Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 13(2016), pp. 145 – 160.

The Ljusternick-Schnirelmann Theory applied to a Schrödinger-Poisson system

Gaetano SICILIANO[1]

Abstract. This is a survey paper in which we present some existence and multiplicity results for solutions of a Schrödinger-Poisson system in a bounded domain with homogeneous boundary conditions on both the unknowns. Depending on the values of the parameters appearing in the system, different tools in classical Ljusternick-Schinirelmann Theory are used.

1. INTRODUCTION

The classical Ljusternick-Schnirelmann Theory (LS Theory) concerns with the existence and multiplicity of critical points of functions defined on compact manifolds, explored with a topological invariant called the *LS category*. However after the "introduction" of the well-known Palais-Smale condition, the LS Theory has been widely applied to functionals defined in infinite dimensional spaces, or manifolds, obtaining beautiful multiplicity results for the existence of solutions to certain (so called "variational") equations.

Moreover it has been observed that it deserve some generalizations, indeed an axiomatic formulation of the LS Theory can be given and other *indices* appeared in the mathematical literature. One of these, maybe for its simplicity and for the wide range of applicability, is the *Krasnoselskii genus*, used to prove, e.g, the famous Symmetric Mountain Pass Theorem of Ambrosetti and Rabinowitz.

In this note we review some results concerning the existence of multiple solutions for a well-studied and interesting, both for a mathematical and physical point of view, elliptic system by means of the LS Theory. The system under study is

$$
\begin{cases}
-\Delta u + u + \lambda\phi u = |u|^{p-2}u & \text{in } \Omega\ , \\
-\Delta\phi = u^2 & \text{in } \Omega\ , \\
u = \phi = 0 & \text{on } \partial\Omega\ ,
\end{cases}
\tag{1.1}
$$

where Ω is a (smooth and) bounded domain in $\mathbb{R}^3$, $p \in (2, 2^*)$ and $\lambda > 0$ is a suitable parameter whose meaning will be soon explained.

[1]G. Siciliano, Universidade de São Paulo, Departamento de Matemática, Instituto de Matemática e Estatística, Rua do Matão 1010, 05508-090 São Paulo, SP, Brazil; sicilian@ime.usp.br
The author was partially supported by Fapesp and CNPq, Brazil.

Keywords. Schrödinger-Poisson system, variational methods, Ljusternick-Schnirelmann theory, multiplicity of solutions.

AMS Subject Classification. 35A15, 58E05, 74G35.

This equations appear studying the nonlinear Schrödinger equation which describes quantum (non relativistic) particles where the electromagnetic field generated by the motion is taken into account, so that the particle system is interacting with its own electromagnetic field. This interaction can be described mathematically in the framework of the abelian gauge theories. Without entering in details here, we refer the reader to [7] were the complete system is derived. In particular looking for standing waves solutions $\psi = u(x)e^{i\omega t}, u(x) \in \mathbb{R}, \omega > 0$ in the purely electrostatic situation (i.e. the magnetic field is identically zero and the electric field is static, $\phi(x,t) = \phi(x)$), after a suitable "normalization" of the constants we arrive at system (1.1). The boundary conditions $u = \phi = 0$ on $\partial\Omega$ can be interpreted by saying that the particle is constraint to "live" in Ω. The parameter $\lambda > 0$ is introduced motivated by some perturbation results (see e.g. [10, 18] in which the case with $\Omega = \mathbb{R}^3$ and $\lambda \to 0^+$ is considered). However it takes a role also in a bounded domain as we will see.

Because of its importance in many different physical framework, the Schrödinger-Poisson-Slater system (sometimes called Schrödinger-Maxwell system) has been extensively studied in the past years: besides the results on bounded domains (see e.g. [7, 16, 17, 20]), there are also many papers on $\mathbb{R}^3$ which treat different aspects of the SP system, even with an additional external and fixed potential $V(x)$. In particular ground states, radially and non-radially solutions or semiclassical limit and concentration of solutions are studied, see e.g. [3, 4, 8, 9, 11, 14, 15, 19, 24] and the references therein.

In this paper we show as, depending on the values of the exponent p in the local nonlinearity and the parameter λ, the LS Theory combined with different methods can be used to show the existence of solutions of (1.1). Indeed we will resume some results proved in [17, 20, 21] based on the Krasnoselki genus, the Ljusternick-Schnirelmann category, the monotonicity trick of Jeanjean and Struwe, suitable mini-max schemes and the "photography" method of Benci and Cerami.

2. Statement of the results

The solutions of (1.1) will be found as critical points of a suitable functional whose "shape" strongly depend on the values of λ and p. For this reason it is reasonable to expect that the existence and multiplicity of critical points of the functional (that is, of solutions of the problem) is affected by the values of these parameters. Observe that the parameter λ can be interpreted as the strength of the interaction with the electric field, while p represents the strength of the nonlinearity.

The main results presented here are the following ones. It has to be noted that some cases remain open.

Theorem 2.1. *Let $p \in (2, 3]$. Then for λ sufficiently large system* (1.1) *has no nontrivial solutions.*

This is the unique result concerning nonexistence. The next ones deal with existence and multiplicity.

Theorem 2.2. *Let $p \in (2, 3)$. Then for λ small enough system* (1.1) *has (at least) two different positive nontrivial solutions.*

We point out that the case $p = 3$ is critical, in some sense, for problem (1.1).

Theorem 2.3. *Let $p \in (3, 4]$. Then for almost every value of $\lambda > 0$ there exist infinitely many (possibly sign changing) solutions of the problem.*

Theorem 2.4. *Let $p \in (4, 6)$. Then for every $\lambda > 0$ problem* (1.1) *has infinitely many (possibly sign changing) solutions.*

We remark that in [17] a more general case of this theorem is treated.

It is also worth to point out that by means of the Topological Degree, it is possible to show that for any $p \in (2, 6)$, the existence of a positive solution is guaranteed for *any* λ smaller then a certain value. Since the proof does not use variational arguments, it is not presented here (see [20, Theorem 2.1]).

The next theorem gives a multiplicity result of positive solutions. This result can be achieved by taking into account that the energy functional associated to the problem is bounded below on a suitable manifold and the Palais-Smale condition is satisfied.

Theorem 2.5. *There exists a $\bar{p} \in (4, 2^*)$ such that for every $p \in [\bar{p}, 2^*)$ and for every $\lambda > 0$ problem* (1.1) *has at least* $\mathrm{cat}_{\bar{\Omega}}(\bar{\Omega})$ *positive solutions. Moreover if Ω is not contractible in itself, the solutions are at least* $\mathrm{cat}_{\bar{\Omega}}(\bar{\Omega}) + 1$.

