'(s)s$ for all $s > 0$.*

Then for any ω, $\lambda > 0$, the equation (1.1) *with the nonlinearity $V'(|\phi|^2)\phi$ replacing $\lambda|\phi|^2\phi$ admits a nontrivial standing wave of the form* (4.1) *with frequency ω. Moreover,the standing wave satisfies $|\phi(0)| = 0$ and $|\phi| > 0$ in $\mathbb{R}^2 \setminus \{0\}$.*

Jiang, Pomponio and Ruiz [4] studied the case $p \in (2,\,4)$ and obtained a results analogous to their former one. We recall that in their result it is assumed that $\lim_{x\to\infty} A_0(x) = 0$.

Theorem 4.3 ([14])**.** *If $p \in (2,4)$, then there exist $\bar{\omega} > \tilde{\omega} > \omega_0$ such that for any $\lambda > 0$, the equation* (1.1) *admits no nontrivial standing wave of the form* (4.1) *for frequency $\omega > \bar{\omega}$, admits at least two standing waves of the form* (3.2) *for frequency $\omega \in (\omega_0,\,\tilde{\omega})$ and admits a standing wave of the form* (3.2) *for almost every frequency $\omega \in (0,\,\omega_0)$.*

Here still ω_0 is given by (3.12).

Remark 4.1. The explicit solution also can be given for $p = 4$, $\lambda = 1$ as in Remark 3.1. Any solution $(\phi,\,A_0, A_1, A_2)$ to (1.1) of the form (4.1) satisfying $|\phi(0)| = 0$ and $|\phi| > 0$ in $\mathbb{R}^2 \setminus \{0\}$ is given by

$$(\phi, A_0, A_1, A_2) = \left(\frac{\sqrt{8}\,l\,(N+1)|lx|^N}{1+|lx|^{2(N+1)}}\,e^{i(N\theta+\omega t)},\ \left(\frac{2l(N+1)|lx|^N}{1+|lx|^{2(N+1)}}\right)^2 - \omega\,,\right.$$

$$\left.\frac{2(N+1)l^2x_2|lx|^{2N}}{1+|lx|^{2(N+1)}}\,,\ -\frac{2(N+1)l^2x_1|lx|^{2N}}{1+|lx|^{2(N+1)}}\right)\,, \tag{4.10}$$

for some $l > 0$ when $p = 4$ and $\lambda = 1$.

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