

Liouville theorems for critical points of the p -Ginzburg-Landau type functional

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Abstract. In this paper, we consider the smooth map from a Riemannian manifold to the standard Euclidean space and the p -Ginzburg-Landau energy ($p \geq 1$). Under suitable curvature conditions on the domain manifold, some Liouville type theorems are established by assuming either growth conditions of the p -Ginzburg-Landau energy or an asymptotic condition at the infinity for the maps. In the end of paper, we obtain the unique constant solution of the constant Dirichlet boundary value problems on starlike domains.

1. INTRODUCTION

One of the important problems for harmonic maps or generalized harmonic maps is to study their Liouville type results. (cf. [9, 10, 20, 13]). It is well known that the stress-energy tensor is a useful tool to investigate the energy behavior and some vanishing results of related energy functional. Most Liouville results have been established by assuming either the finiteness of the energy of the map or the smallness of the whole image of the domain manifold under the map. In [13], Z.R. Jin has shown several interesting Liouville theorems for harmonic maps from complete manifolds with assumptions on the asymptotic behavior of the maps at infinity.

Let $\Omega \subseteq \mathbb{R}^2$ be a smooth bounded simply connected domain. Consider the following functional defined for maps $u \in H^1(\Omega, \mathbb{C})$:

$$E_\epsilon(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{4\epsilon^2} \int_{\Omega} (|u|^2 - 1)^2.$$

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Keywords. p -Ginzburg-Landau energy functional, monotonicity formulae, Liouville theorem.
AMS Subject Classification. 53C20, 53C21, 53C55.

The second author is supported by NSFC Grant No. 12001360.

The third author is supported by NSFC Grant No. 12171091.

Ginzburg-Landau introduced this Functional in the study of phase transition problems and it plays an important role ever since, especially in superconductivity, superfluidity and XY-magnetism (see details for [15, 17, 21]). A lot of papers devote to the asymptotic behavior of minimizers u_ϵ of $E_\epsilon(u, \Omega)$ in H as $\epsilon \rightarrow 0$. It was shown in those cases that u_ϵ converges strongly to a harmonic map u_0 on any compact subset away from the zeros. Readers can refer to [2, 3, 4, 22] for the progress in this field. In the past decades, p -Ginzburg-Landau functionals have been introduced. In [11, 16], the authors investigated the convergence of a p -Ginzburg-Landau type functional when the parameter goes to zero.

In this paper, we consider a smooth map $u : (M^m, g) \rightarrow (R^n, h)$ from a Riemannian manifold to the standard Euclidean space and the following p -Ginzburg-Landau energy

$$E_{GL}^p(u) = \int_M \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g ,$$

where $p \geq 1$ and ϵ is any small positive number. To generalize the Liouville type results for harmonic maps to the critical points of p -Ginzburg-Landau energy functional, we introduce the stress energy tensor $S_{u,p}^{GL}$ associated with the p -Ginzburg-Landau functional $E_{GL}^p(u)$. It is easy to show that any critical point of the p -Ginzburg-Landau functional satisfies the conservation law, that is, $div S_{u,p}^{GL} = 0$. Using a basic integral formula linked naturally to the conservation law enables us to establish some monotonicity formulae for these critical points of the p -Ginzburg-Landau energy functional. Consequently, several Liouville type results can be deduced from these monotonicity formulae under suitable growth conditions on the energy. We also build a Liouville type result under the condition of slowly divergent energy.

Next we want to generalize Jin's results in [13] to the critical point of the p -Ginzburg-Landau energy functional. The methods we use for proving the result is very similar to Jin's. Firstly, we may use the stress-energy tensor to establish the monotonicity formula which gives a lower bound for the growth rates of the energy. Secondly, we use the asymptotic assumption of the map at infinity to obtain the upper energy growth rates. Under suitable conditions on u and the Hessian of the distance functions of the domain manifolds, one may show that these two growth rates are contradictory unless the critical point is constant. In this way, we establish some Liouville theorems for the critical points of the p -Ginzburg-Landau energy functional with the asymptotic property at infinity from some complete manifolds.

In addition to establishing Liouville type results, the monotonicity formulae may be used to investigate the constant Dirichlet boundary value problem as well. We obtain the unique constant solution of the constant Dirichlet boundary value problem on starlike domains for the critical point of p -Ginzburg-Landau energy functional.