It is understood that $\bar{p}$ does not depend on the strength of the interaction λ.

The paper is organized as follow: in the next Section 3 we fix the notations, recall some useful facts and introduce the functional setting of the problem. Later, each theorem is proved in a separate section: see Sections 4, 5, 6, 7 and 8.

3. Some notations and preliminaries

Without loss of generality we assume in all the paper $0 \in \Omega$. We denote by $|\cdot|_{L^p(A)}$ the L^p-norm of a function defined on the domain A. If the domain is specified (usually Ω) or if there is no confusion, we use the notation $|\cdot|_p$. Moreover let $H_0^1(\Omega)$ be the usual Sobolev space whose (squared) norm is

$$\|u\|^2 = |\nabla u|_2^2 + |u|_2^2$$

and dual $H^{-1}(\Omega)$. The letter c will be used indiscriminately to denote a suitable positive constant whose value may change from line to line and we will use $o(1)$ for a quantity which goes to zero.

As we have already said, we approach problem (1.1) by variational methods. First of all, for fixed $u \in H_0^1(\Omega)$ let $\phi_u \in H_0^1(\Omega)$ be the unique (and positive) solution of

$$-\Delta\phi = u^2 \quad \text{in } \Omega \quad , \quad \phi = 0 \quad \text{on } \partial\Omega$$

(it is easily obtained with the Riesz Theorem) and let us recall that:

- for any $\alpha, \beta \geq 0, t > 0$ let $u_t(\cdot) := t^\alpha u(t^\beta(\cdot))$. Then
$$\phi_{u_t}(\cdot) = t^{2(\alpha-\beta)}\phi_u(t^\beta(\cdot)) ;$$
- $u_n \rightharpoonup u$ in $H_0^1(\Omega) \Longrightarrow \int_\Omega \phi_{u_n} u_n^2 \to \int_\Omega \phi_u u^2$;
- $|\nabla\phi_u|_2 \leq c|\nabla u|_2^2$ for some constant $c > 0$;
- $\int_\Omega |\nabla\phi_u|^2 = \int_\Omega \phi_u u^2$.

For a proof of these facts see e.g. [19].

It is standard to check that the weak solutions of (1.1) are characterized as the critical points of the C^1 functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + u^2) + \frac{\lambda}{4}\int_\Omega \phi_u u^2 - \frac{1}{p}\int_\Omega |u|^p \tag{3.1}$$

defined on the Sobolev space $H_0^1(\Omega)$; see e.g. [7].

Remark 1. Observe that the weak solutions found by means of the variational method are indeed classical solutions, by standard regularity results.

In the functional I_λ two kinds of nonlinearities appear: the first one is nonlocal, $\int_\Omega \phi_u u^2$, and concerns the interaction with the electric field. The second nonlinearity is $\int_\Omega |u|^p$ which is local. Physically speaking this represents the interaction among many particles and is in competition with the nonlinearity generated by ϕ_u, which actually couples the Schrödinger equation with the Poisson equation.

A fundamental tool to apply variational techniques is the so-called *Palais-Smale condition* (PS for brevity). It is said that a functional I on a manifold M satisfies the PS condition, if every sequence $\{u_n\}$ such that

$$\{I(u_n)\} \text{ is bounded} \quad \text{and} \quad I'(u_n) \to 0\ , \tag{3.2}$$

admits a converging subsequence. Sequences which satisfy (3.2) are called *Palais-Smale sequences.* This condition will be fundamental in order to prove our theorems. Indeed in the cases in which it holds the proofs are simpler and stronger results are obtained, in comparison with Theorem 2.3 where the PS condition is not known to hold.

4. Proof of Theorem 2.1

This proof is taken from [20]. Take a sequence $\lambda_n \to +\infty$ and (u_n, ϕ_n) solutions of (1.1) for $\lambda = \lambda_n$. Then

$$\|u_n\|^2 + \lambda_n \int_\Omega \phi_n u_n^2 = \int_\Omega |u_n|^p\ .$$

On the other hand, multiplying the second equation in (1.1) by $|u_n| \in H_0^1(\Omega)$ and integrating we get

$$\int_\Omega |u_n|^3 = \int_\Omega \nabla\phi_n \nabla|u_n| \le \frac{1}{2}\int_\Omega |\nabla u_n|^2 + \frac{1}{2}\int_\Omega |\nabla \phi_n|^2 \le \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\int_\Omega |\nabla\phi_n|^2\ .$$

We combine both previous expressions:

$$\frac{1}{2}\|u_n\|^2 + \left(\lambda_n - \frac{1}{2}\right)\int_\Omega \phi_n u_n^2 + \int_\Omega |u_n|^3\,dx \le \int_\Omega |u_n|^p\ .$$

In particular, for $\lambda_n \ge 1/2$, we have

$$\frac{1}{2}\|u_n\|^2 \le \int_\Omega |u_n|^p - \int_\Omega |u_n|^3\ . \tag{4.1}$$

By (4.1) we conclude, since $p \le 3$, that $\{u_n\}$ must be bounded as $\lambda_n \to +\infty$. So we can assume that $u_n \rightharpoonup u$. Standard regularity results imply that ϕ_n converges strongly to a limit ϕ, that satisfies $-\Delta\phi = u^2$. Now we distinguish two cases:

Case a) $u \ne 0$.

Clearly, $I'_{\lambda_n}(u_n)[u] = 0$, that is:

$$\int_\Omega \nabla u_n \nabla u + \int_\Omega u_n u + \lambda_n \int_\Omega \phi_n u_n u = \int_\Omega |u_n|^{p-2} u_n u \ .$$

As $\lambda_n \to +\infty$, we get a contradiction since

$$\int_\Omega \nabla u_n \nabla u + \int_\Omega u_n u \to \int_\Omega |\nabla u|^2 + \int_\Omega u^2 \quad , \quad \int_\Omega \phi_n u_n u \to \int_\Omega \phi u^2 > 0 \ ,$$

$$\int_\Omega |u_n|^{p-2} u_n u \to \int_\Omega |u|^p \ .$$

Case b) $u = 0$.

By using the compactness of the embedding $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$, we obtain

$$\|u_n\|^2 \le \|u_n\|^2 + \lambda_n \int_\Omega \phi_n u_n^2 = \int_\Omega |u_n|^p \to 0 \ .$$

Therefore, $u_n \to 0$ strongly. Then

$$0 = \|u_n\|^2 + \lambda_n \int_\Omega \phi_n u_n^2 - \int_\Omega |u_n|^p \ge \|u_n\|^2 - \int_\Omega |u_n|^p \ .$$

We now use Sobolev embedding to conclude: $0 \ge \|u_n\|^2 - c\|u_n\|^p$. Since $\|u_n\| \to 0$, this readily implies that $\|u_n\| = 0$ for all $n \ge n_0$. This finishes the proof.

