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Factors of edge-chromatic critical graphs: a brief survey and some equivalences

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Abstract. This article discusses the main results that have been derived till date on some conjectures on factors of critical graphs. It provides some equivalent formulations of Vizing's 2-factor conjecture, and states some problems on factors of critical graphs.

1. Introduction

The conjectures and results to be discussed in this article are derived on simple graphs. However, in some parts of the article we will need multigraphs in intermediate steps in the construction of graphs with specific properties. If we allow multiedges, then we explicitly will use the term multigraph.

If G is a multigraph, then $V(G)$ denotes its vertex set and $E(G)$ denotes its edge set. If v is a vertex of G, then $E_G(v)$ denotes the set of edges which are incident to v and $N_G(v)$ denotes the set of its neighbors. The *degree* of v is $|E_G(v)|$ and it is denoted by $d_G(v)$. The maximum degree and the minimum degree of a vertex of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. If it clear from the context, we will also use Δ and δ instead of $\Delta(G)$ and $\delta(G)$, respectively. Clearly, if G is a simple graph, then $|E_G(v)| = |N_G(v)|$. A multigraph is *regular* and r-regular if all vertices have degree r, where $r \geq 0$ is an integer. A multigraph has a *k-factor* if it has a spanning k-regular submultigraph.

A *k-edge-coloring* of G is a function $\phi : E(G) \longrightarrow \{1, \cdots, k\}$ such that $\phi(e) \neq \phi(f)$ for adjacent edges e and f. Since we do not consider any other kind of colorings we will use the term k-coloring instead of k-edge-coloring in the following. The *chromatic index* $\chi'(G)$ is the smallest number k such that there is k-coloring of G.

In 1965, Vizing proved the fundamental result on the chromatic index of simple graphs.

Theorem 1.1 ([25]). *If G is a graph, then $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$.*

Theorem 1.1 classifies the set of all simple graphs into two classes namely, into class 1 and class 2 graphs depending upon whether their edge chromatic number is

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Δ and $\Delta + 1$, respectively. There is no known characterization for class 2 graphs. Indeed, the decision problem whether a graph is a class 2 graph is an NP-complete problem even for 3-regular graphs [17]. However, it is easy to decide whether the graph has too many edges to be in class 1. A multigraph G is *overfull* if $|E(G)|/\lfloor |V(G)|/2 \rfloor > \Delta$. Clearly, overfull multigraphs always have odd order and they are class 2. A class 2 multigraph G is *critical*, if $\chi'(H) < \chi'(G)$ for every proper subgraph H of G. A critical multigraph G with maximum degree $\Delta(G)$ is often referred to as a $\Delta(G)$-critical graph or Δ-critical graph if there is no confusion regarding the context of G.

We stress the fact that all multigraphs which are studied in this article have chromatic index at most $\Delta + 1$.

It is easy to see that critical graphs are 2-connected and the only regular critical graphs are the odd circuits. Indeed, these are the 2-critical graphs.

Theorem 1.2 ([24])**.** *If G is a class 2 graph, then G contains a Δ-critical subgraph for each $k \in \{2, \cdots, \Delta\}$.*

Beineke and Fiorini [1] determined all critical graphs with order at most 7, and they showed that there is no critical graph of even order with less than 10 vertices and no 3-critical graph with 12 vertices. Jakobsen [18] determined all 3-critical graphs up to 9 vertices. Brinkmann and Steffen [4] showed that the smallest example of a non-overfull critical graph is the Petersen graph minus a vertex, and there are precisely two non overfull graphs of order 11 and none of order 12. These results were further extended to graphs up to 14 vertices in [3].

These early results on small critical graphs might had been the motivation for the following conjecture of Beineke and Wilson [2] and by Jakobsen [18]. This conjecture was known as the *Critical Graph Conjecture*, see e.g. [9].

Conjecture 1.3 ([2, 18])**.** *For all $\Delta \geq 2$: Every Δ-critical graph has odd order.*

Conjecture 1.3 is certainly true for $\Delta = 2$. Goldberg disproved it for $\Delta = 3$, and Chetwynd and Fiol independently disproved it for $\Delta = 4$, see e.g. [7]. Goldberg [12] constructed an infinite family of 3-critical graphs of even order. The smallest graph of this family has 22 vertices. Later, Grünewald and Steffen [15] constructed Δ-critical graphs of even order for each $\Delta \geq 3$. Grünewald [14] gave further constructions of small critical graphs of even order, which we will use in Section 3. To understand more about the actual reasons of a Δ-critical graph G to belong to class 2, other than that of G being overfull the structures of critical graphs must be studied in further detail. With this motivation, the following conjectures were formulated.