5. Proof of Theorem 2.2

We present first a useful lemma.

Lemma 5.1. *Assume $p \in (2, 3)$ and $\lambda > 0$. Then the functional I_λ given in* (3.1) *satisfies the following properties:*

(1) *I_λ is weakly lower semi-continuous and coercive.*
(2) *I_λ satisfies the Palais-Smale condition.*

Proof. Take $u_n \rightharpoonup u$; the compactness of the Sobolev embedding show that

$$\int_\Omega |u_n|^p \to \int_\Omega |u|^p \quad , \quad \int_\Omega \phi_{u_n} u_n^2 \to \int_\Omega \phi_u u^2 \ .$$

Then, it follows that I_λ is w.l.s.c.

Let us now show that I_λ is coercive. By multiplying the second equation of (1.1) by $|u|$ we get:

$$\frac{\sqrt{\lambda}}{2} \int_\Omega |u|^3 \le \frac{1}{4} \int_\Omega |\nabla u|^2 + \frac{\lambda}{4} \int_\Omega |\nabla \phi|^2 \ ,$$

and hence

$$I_\lambda(u) \ge \frac{1}{4} \|u_n\|^2 + \int_\Omega \left(\frac{\sqrt{\lambda}}{2} |u|^3 - \frac{1}{p} |u|^p \right) \ge \frac{1}{4} \|u\|^2 - c \ .$$

To prove (2), take a Palais-Smale sequence $\{u_n\} \subset H^1_0(\Omega)$, hence

$$I_\lambda(u_n) \text{ is bounded and } I'_\lambda(u_n) \to 0 \ .$$

Let us write $\phi_n = \phi_{u_n}$. We show that $\{u_n\}$ admits a converging subsequence. First, since $\{I_\lambda(u_n)\}$ is bounded, this implies that $\{u_n\}$ is also bounded (by coercivity).

Hence, up to subsequence, $u_n \rightharpoonup u$; by the compactness of the Sobolev embedding, we have that $u_n \to u$ strongly in L^q for any $q \in (2,6)$. We evaluate:

$$I'_\lambda(u_n)[u_n - u] = \int_\Omega \nabla u_n \nabla(u_n - u) + \int_\Omega u_n(u_n - u) +$$

$$+\lambda \int_\Omega \phi_n u_n(u_n - u) - \int_\Omega |u_n|^{p-2} u_n(u_n - u) \ .$$

By using Holder inequality, we have:

$$\int_\Omega \phi_n u_n(u_n - u) \leq |\phi_n|_3 |u_n|_3 |u_n - u|_3$$

and the above expression tends to zero as $n \to +\infty$. Moreover,

$$\int_\Omega |u_n|^{p-2} u_n(u_n - u) \to 0 \ .$$

Then we conclude that $u_n \to u$ strongly in $H^1_0(\Omega)$.

□

Now we can give the proof of Theorem 2.2

By Lemma 5.1, I_λ is bounded from below and attains its minimum. Moreover, since I_0 is unbounded from below, we can choose λ small such that $\min I_\lambda < 0$; then, the minimizer is different from zero and corresponds to a nontrivial solution of (1.1). Observe also that 0 is a local minimum of the functional I_λ, and I_λ satisfies the PS condition. Then the Mountain Pass Theorem applies and I_λ admits another critical point at a positive level.

In order to obtain the positivity of the solutions, it suffices to observe that we can apply the previous procedure to the functional

$$I^+_\lambda(u) = \frac{1}{2}\ \|u\|^2 + \frac{\lambda}{4}\ \int_\Omega \phi_u u^2 - \frac{1}{p}\ \int_\Omega (u^+)^p \ .$$

In this way we obtain two different nontrivial critical points of I^+_λ. Thanks to the maximum principle, those solutions are positive and hence they solve (1.1).

6. Proof of Theorem 2.3

The main difficulty here is that we do not know whether the PS condition holds or not. Hence, we cannot simply apply classical variational arguments and we will use an argument which dates back to Struwe [22] (see also Jeanjean [13]). Basically, we introduce a parameter μ in our problem and solve it for almost all values of this parameter.

Consider a positive real parameter μ and define the functional:

$$I_{\lambda,\mu}(u) = \frac{1}{2}\ \|u\|^2 + \frac{\lambda}{4}\ \int_\Omega \phi u^2 - \frac{\mu}{p}\ \int_\Omega |u|^p \ . \tag{6.1}$$

It is not difficult to show that for every $\lambda, \mu > 0$ the functional $I_{\lambda,\mu}$ is unbounded from below (see [20, Proposition 3.1]). We argue as follows. Let us fix $k \in \mathbb{N}$, $J_k = [1/k, k]$, take $\mu \in J_k$ a real parameter, and $I_{\lambda,\mu}$ as given in (6.1). Take $B \subset \Omega$ a ball centered at zero (recall we are supposing that $0 \in \Omega$). Take $E_1 \subset E_2 \subset \cdots \subset H^1_0(B)$ a nested sequence of subspaces, $dim\ E_j = j$, and consider $H^1_0(B) \subset H^1_0(\Omega)$ via the trivial extension. Define B_j the unit ball in E_j, and S_j its boundary. Unless otherwise stated, in what follows we consider j fixed and $\lambda = 1$.

For any $u \in S_j$ and $t \geq 1$, we take $u_t(x) = t^2 u(tx)$. Define $T > 1$ such that

$$I_{1,1/k}(u_t) < 0 \qquad \forall\, t \geq T,\ \forall\, u \in S_j\ . \tag{6.2}$$

The choice of such T is always possible, see [20, proof of Proposition 3.1]. Define $h : B_j \to H_0^1(\Omega)$, $h(u) = u_{s(\|u\|)}$, where:

$$s(r) = \begin{cases} 1 & \text{if } r \leq 1/T\ , \\ Tr & \text{if } r \in (1/T, 1]\ . \end{cases}$$

We now define the min-max scheme by following [2, Theorem 2.11]. Let

$$\tilde{\Gamma}_j = \{g \in C(B_j, H_0^1(\Omega))\ :\ g \text{ is odd, injective and } I_{1,1/k}(g(S_j)) \subset (-\infty, 0]\}\ .$$

Clearly, $\tilde{\Gamma}_j$ is not empty since h belongs to it. We now define the values:

$$\tilde{b}_{j,\mu} = \inf_{g \in \tilde{\Gamma}_j} \max_{u \in B_j} I_{1,\mu}(g(u)) \quad , \quad \mu \in J_k\ .$$

Observe that $\tilde{b}_{j,\mu} \geq \delta > 0$ since $g(B_j)$ always intersects a small sphere around zero, and 0 is a local minimum of $I_{1,\mu}$.

By comparing our situation with that [2, Theorem 2.11], we see that condition (I_5) here does not hold, but such condition is used in [2] only to show that $\tilde{\Gamma}_j$ is not empty. Moreover, condition (I_3) (the Palais-Smale condition) is not known in our case: we will overcome this difficulty again by using the monotonicity trick of Struwe. Specifically, we are under the conditions of [3, Proposition 2.3]. Therefore, there exists $A_{k,j} \subset J_k$ such that:

(1) The set $J_k \setminus A_{k,j}$ has Lebesgue measure equal to zero.
(2) For any $\mu \in A_{k,j}$, there exists a bounded sequence $\{u_n\}$ such that:

$$I_{1,\mu}(u_n) \to \tilde{b}_{j,\mu} \quad , \quad I'_{1,\mu}(u_n) \to 0\ .$$

Now it is possible to pass to the limit of in n and find a critical point at level $\tilde{b}_{j,\mu}$ for any $\mu \in A_{k,j}$.

We now claim that $\tilde{b}_{j,\mu} \to +\infty$ as $j \to +\infty$. Indeed, $I_{0,k} \leq I_{1,\mu}$, and if we define:

$$\hat{\Gamma}_j = \{g \in C(B_j, H_0^1(\Omega))\ :\ g \text{ is odd, injective and } I_{0,k}(g(S_j)) \subset (-\infty, 0]\}\ ,$$

we have that $\tilde{\Gamma}_j \subset \hat{\Gamma}_j$. Therefore,

$$\hat{b}_j = \inf_{g \in \hat{\Gamma}_j} \max_{u \in B_j} I_{0,k}(g(u)) \leq \tilde{b}_{j,\mu}\ .$$

Since $\hat{b}_j \to +\infty$ (see [2], in particular Remark 3.19), we conclude the claim.

Therefore, for any $\mu \in A_k = \cap_{j=1}^{\infty} A_{k,j}$, there exist infinitely many different critical points for $I_{1,\mu}$. If we now take $A = \cup_{k=2}^{\infty} A_k$, we have that $(0,\infty) \setminus A$ has zero Lebesgue measure. Finally, given any $\mu \in A$ and any critical point u of $I_{1,\mu}$, the function $v = \mu^{1/(p-2)} u$ solves

$$\begin{cases} -\Delta v + v + \mu^{2/(2-p)} \phi v = |v|^{p-2} v & \text{in } \Omega\ , \\ -\Delta \phi = v^2 & \text{in } \Omega\ , \\ v = \phi = 0 & \text{on } \partial\Omega\ . \end{cases}$$

This concludes the proof.

□

Remark 2. Observe that the method used in the proof works in the whole range $p \in (3,6)$; however for $p \in (4,6)$ we have a better result with a simpler proof (see the next Section).

7. Proof of Theorem 2.4

To prove our Theorem the Symmetric Mountain Pass Theorem is used, which is strongly based on the notion of genus. Notice that the value of $\lambda > 0$ is insignificant in this case. Moreover we can take advantage now ($p \in (4,6)$) of the PS condition.

Lemma 7.1. *The functional I_λ satisfies the Palais-Smale condition.*

The proof of this fact is omitted, since a little bit technical. The interested reader can see [17, Theorem 2], where a more general case is treated.

Now we are going to study the geometrical properties of I_λ. Of course I_λ is even and $I_\lambda(0) = 0$. Furthermore we have

$$I_\lambda(u) \leq \frac{1}{2}\,\|u\|^2 + c\|u\|^4 - \frac{1}{p}\,\|u\|_p^p$$

hence $I_\lambda(u) \to -\infty$, as $\|u\| \to \infty$, on every finite dimensional subspace of $H_0^1(\Omega)$. It is also easy to see that the functional has a strict local minimum in 0 and then by the well known Symmetric Mountain Pass Theorem we deduce the existence of infinitely many solutions.

8. Proof of Theorem 2.5

Also in this case the role of $\lambda > 0$ is insignificant, so we will assume for simplicity $\lambda = 1$. On the other hand it is important to write explicitly the dependence on p: so we will write I_p for our functional (3.1) (with $\lambda = 1$).

The proof of Theorem 2.5 uses the LS category and is quite involved and technical. We give here the main ideas of the proof, which is, in turn based on some ideas and the barycenter method of [5]. All the details of the proofs can be found in [21].

First of all, since the functional is unbounded from above and from below on $H_0^1(\Omega)$ to use the LS Theory we restrict the functional to a suitable manifold.

We use $B_r(y)$ for the closed ball of radius $r > 0$ centered in y. If $y = 0$ we simply write B_r.

8.1. The Nehari manifold.

We recall here some known facts about the Nehari manifold that will be used throughout the paper.

The Nehari manifold associated to (3.1) is defined by

$$\mathcal{N}_p = \left\{ u \in H_0^1(\Omega) \setminus \{0\} \,:\, G_p(u) = 0 \right\}$$

where

$$G_p(u) := I_p'(u)[u] = \|u\|^2 + \int_\Omega \phi_u u^2 - |u|_p^p\,.$$

On $\mathcal{N}_p$ the functional (3.1) has the form

$$I_p(u) = \frac{p-2}{2p}\,\|u\|^2 + \frac{p-4}{4p}\,\int_\Omega \phi_u u^2\,. \tag{8.1}$$

Sometimes we will refer to (8.1) as the constraint functional, also denoted with $I_p|_{\mathcal{N}_p}$.

In the next Lemma we recall the basic properties of the Nehari manifold.

Lemma 8.1. *We have*

1. $\mathcal{N}_p$ *is a* C^1 *manifold,*
2. *there exists* $c > 0$ *such that for every* $u \in \mathcal{N}_p : c \leq \|u\|$,
3. *for every* $u \neq 0$ *there exists a unique* $t > 0$ *such that* $tu \in \mathcal{N}_p$,
4. *the following equalities are true*

$$m_p = \inf_{u \neq 0} \max_{t>0} I_p(tu) = \inf_{g \in \Gamma_p} \max_{t \in [0,1]} I_p(g(t))$$

where

$$\Gamma_p = \left\{ g \in C([0,1]; H_0^1(\Omega)) : g(0) = 0, \ I_p(g(1)) \leq 0, \ g(1) \neq 0 \right\} .$$

Then recalling that $p > 4$, we have

$$m_p := \inf_{u \in \mathcal{N}_p} I_p(u) > 0 .$$

Moreover the manifold $\mathcal{N}_p$ is a natural constraint for I_p (given by (3.1)) in the sense that any $u \in \mathcal{N}_p$ critical point of $I_p|_{\mathcal{N}_p}$ is also a critical point for the free functional I_p (for a proof of these facts, see e.g. Section 6.4 in [1]). Hence the (constraint) critical points we find are solutions of our problem since no Lagrange multipliers appear.