Conjecture 1.4 ([11])**.** *For all $\Delta \geq 2$,*

1. *every Δ-critical graph of even order has a 1-factor.*
2. *Let G be a Δ-critical graph of odd order and v be a vertex of minimum degree in G. Then $G - v$ has a 1-factor.*

Conjecture 1.5 ([23])**.** *Every critical graph has a 2-factor.*

The first part of Conjecture 1.4 was proposed by Fiorini and Wilson, see [11]. The second part was stated by Chetwynd and Yap [8]. Conjecture 1.5 is due to Vizing and it is known as *Vizing's 2-factor conjecture.*

This survey is devoted to these conjectures and similar topics. In Section 2, we discuss some results and constructions of graphs which were used to disprove both statements of Conjecture 1.4. Some of these results are used to give some equivalent formulations of Conjecture 1.5 in Section 3. This is a new result in addition to the known results on the above mentioned conjectures. For instance, it might seem intuitive that verifying Vizing's 2-factor conjecture for graphs of even order could be easier than proving the conjecture in general. However in contrary to the intuition, it turns out that proving the conjecture for this particular case is as difficult as proving the conjecture in general. In total, we give seven statements which are equivalent to Vizing's 2-factor conjecture. Clearly, a 3-critical graph of even order cannot have an even 2-factor. In particular, they are not Hamiltonian. The situation changes for $\Delta \geq 4$. We show that for each $\Delta \geq 4$ there are Δ-critical graphs of even order which are Hamiltonian. In the last section we discuss some other kinds of factors of critical graphs and we state some questions.

2. 1-factors

This section gives some constructions of critical graphs, which we will use in Section 3 as well.

Let H be a multigraph which is obtained from two multigraphs G and G' as follows:

1. choose vertices $v \in V(G)$ and $v' \in V(G')$,
2. remove edges vw and $v'w'$ from G and G', respectively,
3. identify v and v',
4. add the edge ww'.

If $\Delta(G) = \Delta(G')$, and $d_G(v) + d_{G'}(v') \leq \Delta + 2$, then H is called the *Hajós union* of G and G'. It is well known that the Hajós-union of two Δ-critical graphs is Δ-critical whenever the sum of the degrees of the two vertices used in the construction is smaller or equal to $\Delta + 2$, cf. [26], Theorem 4.6. The corresponding statement for multigraphs appeared first in [15].

Theorem 2.1. *Let $\Delta \geq 2$. If H is the Hajós union of two Δ-critical multigraphs, then H is Δ-critical.*

For a vertex v of a graph G let $s(v) = \Delta(G) - d_G(v)$ and let $s(G) = \sum_{v \in V(G)} s(v)$; $s(v)$ is the *deficiency of* v, and $s(G)$ is the *deficiency of* G.

Lemma 2.2 ([15])**.** *Let $\Delta \geq 2$, let M be a Δ-critical multigraph and let N be an induced proper submultigraph of odd order. If $s(N) = \Delta$, then the multigraph M^- obtained from M by contracting N to a single vertex is Δ-critical.*

Next, we describe two constructions which allow us to construct simple Δ-critical graphs from Δ-critical multigraphs. Let M be a Δ-critical multigraph, let $v \in V(M)$ with $d_M(v) = d$, and let $v_1, v_2, \cdots, v_d$ be the not necessarily distinct neighbors of v. Let $u_1, \cdots, u_\Delta$ be vertices of degree $\Delta - 1$ in a complete bipartite graph $K_{\Delta,\Delta-1}$. Multigraph M is a *Type I extension* (also known as *Meredith extension* [20]) of M (applied on v), if it is obtained from $M - v$ and $K_{\Delta,\Delta-1}$ by adding edges $v_i u_i$ for each $i \in \{1, \cdots, d\}$.