It is also clear that the Nehari manifold well-behaves with respect to the PS sequences:

Lemma 8.2. *Let* $\{u_n\} \subset \mathcal{N}_p$ *be a PS sequence for* $I_p|_{\mathcal{N}_p}$. *Then it is a PS sequence for the free functional* I_p *on* $H_0^1(\Omega)$.

As we have seen (Lemma (7.1)) for $p \in (4, 2^*)$ the free functional I_p given by (3.1) satisfies the PS condition on $H_0^1(\Omega)$ (see e.g. [17]). The fact that the PS condition follows also for the functional restricted to $\mathcal{N}_p$ is standard.

In the following we will deal always with the restricted functional on the Nehari manifold; this will be denoted simply with I_p.

As a consequence of the PS condition we deduce that

$$\forall\, p \in (4, 2^*) : m_p = \min_{\mathcal{N}_p} I_p = I_p(u_p) ,$$

i.e. m_p is achieved on a function, hereafter denoted with u_p, in $\mathcal{N}_p$. Since u_p minimizes the energy I_p, it will be called a *ground state*.

Observe that the sequence of minimizers $\{u_p\}_{p \in (4,2^*)}$ is bounded away from zero; indeed, since $u_p \in \mathcal{N}_p$,

$$\|u_p\|^2 \leq |u_p|_p^p \leq C \|u_p\|^p \tag{8.2}$$

where C is a positive constant which can be made independent of p. Hence

$$\exists\, c > 0 \quad \text{such that } \forall\, p \in (4, 2^*) : 0 < c \leq \|u_p\| .$$

Remark 3. Turning back to (8.2), we have that $\{|u_p|_p\}_{p \in (4,2^*)}$ is bounded away from zero. Moreover, denoting with $|\Omega|$ the Lebesgue measure of Ω, by the Hölder inequality,

$$|u_p|_p \leq |\Omega|^{(2^*-p)/2^*p} |u_p|_{2^*}$$

and so also $\{|u_p|_{2^*}\}_{p \in (4,2^*)}$ is bounded away from zero.

Clearly, all we have stated until now is true also in the case $\lambda = 0$. Moreover also the case $p = 2^*$ is covered for those results which do not require compactness (in particular Lemma 8.1 and 8.2).

8.2. **The limit cases.** We consider in this subsection two limit cases related to (1.1); the aim is to evaluate the limit of the sequence $\{m_p\}_{p\in(4,2^*)}$ when $p \to 2^*$.

The first case is the critical problem. Let us introduce the functional

$$I_*(u) = \frac{1}{2}\, \|u\|^2 - \frac{1}{2^*}\, |u|_{2^*}^{2^*}$$

whose critical points are the solutions of

$$\begin{cases} -\Delta u + u = |u|^{2^*-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{8.3}$$

It is known that the lack of compactness of the embedding of $H_0^1(\Omega)$ in $L^{2^*}(\Omega)$ implies that I_* does not satisfies the PS condition at every level. This is due to the invariance with respect to the conformal scaling

$$u(\,\cdot\,) \longmapsto u_R(\,\cdot\,) := R^{1/2}u(R(\,\cdot\,)) \qquad (R > 1)$$

which leaves invariant the L^2-norm of the gradient an the $L^{2^*}-$norm, i.e. $|\nabla u_R|_2^2 = |\nabla u|_2^2$ and $|u_R|_{2^*}^{2^*} = |u|_{2^*}^{2^*}$. As a consequence, if

$$\mathcal{N}_* = \{u \in H_0^1(\Omega) \,:\, G_*(u) = 0\} \quad , \quad G_*(u) = \|u\|^2 - |u|_{2^*}^{2^*}$$

is the Nehari manifold associated, it can be proved that $m_* := \inf_{\mathcal{N}_*} I_*$ is not achieved and $m_* = (1/3)S^{3/2}$, where $S = \inf_{u\in H_0^1(\Omega),u\neq 0} \|u\|^2/|u|_{2^*}^2$ is the best Sobolev constant.

The value m_* turns out to be an upper bound for the sequence of ground states levels $\{m_p\}_{p\in(4,2^*)}$. The proof of the next lemma is technical, and will be omitted. However it uses the identities

$$|u_R|_p^p = R^{(p-2^*)/2}|u|_p^p \quad \text{and} \quad \int_\Omega \phi_{u_R} u_R^2 = R^{-3} \int_\Omega \phi_u u^2\ .$$

Lemma 8.3. *We have* $\limsup_{p\to 2^*} m_p \leq m_*$.

Note that by (8.1), the boundedness of $\{m_p\}_{p\in(4,2^*)}$ implies the boundedness of the ground state solutions, namely

$$\exists\, c > 0 \quad \text{such that} \quad \forall\, p \in (4, 2^*) \,:\, \|u_p\| \leq c\ . \tag{8.4}$$

We need now another lemma.

Lemma 8.4. *Let* $p \in (4, 2^*)$ *and* $t_p > 0$ *the unique value such that* $t_p u_p \in \mathcal{N}_*$. *Then* $\{t_p\}$ *is bounded away from zero and* $\limsup_{p\to 2^*} t_p \leq 1$.

Remark 4. Again note that Proposition 8.3, (8.4) and Lemma 8.4 hold also for problem (1.1) with $\lambda = 0$.

The other limit case we consider is that related to problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega\ , \\ u = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{8.5}$$

For any $p \in (4, 2^*)$ let $\tilde{I}_p(u) = (1/2)\|u\|^2 - (1/p)|u|_p^p$ be the functional on $H_0^1(\Omega)$ whose critical points solve (8.5).