Let $d(v) = k$ for some vertex v of a Δ-critical graph M, and let $w_1, \cdots, w_k$ be the vertices of the complete graph K_k. Multigraph M'' is a *Type II extension* of M (applied on v) if it is obtained from $M - v$ and K_k by adding edges $v_i w_i$ for each $i \in \{1, \cdots, k\}$.

Theorem 2.3 ([15]). *Let $\Delta \geq 2$ and let M be a multigraph. If M' is a Type I extension of M and M'' is a Type II extension of M applied on a vertex of degree Δ, then*

1. *M is Δ-critical if and only if M' is Δ-critical,*
2. *furthermore, if Δ is odd, then M is Δ-critical if and only M'' is Δ-critical.*

Lemma 2.4 ([26] (Yap's Construction)). *For $\Delta \geq 4$ let G be a Δ-critical graph with two divalent vertices v and w. Let G' be a copy of G with corresponding vertices v' and w'. Let H' be obtained by performing Hajós union on G and G' where v and v' are identified. Then the graph H formed from H' by identifying w and w' is Δ-critical.*

The following result of Choudum was the door opener for the construction of counterexamples to Conjecture 1.4.

Theorem 2.5 ([10]). *Let $\Delta \geq 3$. If there is a Δ-critical graph of even order with a vertex of degree two, then*

1. *there is a Δ-critical graph of even order without a 1-factor and,*
2. *for every $n \in \{2, \cdots, \Delta - 1\}$ there is a Δ-critical graph G of odd order that has a vertex v with $n = d_G(v) = \delta(G)$ for which $G - v$ has no 1-factor.*

Since Goldberg's counterexamples to Conjecture 1.3 are 3-critical graphs of even order that have four divalent vertices, both statements of Conjecture 1.4 are false for $\Delta = 3$. We will describe the construction of counterexamples to Conjecture 1.3 for every $\Delta \geq 3$.

Lemma 2.6 ([15]). *For $k \geq 0$, let M be a multigraph obtained from the Petersen graph P by adding k 1-factors of P. Then $M - v$ is a $(k + 3)$-critical multigraph for any $v \in V(M)$.*

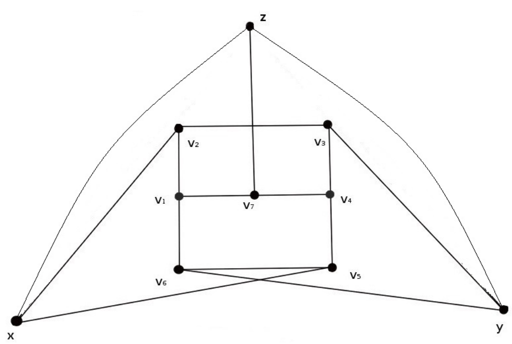

FIGURE 1. Petersen graph.

The following lemma is a direct consequence of Theorem 3.2 in [15]. Since we will use it for proving some new statements in Section 3, we add the proof for the sake of self-containment.

Lemma 2.7 ([15]). *For every $\Delta \geq 4$ there is a simple Δ-critical graph of even order with at least two divalent vertices.*

Proof. Let the vertices and edges of the Petersen graph P be labeled as in Figure 1. Then $F_1 = \{v_2v_3, v_1v_6, v_4v_7, v_5x, zy\}$ and $F_2 = \{v_2v_3, v_1v_7, v_4v_5, v_6y, zx\}$ are two 1-factors of P. For $\Delta \geq 3$ and $j \in \{1, \cdots, \Delta-2\}$, let P^j_Δ be the multigraph obtained from P by adding $j-1$ copies of F_1 and $\Delta - j - 2$ copies of F_2. Let $P^j_\Delta - z = M^j_\Delta$. From Lemma 2.6, it follows that M^j_Δ is a Δ-critical multigraph. Furthermore, $d_{M^j_\Delta}(v_7) = \Delta - 1, d_{M^j_\Delta}(x) = j+1, d_{M^j_\Delta}(y) = \Delta - j$, and all other vertices of M^j_Δ have degree Δ.