As usual, we can define $\tilde{\mathcal{N}_p} = \{u \in H_0^1(\Omega) \setminus \{0\} \,:\, \|u\|^2 = |u|_p^p\}$ on which the functional is $\tilde{I}_p(u) = [(p-2)/2p]\|u\|^2$ and we denote with

$$\tilde{m}_p := \min_{\tilde{\mathcal{N}_p}} \tilde{I}_p = \tilde{I}_p(\tilde{u}_p) \,.$$

By Remark 4 we have

$$\{\|\tilde{u}_p\|\}_{p\in(4,2^*)} \quad \text{is bounded} \,. \tag{8.6}$$

Moreover, if $t_p > 0$ is such that $t_p u_p \in \tilde{\mathcal{N}_p}$, by (8.2) we get $t_p^{p-2} = \|u_p\|^2/|u_p|_p^p \le 1$ and so

$$\tilde{m}_p \le \tilde{I}_p(t_p u_p) = \frac{p-2}{2p}\, t_p^2 \|u_p\|^2 \le \frac{p-2}{2p}\, \|u_p\|^2 < I_p(u_p) \,.$$

This means

$$\tilde{m}_p < m_p \,. \tag{8.7}$$

Now we are ready to compute the limit of m_p when p tends to 2^*.

Proposition 8.5. *For any bounded domain we have*

$$\lim_{p\to 2^*} m_p = m_* \,.$$

Proof. By (8.7) and Lemma 8.3 it is sufficient to prove that

$$m_* \le \liminf_{p\to 2^*} \tilde{m}_p \,.$$

Let $t_p > 0$ the unique value such that $t_p \tilde{u}_p \in \mathcal{N}_*$. Applying Lemma 8.4 (with $\lambda = 0$) we know $\limsup_{p\to 2^*} t_p \le 1$. Finally, using (8.6) we derive

$$\begin{aligned} m_* \le I_*(t_p\tilde{u}_p) &= \left(\frac{1}{2} - \frac{1}{2^*}\right) t_p^2 \|\tilde{u}_p\|^2 = \\ &= \tilde{I}_p(\tilde{u}_p) t_p^2 + \left(\frac{1}{p} - \frac{1}{2^*}\right) \|\tilde{u}_p\|^2 t_p^2 = \\ &= \tilde{m}_p\, t_p^2 + o(1) \end{aligned}$$

where $o(1) \to 0$ for $p \to 2^*$. Hence the conclusion follows.

□

8.3. **The barycenter map.** In this subsection we introduce the barycenter map that will allow us to compare the topology of Ω with the topology of suitable sublevels of I_p; precisely sublevels with energy near the minimum level m_p.

Before to proceed, some other notations are in order. For $u \in H^1(\mathbb{R}^3)$ with compact support, let us denote with the same symbol u its trivial extension out of supp u. The barycenter of u (see [5]) is defined as

$$\beta(u) = \frac{\displaystyle\int_{\mathbb{R}^3} x|\nabla u|^2}{\displaystyle\int_{\mathbb{R}^3} |\nabla u|^2} \in \mathbb{R}^3 \,.$$

From now on, we fix $r > 0$ a radius sufficiently small such that $B_r \subset \Omega$ and the sets

$$\Omega_r^+ = \{x \in \mathbb{R}^3 \,:\, d(x,\Omega) \le r\}$$

$$\Omega_r^- = \{x \in \Omega \,:\, d(x,\partial\Omega) \ge r\}$$

are homotopically equivalent to Ω. In particular we denote by

$$h : \Omega_r^+ \to \Omega_r^- \tag{8.8}$$

the homotopic equivalence map such that $h|_{\Omega_r^-}$ is the identity.

Let us introduce the space $D^{1,2}(\mathbb{R}^3) = \{u \in L^{2^*}(\mathbb{R}^3) : \nabla u \in L^2\}$ which can also be characterized as the closure of $C_0^\infty(\mathbb{R}^3)$ with respect to the (squared) norm

$$\|u\|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla u|^2 \ .$$

A function in $H_0^1(\Omega)$ can be thought as an element of $D^{1,2}(\mathbb{R}^3)$.

It will be useful the following "global compactness" result (see [23, Theorem 3.1]).

Theorem 8.6. *Let $\{v_n\}$ be a PS sequence for I_* in $H_0^1(\Omega)$. Then there exist a number $k \in \mathbb{N}_0$, sequences of points $\{x_n^j\} \subset \Omega$ and sequences of radii $\{R_n^j\}$ $(1 \le j \le k)$ with $R_n^j \to +\infty$ for $n \to +\infty$, there exist a positive solution $v \in H_0^1(\Omega)$ of (8.3) and non trivial solutions $v^j \in D^{1,2}(\mathbb{R}^3)$ $(1 \le j \le k)$ of*

$$-\Delta u = |u|^{2^*-2} \qquad \text{in } \mathbb{R}^3 \ , \tag{8.9}$$

such that, a (relabeled) subsequence $\{v_n\}$ satisfies

$$v_n - v - \sum_{j=1}^k v^j_{R_n}(\,\cdot\, - x_n^j) \to 0 \qquad \text{in } D^{1,2}(\mathbb{R}^3)$$

$$I_*(v_n) \to I_*(v) + \sum_{j=1}^k \hat{I}(v^j)$$

where $\hat{I} : H_0^1(\mathbb{R}^3) \to \mathbb{R}$ is given by

$$\hat{I}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 - \frac{1}{2^*}\int_{\mathbb{R}^3} |u|^{2^*} \ .$$

Basically the theorem states that if the PS condition fails, it is due to the solutions of (8.9). For what concerns $\hat{I}$, it is known that it achieves its minimum on functions of type

$$U_R(x-a) = \frac{(3R^2)^{1/4}}{(R^2 + |x-a|^2)^{1/2}} \qquad R > 0,\ a \in \mathbb{R}^3 \tag{8.10}$$

and the minimum value is exactly $\hat{I}(U_R(\,\cdot - a)) = (1/3)\int_{\mathbb{R}^3} |\nabla U|^2 = m_*$, namely the infimum of I^*. On the other hand, the value of $\hat{I}$ on solutions of (8.9) which do not belong to the family (8.10) is greater than $2m_*$. As a consequence, if the sequence $\{v_n\}$ of Theorem 8.6 is a PS sequence for I_* at level m_*, we deduce $I_*(v) = 0, k = 1$ and $v^1 = U$. Furthermore, since v is a solution of (8.3) and I_* is positive on the solutions, necessarily $v = 0$ and so Theorem 8.6 gives

$$v_n - U_{R_n}(\,\cdot\, - x_n) \to 0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^3) \ .$$

Now we have the fundamental

Proposition 8.7. *There exists $\varepsilon > 0$ such that if $p \in (2^* - \varepsilon 2^*)$, it follows*

$$u \in \mathcal{N}_p \quad \text{and} \quad I_p(u) < m_p + \varepsilon \implies \beta(u) \in \Omega_r^+ \ .$$