We now apply Hajós union on the multigraphs M^j_Δ for fixed $\Delta \geq 3$. To avoid confusion, we indicate the involved vertices of M^j_Δ by an upper index j. By definition, $d_{M^j_\Delta}(y^j) + d_{M^{j+1}_\Delta}(x^{j+1}) = \Delta + 2$. Identify y^{j+1} and x^j to a new vertex, which will be labeled y^j in the new multigraph, and all other labellings remain unchanged. Substitute $v^j_3v^{j+1}_2$ for $y^jv^j_3$ and $x^{j+1}v^{j+1}_2$ and call this multigraph M. Clearly, M is the Hajós union of M^j_Δ and M^{j+1}_Δ, and we write $M = M^j_\Delta +_{(y^jx^{j+1})} M^{j+1}_\Delta$. Define, $M_\Delta = (\cdots(M^1_\Delta +_{(y^1x^2)} M^2_\Delta) +_{(y^2x^3)} M^3_\Delta)\cdots +_{(y^{\Delta-3}x^{\Delta-2})} M^{\Delta-2}_\Delta)$.

For $\Delta \geq 4$, the multigraph M_Δ is the Hajós union of Δ-critical multigraphs and hence it is Δ-critical. It has two divalent vertices, namely x^1 and $y^{\Delta-2}$. Appropriate applications of Type I extensions on M_Δ yield a simple Δ-critical graph with two divalent vertices.

□

Hence, Theorem 2.5, Lemma 2.7 and the Goldberg graphs allow to disprove Conjecture 1.4 for all $\Delta \geq 3$.

Theorem 2.8 ([15]). *For every $\Delta \geq 3$,*

1. *there is a Δ-critical graph of even order without a 1-factor and,*
2. *for every $n \in \{2, \cdots, \Delta-1\}$ there is a Δ-critical graph G of odd order that has a vertex v with $n = d_G(v) = \delta(G)$ for which $G - v$ has no 1-factor.*

We close this section with some remarks on k-factors of critical graphs.

Lemma 2.9. *Let $\Delta \geq 2$, M' be a Type I extension of a multigraph M and $k \leq \delta(G) \leq \Delta(M) = \Delta$. Then M has a k-factor if and only if M' has a k-factor.*

Proof. Assume that the Type I extension is applied on v where $v_1, \cdots, v_d$ are the not necessarily distinct neighbors of v. Let A and B be the two partition sets of $K_{\Delta,\Delta-1}$ where $A = \{a_1, \cdots, a_\Delta\}$ is the set of vertices of degree $\Delta - 1$ and $B = \{b_1, \cdots, b_{\Delta-1}\}$ is the set of vertices of degree Δ. For $i \in \{1, \cdots d\}$ let $e_i = v_ia_i$ be the edges which are added to $M - v$ and $K_{\Delta,\Delta-1}$ to obtain M'. Since $\delta(G) \geq k$, it follows that $d \geq k$.

Let F be the edge set of a k-factor of M. Let $vv_{i_j} \in F$ for each $j \in \{1, \cdots, k\}$. Let c be a Δ-coloring of $K_{\Delta,\Delta-1}$. Then precisely one color is missing at every vertex of A and the missing colors are pairwise different. Hence, we may assume that color i_j is missing at a_{i_j}. Then $(F \cap E(M-v)) \cup \{e_{i_1} \cdots e_{i_k}\} \cup \bigcup_{j=1}^k c^{-1}(i_j)$ is the edge set of a k-factor of M'.

Let F' be the edge set of a k-factor of M'. Since each $b \in B$ is incident to k edges of F', it follows that precisely $k(\Delta - 1)$ edges of $K_{\Delta,\Delta-1}$ belong to F. Hence, $|F' \cap \{e_1, \cdots, e_d\}| = k$. Therefore, the contraction of $K_{\Delta,\Delta-1}$ to a single vertex v yields M and a k-factor of M.

□

If M' is obtained from M by a Type I extension applied on a vertex v, then M and M' have the same deficiency, since v is replaced by the subgraph $K_{\Delta(M),\Delta(M)-1}$ which contains $s(v)$ vertices of degree $\Delta(M) - 1$. Hence, we easily deduce the following corollary.

Corollary 2.10. *For every $\Delta \geq 3$, there is a Δ-critical graph of even order and $\delta = \Delta - 1$ without a 1-factor.*

3. 2-factors

Conjecture 1.5 has been verified for some graphs. Grünewald and Steffen proved it for graphs with many edges. Major tools for the proofs are the parity lemma and Vizing's Adjacency Lemma (VAL).