Proof. The proof is by contradiction. Assume that there exist sequences $\varepsilon_n \to 0$, $p_n \to 2^*$ and $u_n \in \mathcal{N}_{p_n}$ such that

$$I_{p_n}(u_n) \le m_{p_n} + \varepsilon_n \quad \text{and} \quad \beta(u_n) \notin \Omega_r^+ \ . \tag{8.11}$$

Then, by Proposition 8.5

$$I_{p_n}(u_n) \to m_* \tag{8.12}$$

and $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Let $t_n > 0$ such that $t_n u_n \in \mathcal{N}_*$. By Lemma 8.4 we may assume (up to subsequence) that $t_n \to t_0 \in (0, 1]$ and we evaluate

$$\begin{aligned}
I_{p_n}(u_n) - I_*(t_n u_n) &= \left(\frac{1}{2} - \frac{1}{p_n}\right) \|u_n\|^2 + \lambda\, \frac{p_n - 4}{4p_n} \int_\Omega \phi_{u_n} u_n^2\, dx - \\
&\quad - \left(\frac{1}{2} - \frac{1}{2^*}\right) t_n^2 \|u_n\|^2 \ge \\
&\ge \left(\frac{1}{2} - \frac{1}{p_n}\right) \|u_n\|^2 - \left(\frac{1}{2} - \frac{1}{2^*}\right) t_n^2 \|u_n\|^2 = \\
&= \left(\frac{1}{2} - \frac{1}{p_n}\right) \|u_n\|^2 \left(1 - t_n^2\right) - \left(\frac{1}{p_n} - \frac{1}{2^*}\right) t_n^2 \|u_n\|^2 = \\
&= o(1)
\end{aligned}$$

which gives

$$m_* \le I_*(t_n u_n) \le I_{p_n}(u_n) + o(1) \ .$$

By (8.12), $I_*(t_n u_n) \to m_*$ for $n \to +\infty$. The Ekeland's variational principle implies that there exist $\{v_n\} \subset \mathcal{N}_*$ and $\{\mu_n\} \subset \mathbb{R}$ such that

$$\begin{aligned}
&\|t_n u_n - v_n\| \to 0 \\
&I_*(v_n) = \frac{1}{3}\, \|v_n\|^2 \to m_* \\
&I_*'(v_n) - \mu_n G_*'(v_n) \to 0
\end{aligned}$$

and Lemma 8.2 (in the case $\lambda = 0$) ensures that $\{v_n\}$ is a PS sequence for the free functional I_* at level m_*. By the arguments after Theorem 8.6,

$$v_n - U_{R_n}(\,\cdot\, - x_n) \to 0 \qquad \text{in } D^{1,2}(\mathbb{R}^3)$$

where $\{x_n\} \subset \Omega, R_n \to +\infty$ and we can write

$$v_n = U_{R_n}(\,\cdot\, - x_n) + w_n$$

with a remainder w_n such that $\|w_n\|_{D^{1,2}(\mathbb{R}^3)} \to 0$. It is clear that $t_n u_n = v_n + t_n u_n - v_n$; so, renaming the remainder again w_n, we have

$$t_n u_n = U_{R_n}(\,\cdot\, - x_n) + w_n \ .$$

Now writing $x \in \mathbb{R}^3$ as $x = (x^1, x^2, x^3)$, the $i-$th coordinate of the barycenter of u_n satisfies

$$\beta(u_n)^i \|t_n u_n\|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} x^i |\nabla U_{R_n}(x - x_n)|^2 \, dx +$$

$$+ \int_{\mathbb{R}^3} x^i |\nabla w_n(x)|^2 \, dx + 2 \int_{\mathbb{R}^3} x^i \nabla U_{R_n}(x - x_n) \nabla w_n(x) \, dx \, .$$

The aim is to localize the sequence of barycenters, so we pass to the limit in the above expression evaluating $\|t_n u_n\|^2_{D^{1,2}(\mathbb{R}^3)}$ and the right hand side. After straightforward computations (see [21, Proposition 4.2]) we find

$$\beta(u_n)^i = \frac{x_n^i \displaystyle\int_{\mathbb{R}^3} |\nabla U(y)|^2 \, dy + o(1)}{\|U\|^2_{D^{1,2}(\mathbb{R}^3)} + o(1)} \, . \tag{8.13}$$

Since $\{x_n\} \subset \Omega$, (8.13) implies that definitively $\beta(u_n) \in \bar{\Omega}$ which is in contrast with (8.11) and proves the proposition.

□

8.4. **Concluding the proof.** Here we complete the proof of our theorem but first we need a slight modification to the previous notations. We add a subscript r ($r > 0$ and small as before) to denote the same quantities defined in the previous sections when the domain Ω is replaced by B_r; namely integrals are taken on B_r and norms are taken for functional spaces defined on B_r. Hence

$$\mathcal{N}_{p,r} = \left\{ u \in H_0^1(B_r) \, : \, \|u\|^2_{H_0^1(B_r)} + \int_{B_r} \phi_u u^2 = |u|^p_{L^p(B_r)} \right\}$$

and, for $u \in \mathcal{N}_{p,r}$

$$I_{p,r}(u) = \frac{p-2}{2p} \, \|u\|^2_{H_0^1(B_r)} + \frac{p-4}{4p} \int_{B_r} \phi_u u^2 \, ,$$

$$m_{p,r} = \min_{\mathcal{N}_{p,r}} I_{p,r} = I_{p,r}(u_{p,r}) \, .$$

Moreover let

$$I_p^{m_{p,r}} = \{ u \in \mathcal{N}_p \, : \, I_p(u) \le m_{p,r} \}$$

which is non empty since $m_p < m_{p,r}$.

Define also, for $p \in (4, 2^*)$ the map $\Psi_{p,r} : \Omega_r^- \to \mathcal{N}_p$ such that

$$\Psi_{p,r}(y)(x) = \begin{cases} u_{p,r}(|x-y|) & \text{if } x \in B_r(y) \\ 0 & \text{if } x \in \Omega \setminus B_r(y) \end{cases}$$

and note that we have

$$\beta(\Psi_{p,r}(y)) \in B_r(y) \quad \text{and} \quad \Psi_{r,p}(y) \in I_p^{m_{p,r}} \, .$$

Moreover, since $m_p + k_p = m_{p,r}$ where $k_p > 0$ and tends to zero if $p \to 2^*$ (see Proposition 8.5), in correspondence of $\varepsilon > 0$ provided by Proposition 8.7, there exists a $\bar{p} \in [4, 2^*)$ such that for every $p \in [\bar{p}, 2^*)$ it results $k_p < \varepsilon$; so if $u \in I_p^{m_{p,r}}$ we have

$$I_p(u) \le m_{p,r} < m_p + \varepsilon \, ,$$

at least for p near 2^*. Hence the following maps are well-defined:

$$\Omega_r^- \xrightarrow{\Psi_{p,r}} I_p^{m_{p,r}} \xrightarrow{h\circ\beta} \Omega_r^-$$

where h is given by (8.8). Since the composite map $h \circ \beta \circ \Psi_{p,r}$ is homotopic to the identity of Ω_r^-,

$$\mathrm{cat}_{I_p^{m_{p,r}}}(I_p^{m_{p,r}}) \geq \mathrm{cat}_{\Omega_r^-}(\Omega_r^-)$$

(see e.g. [12]) and our choice of r gives $\mathrm{cat}_{\Omega_r^-}(\Omega_r^-) = \mathrm{cat}_{\bar\Omega}(\bar\Omega)$. In conclusion, we have found a sublevel of I_p on $\mathcal{N}_p$ with category greater than $\mathrm{cat}_{\bar\Omega}(\bar\Omega)$. Since, as we have already said, the PS condition is verified on $\mathcal{N}_p$, applying the Ljusternik-Schnirelmann Theory we get the existence of at least $\mathrm{cat}_{\bar\Omega}(\bar\Omega)$ critical points for I_p on the manifold $\mathcal{N}_p$ which give rise to solutions of (1.1).

Whenever $\mathrm{cat}_{\bar\Omega}\bar\Omega > 1$ the existence of another solution is obtained with the same arguments of [6]. Since by hypothesis Ω is not contractible in itself, by the choice of r it results $\mathrm{cat}_{\Omega_r^+}(\Omega_r^-) > 1$, namely Ω_r^- is not contractible in Ω_r^+. This redly implies that $\Psi_{p,r}(\Omega_r^-)$ can not be contractible in $I_p^{m_{p,r}}$, but on the other hand is contractible in a bigger sublevel $I_p^{M_p}$ for a suitable $M_p > m_{p,r}$; see [21] for all the details. Since the PS condition is satisfied, this implies the existence of another critical point with critical level between $m_{p,r}$ and M_p.

It remains to prove that these solutions are positive. Note that we can apply all the previous machinery replacing the functional (3.1) with

$$I_p^+(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + u^2) + \frac{1}{4}\int_\Omega \phi_u u^2 - \frac{1}{p}\int_\Omega (u^+)^p$$

obtaining again at least $\mathrm{cat}_{\bar\Omega}(\bar\Omega)$ (or $\mathrm{cat}_{\bar\Omega}(\bar\Omega) + 1$) nontrivial solutions. Finally the maximum principle ensures that these solutions are positive, hence they solve (1.1).

References

[1] A. Ambrosetti & A. Malchiodi, *Nonlinear analysis and semilinear elliptic problems*, Cambridge Studies in Advanced Mathematics, 104, Cambridge University Press, 2007.

[2] A. Ambrosetti & P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., 14(1973), 349–381.

[3] A. Ambrosetti & D. Ruiz, *Multiple bound states for the Schrödinger-Poisson problem*, Comm. Contemp. Math., 10(2008), 391–404.

[4] A. Azzollini & A. Pomponio, *Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl., 345(1)(2008), 90–108.

[5] V. Benci & G. Cerami, *The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems*, Arch. Rational Mech. Anal., 114(1991), 79–93.

[6] V. Benci, G. Cerami & D. Passaseo, *On the number of positive solutions of some nonlinear elliptic problem*, Nonlinear Analysis, 93–107, Sc. Norm. Super. di Pisa Quaderni, Scuola Norm. Sup., Pisa, 1991.

[7] V. Benci & D. Fortunato, *An eigenvalue problem for the Schrödinger-Maxwell equations*, Topol. Methods Nonlinear Anal., 11(1998), 283–293.

[8] G.M. Coclite, *A multiplicity result for the nonlinear Schrödinger-Maxwell equations*, Commun. Appl. Anal., 7(2003), 417–423.

[9] T. D'Aprile & D. Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. Royal Soc. Edinburgh, 134 A(2004), 893–906.

[10] T. D'Aprile & J. Wei, *On bound states concentrating on spheres for the Maxwell-Schrödinger equation*, SIAM J. Math. Anal., 37(2005), 321–342.

[11] P. d'Avenia, *Non-radially symmetric solutions of nonlinear Schrödinger equation coupled with Maxwell equations*, Adv. Nonl. Studies, 2, 2002, 177–192.

[12] I.M. James, *On category, in the sense of Lusternik-Schnirelmann*, Topology, 17(1978), 331–348.

[13] L. Jeanjean, *On the existence of bounded Palais-Smale sequences and applications to a Landesman-Lazer type problem set on $\mathbb{R}^N$*, Proc. Roy. Soc. Edinburgh, A 129(1999), 787–809.

[14] H. Kikuchi, *On the existence of solutions for a elliptic system related to the Maxwell-Schrödinger equations*, Nonl. Anal., 67(2007), 1445–1456.

[15] I. Ianni & G. Vaira, *On concentration of positive bound states for the Schrödinger-Poisson problem with potentials*, Adv. Nonlinear Stud., 8(2008), 573–595.

[16] L. Pisani & G. Siciliano, *Neumann condition in the Schrödinger-Maxwell system*, Topol. Methods Nonlinear Anal., 29(2007), 251–264.

[17] L. Pisani & G. Siciliano, *Note on a Schrödinger-Poisson system in a bounded domain*, Appl. Math. Lett., 21(2008), 521–528.

[18] D. Ruiz, *Semiclassical states for coupled Schrödinger-Maxwell equations: concentration around a sphere*, Math. Models Methods Appl. Sci., 15(2005), 141–164.

[19] D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*, J. Funct. Anal., 237(2006), 655–674.

[20] D. Ruiz and G. Siciliano, *A note on the Schrödinger-Poisson-Slater equation on bounded domains*, Adv. Nonlinear Stud., 8(2008), 179–190.

[21] G. Siciliano, *Multiple positive solutions for a Schrdinger-Poisson-Slater system*, J. Math. Anal. Appl., 365(1)(2010), 288–299.

[22] M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helvetici, 60(1985), 558–581.

[23] M. Struwe, *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems*, Springer, Berlin, 1996.

[24] Z. Wang & H.S. Zhou, *Positive solution for a nonlinear stationary Schrödinger-Poisson system in $\mathbb{R}^3$*, Discrete Contin. Dyn. Syst., 18(2007), 809–816.