Lemma 3.1 (Parity Lemma)**.** *Let G be a (multi-)graph whose edges are colored with colors $1, \cdots, k$, and let a_i be the number of vertices where color i is missing. Then $a_i \equiv |V(G)| \mod 2$.*

Lemma 3.2 (VAL [24])**.** *Let G be a Δ-critical graph and $vw \in E(G)$. If $d_G(v) = k$, then at least $\Delta - k + 1$ vertices of $N(w) \setminus \{v\}$ have degree Δ.*

The following statement is a simple consequence of Lemma 3.2.

Lemma 3.3. *If G is a Δ-critical graph and $vw \in E(G)$, then $d(v) + d(w) \geq \Delta + 2$.*

Theorem 3.4 ([16])**.** *Let $\Delta \geq 3$ and G be a Δ-critical graph and $k = \min\{d_G(v) + d_G(w) : vw \in E(G)\}$*

1. *If $|V(G)|$ is odd and $s(G) \leq 5\Delta - 2k - 1$, then G has a 2-factor.*
2. *If $|V(G)|$ is even and $s(G) \leq 2\Delta - 6$, then G has a 2-factor.*

Proof. Since there is a typo in statement 1. in the original article, we will give a short proof. Let $e = vw$ be an edge of G such that $d_G(v) + d_G(w) = k$, and ϕ be a Δ-coloring of $G - e$. Let C_1 be the set of colors present at only one of v and w, and let C_2 be the set of colors which are present at both. We may assume that $C_1 = \{1, \cdots, 2\Delta - k + 2\}$ and $C_2 = \{2\Delta - k + 3, \cdots, \Delta\}$, where $C_2 = \emptyset$ if $k = \Delta + 2$. For $i \in \{1, \cdots, \Delta\}$, let a_i be the number of vertices where color i is missing. Note that a_i is odd.

a) There is $l \in C_1$ such that $a_l = \min\{a_i : i \in \{1, \cdots, \Delta\}\}$.

If there is $m \in C_1$ such that $m \neq l$ and $a_m = 1$, then we can assume that m is missing at v and l is missing at w. Then $\phi^{-1}(l) \cup \phi^{-1}(m) \cup \{e\}$ is the edge set of a 2-factor of G.

If $a_m > 1$ for all $m \neq l$, then $s(G) + 2 = \sum_{i=1}^{\Delta} a_i \geq 3(|C_1| - 1) + 1 + |C_2| = 5\Delta - 2k + 2$. Hence, $s(G) \geq 5\Delta - 2k$, a contradiction.

b) For all $j \in C_1$: $a_j > \min\{a_i : i \in \{1, \cdots, \Delta\}\}$.

Let $l \in C_2$ such that $a_l = \min\{a_i : i \in \{1, \cdots, \Delta\}\}$. Then, $a_i \geq 3$, for all $i \in C_1$, and therefore, $s(G) + 2 \geq 3|C_1| + |C_2|$. Thus $s(G) > 5\Delta - 2k$, a contradiction.

□

With Lemma 3.3, the following corollary is derived.

Corollary 3.5 ([16]). *If G is an overfull critical graph, then G has a 2-factor.*

Luo and Zhao [19] investigated conditions under which a critical graph is Hamiltonian and therefore has a 2-factor as well. Using results of [1, 8, 18, 26] on degree sequences of critical graphs of order at most 10 they proved the following theorem.

Theorem 3.6 ([19]). *A critical graph with at most 10 vertices is Hamiltonian.*

Proving deep refinements of VAL, so-called second neighborhood adjacency lemmas they deduced the following theorem. Clearly, every 2-critical graph is Hamiltonian.

Theorem 3.7 ([19]). *Let $\Delta \geq 3$ and G be a Δ-critical graph of order n. If $\Delta \geq 6n/7$, then G is Hamiltonian and thus has a 2-factor.*

They further deduced the following improvement for overfull graphs.

Theorem 3.8 ([19]). *Let $\Delta \geq 3$ and G be an overfull Δ-critical graph. If $\Delta \geq n/2$, then G is Hamiltonian.*

Luo and Zhao related their results to Hilton and Chetwynd's overfull graph conjecture.

Conjecture 3.9 ([6]). *Let G be a graph with $\Delta(G) > n/3$. Then G is class 2 if and only if G has an overfull subgraph H such that $\Delta(H) = \Delta(G)$.*

Consequently:

Theorem 3.10 ([19]). *If Conjecture 3.9 is true, then every Δ-critical graph G with $\Delta \geq n/2$ is Hamiltonian.*

In the line of these results Chen and Shan [5] proved Conjecture 1.5 for graphs having large maximum degree in relation to their orders. They proved that the condition of Theorem 3.10 on the truth of the overfull graph conjecture can be skipped if we just ask for a 2-factor instead of a Hamiltonian circuit. As far as we know, the article of Chen and Shan [5] is the first article which uses Tutte' 2-factor Theorem, which is a special case of his f-factor Theorem [22].

Theorem 3.11 ([5]). *Let $\Delta \geq 3$ and G be a Δ-critical graph of order n. If $\Delta \geq n/2$, then G has a 2-factor.*

We will now prove some equivalent formulations of Vizing's 2-factor conjecture.

Theorem 3.12. *For $\Delta \geq 3$, the following statements are equivalent:*

1. *Every Δ-critical graph has a 2-factor.*
2. *Every Δ-critical graph of even order has a 2-factor.*
3. *Every Δ-critical graph of odd order has a 2-factor.*
4. *Every Δ-critical graph with $\delta = \Delta - 1$ has a 2-factor.*
5. *Every Δ-critical graph with $\delta = 2$ has a 2-factor.*
6. *For every Δ-critical graph G with a divalent vertex v: $G - v$ has a 2-factor.*

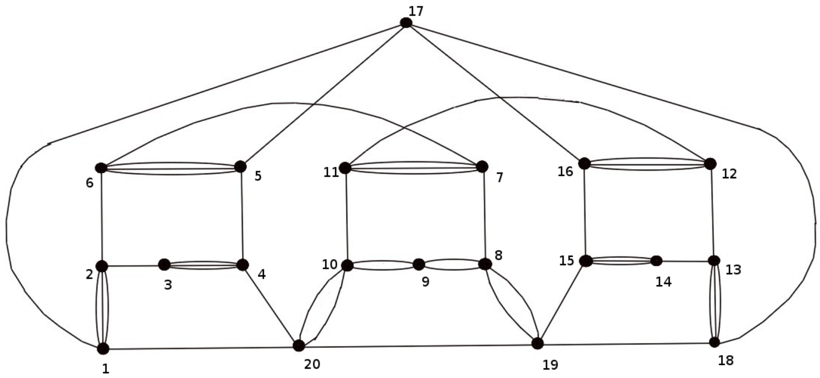

FIGURE 2. The 5-critical Grünewald graph Z_5 on 20 vertices.

7. *For every Δ-critical graph G with a divalent vertex v: G and $G - v$ have a 2-factor.*

Proof. Clearly, statement 1. implies statements 2. - 5.

(1. $\Rightarrow$ 6.) Let G and H be Δ-critical graphs and let v be a divalent vertex of G, and u, w be the neighbors of v. Let $x \in V(H)$ with $d_H(x) = t \geq 3$ and $x_1, \cdots x_t$ be the neighbors of x. Let $G_1, \cdots, G_t$ be t copies of G with divalent vertices $v_1, \cdots, v_t$ and neighbors $u_1, \cdots, u_t, w_1, \cdots, w_t$, respectively. Note that for $i \in \{1, \cdots, t\}$, the vertices v_i, u_i and w_i of G_i correspond to the vertices v, u and w in G, respectively.

Let H_1 be the Hajós union of H and G_1 where the vertices x and v_1 are identified (this vertex is still denoted by x in H_1) and the edge x_1w_1 is added. For $j \in \{2, \cdots, t\}$ let H_j be the Hajós union of H_{j-1} and G_j where the vertices x and v_j are identified and the edge x_jw_j is added. By Theorem 2.1, H_t is Δ-critical and hence, it has a 2-factor F. Thus, there is $q \in \{1, \cdots, t\}$ such that $xu_q \notin F$. Since $\{xu_q, x_qw_q\}$ is a 2-edge-cut in H_t, it follows that F induces a 2-factor in $G_q - v_q$. Therefore, $G - v$ has a 2-factor.

(6. $\Rightarrow$ 1.) and (5. $\Rightarrow$ 1.) Let H be a Δ-critical graph with two vertices v, w of degree two. Such graphs exist by Lemma 2.7. Let G be a Δ-critical graph and let H' be the Hajós union of H and G, where vertex v is used and edge e is removed from G. For the proof of (6. $\Rightarrow$ 1.): Since vertex w has degree 2 in H', it follows that $H' - w$ has a 2-factor F'. For the proof of (5. $\Rightarrow$ 1.): Since vertex w has degree 2 in H', it follows that H' has a 2-factor F'. Further $G - e$ is a subgraph of H' and it is connected by two edges f_1, f_2 to $H'[V(H') - V(G)]$. Since $f_1 \in F'$ if and only if $f_2 \in F'$, it follows that G has a 2-factor F. Note that $e \in F$ if and only if $f_1, f_2 \in F'$.

(4. $\Rightarrow$ 1.) Let G be a Δ-critical graph. Apply Type I extension to each vertex of degree smaller than $\Delta - 1$ to obtain a Δ-critical graph G' with $\delta(G') = \Delta - 1$. Now the statement follows with Lemma 2.9.

(3. $\Rightarrow$ 1.) Let G be a Δ-critical graph of even order and H be a Δ-critical graph of even order which has a divalent vertex v. Such a graph exists by Lemma 2.7. The Hajós union of H and G, where v is used, yields a Δ-critical graph H' of odd

order. By our assumption H' has a 2-factor. Now it easily follows that G has a 2-factor.

The case (2. $\Rightarrow$ 1.) is proved analogously.

The case (1. $\Leftrightarrow$ 7.) is proved analogously to (1. $\Leftrightarrow$ 6.).

$\square$

Not much is known about critical graphs of even order. There is no 2-critical graph of even order, and 3-critical graphs do not have a 2-factor which consists of even circuits, otherwise they would be class 1. In particular, 3-critical graphs of even order are not Hamiltonian. The situation changes for $\Delta \geq 4$.

Grünewald [14] constructed a Δ-critical multigraph on 20 vertices for every $\Delta \geq 5$. The first member Z_5 of this family is shown in Figure 2. For $\Delta > 5$, the graphs Z_Δ are obtained from Z_5 by adding appropriate 1-factors. See [14] for the details.

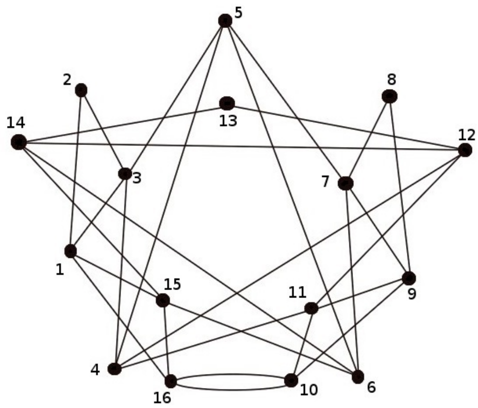

FIGURE 3. The 4-critical graph G_4 on 16 vertices [13].

Theorem 3.13. *For every $\Delta \geq 4$ there are Hamiltonian Δ-critical graphs of even order. Furthermore, these graphs also have a 2-factor which contains odd circuits.*

Proof. For $\Delta \geq 5$, consider the 5-critical graph of Figure 2. A Hamiltonian circuit is given by the vertex sequence $1, 2, 3, \cdots, 19, 20, 1$. An odd 2-factor consists of circuits with vertex sets $\{2, 3, 4, 5, 6\}$, $\{7, 8, 9, 10, 11\}$, $\{12, 13, 14, 15, 16\}$, and $\{17, 18, 19, 20, 1\}$. Since Z_5 is a spanning subgraph of each Z_Δ, it follows that Z_Δ has the above properties as well. Clearly, by Type I extension we obtain simple Δ-critical graphs with the desired properties for each $\Delta \geq 5$.

For $\Delta = 4$, consider the graph G_4 given in Figure 3, which is due to Grünewald [13]. The vertex sequence $1, 2, 3, \cdots, 15, 16, 1$ gives a Hamiltonian circuit of G_4. A 2-factor that consists of four odd circuits is given by the following vertex sets $\{1, 2, 3\}$, $\{7, 8, 9\}$, $\{12, 13, 14\}$ and $\{4, 5, 6, 10, 11, 15, 16\}$. Again, we get simple 4-critical Hamiltonian graphs which also have a 2-factor with four odd circuits by applying Type I extension to some vertices including vertex 10 or 16.

$\square$

4. Other factors and open problems

4.1. **Eulerian factors.** The 2-factor conjecture seems to be very difficult to prove since the structural implications of edge-chromatic criticality of graphs are still not well-understood. To gain a better understanding of the structure of critical graphs it might be helpful to study other spanning subgraphs. If H is a spanning subgraph of a graph G and $d_H(v)$ is even and at least 2 for every $v \in V(G)$, then we say that H is an *Eulerian factor* of G.

Conjecture 4.1. *For every $\Delta \geq 2$: Every Δ-critical graph has an Eulerian factor.*

Clearly, if the 2-factor conjecture (Conj. 1.5) is true, then every critical graph has an Eulerian factor. Indeed, Conjecture 4.1 is true for $\Delta = 2$, and it is equivalent to the 2-factor conjecture for $\Delta = 3$.

In the following theorem we use the terminology of the definition of the Hajós union of two graphs.

Theorem 4.2. *Let H be the Hajós union of G and G'. Then H has an Eulerian factor if and only if G has an Eulerian factor F and G' has an Eulerian factor F' such that $vw \in F$, $v'w' \in F'$ or $vw \notin F$, $v'w' \notin F'$.*

Proof. ($\Leftarrow$) It is easy to see that in both cases F and F' can be modified to obtain an Eulerian factor of H.

($\Rightarrow$) Let x be the vertex of H which is obtained by identifying v and v'. Let F_H be an Eulerian factor of H. If $ww' \notin F_H$, then x is a cut vertex of F_H. Hence, $E_H(x) \cap F_H$ contains an even number of edges which are incident to a vertex of $G - v$ and an even number of edges which are incident to a vertex of $G' - v'$. Thus, G and G' have Eulerian factors F and F' such that $vw \notin F$ and $v'w' \notin F'$.

If $ww' \in F_H$, then there is a circuit C in F_H that contains ww' and x. This implies that x is a cut vertex of $F'_H = F_H - C$, which is Eulerian. Hence, $E_H(x) \cap F_H$ contains an odd number of edges which are incident to a vertex of $G - v$ and an odd number of edges which are incident to a vertex of $G' - v'$. Hence, G and G' have Eulerian factors F and F' such that $vw \in F$ and $v'w' \in F'$.

□

The following equivalences are analogously proved as the corresponding statements of Theorem 3.12.

Theorem 4.3. *For $\Delta \geq 3$, the following statements are equivalent:*

1. *Every Δ-critical graph has an Eulerian factor.*
2. *Every Δ-critical graph of even order has an Eulerian factor.*
3. *Every Δ-critical graph of odd order has an Eulerian factor.*

4.2. **Regular and almost regular factors.** The conjecture on 1-factors turned out to be false. The 2-factor conjecture is open. However, it is a special case of the following conjecture.

Conjecture 4.4. *Every critical graph has a regular factor.*

Let G be a graph and let $f : V(G) \to \mathbb{N}$ be a function. A subgraph H of G is a f-factor of G, if $d_H(v) = f(v)$, for all $v \in V(G)$. Conjecture 4.4 is a special case of the question, for which function f every critical graph has a f-factor.

A graph H is a $[k, k-1]$-graph, if $d_H(v) \in \{k-1, k\}$, for each $v \in V(H)$. We also say that H is *almost regular*. By Theorem 3.12, Conjecture 1.5 is equivalent to its restriction on graphs with $\delta = \Delta - 1$. Thomassen [21] proved that these graphs have a $[r-1, r]$-factor for every $r \in \{1, \cdots, \Delta\}$. A $[r-1, r]$-factor is non-trivial if $r \geq 2$. A natural question is whether every critical graph has a non-trivial almost regular factor. If this question has an affirmative answer, then the following conjecture would by true by the aforementioned theorem of Thomassen.

Conjecture 4.5. *Every critical graph has* $[1, 2]$*-factor.*

Clearly, Conjecture 4.5 is true if Conjecture 4.4 is true. In particular, it is true for critical graphs with $\delta = \Delta - 1$ and therefore, for 3-critical graphs as well.

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