2. p -GINZBURG-LANDAU ENERGY FUNCTIONAL AND STRESS-ENERGY TENSOR

Let $u : (M^m, g) \rightarrow (R^n, h)$ be a smooth map from a Riemannian manifold to the standard Euclidean space. We consider the following p -Ginzburg-Landau energy

$$E_{GL}^p(u) = \int_M \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g ,$$

where $p \geq 1$ and ϵ is any small positive number. Let $\{u_t\}(|t| < \kappa)$ with $u_0 = u$ and $v = \frac{\partial u_t}{\partial t} \Big|_{t=0}$ be a one parameter variation, we have the following lemma.

Lemma 2.1. *The first variation formula for p -Ginzburg-Landau energy functional*

$$\frac{d}{dt} \Big|_{t=0} E_{GL}^p(u_t) = - \int_M \langle \operatorname{div}(|du|^{p-2} du) + \frac{1}{\epsilon^n} (1 - |u|^2)u, v \rangle dv_g .$$

Proof. Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame of TM . Since the target manifold is Standard Euclidean space, we can perform the following calculations

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} E_{GL}^p(u_t) &= \int_M \frac{\partial}{\partial t} \Big|_{t=0} \left(\frac{|du_t|^p}{p} \right) dv_g + \int_M \frac{\partial}{\partial t} \Big|_{t=0} \left[\frac{1}{4\epsilon^n} (1 - |u_t|^2)^2 \right] dv_g \\ &= \int_M |du|^{p-2} \sum_{i=1}^m \langle \nabla_{\frac{\partial}{\partial t}} du_t(e_i), du_t(e_i) \rangle \Big|_{t=0} dv_g \\ &\quad - \frac{1}{2\epsilon^n} \int_M (1 - |u|^2) \frac{\partial}{\partial t} \Big|_{t=0} |u_t|^2 dv_g \\ &= \int_M |du|^{p-2} \sum_{i=1}^m \langle \nabla_{e_i} du_t \left(\frac{\partial}{\partial t} \right), du_t(e_i) \rangle \Big|_{t=0} dv_g \\ &\quad - \frac{1}{\epsilon^n} \int_M (1 - |u|^2) v \cdot u dv_g \\ &= \int_M |du|^{p-2} \sum_{i=1}^m \langle \nabla_{e_i} v, du(e_i) \rangle dv_g - \frac{1}{\epsilon^n} \int_M (1 - |u|^2) v \cdot u dv_g \\ &= - \int_M \langle \operatorname{div}(|du|^{p-2} du), v \rangle dv_g - \frac{1}{\epsilon^n} \int_M (1 - |u|^2) v \cdot u dv_g \\ &= - \int_M \langle \operatorname{div}(|du|^{p-2} du) + \frac{1}{\epsilon^n} (1 - |u|^2)u, v \rangle dv_g . \end{aligned}$$

□

Definition 2.1. u is called a *critical point* of p -Ginzburg-Landau energy functional if

$$\operatorname{div}(|du|^{p-2} du) + \frac{1}{\epsilon^n} (1 - |u|^2)u = 0 .$$

When $p = 2$, above equation is reduced to $\Delta u + \frac{1}{\epsilon^n} (1 - |u|^2)u = 0$.

In [BE], Baird-Eells introduced the stress-energy tensor associated with the usual energy and proved that harmonic maps satisfy the conservation law. We can also define the stress-energy tensor $S_{u,p}^{GL}$ associated with the p -Ginzburg-Landau energy functional $E_{GL}^p(u)$ and prove that the critical points satisfy the conservation law, i.e. $\operatorname{div} S_{u,p}^{GL} = 0$.

Definition 2.2. Let $u : (M^m, g) \rightarrow (R^n, h)$ be a smooth map from a Riemannian manifold to the standard Euclidean space. The *stress-energy tensor* of u is the symmetric 2-tensor on M given by

$$S_{u,p}^{GL} = \left[\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right] g - |du|^{p-2} u^* h .$$

Theorem 2.1. $\operatorname{div} S_{u,p}^{GL}(X) = -\langle \operatorname{div}(|du|^{p-2}du) + \frac{1}{\epsilon^n}(1-|u|^2)u, du(X) \rangle$, for any $X \in \Gamma(TM)$.

Proof. For any 2-tensor field $W \in \Gamma(T^*M \otimes T^*M)$, the divergence of W is defined by

$$(2.1) \quad (\operatorname{div} W)(X) = \sum_{i=1}^m (\nabla_{e_i}^M W)(e_i, X),$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal basis of M . Then we have

$$\begin{aligned} \operatorname{div} S_{u,p}^{GL}(X) &= \sum_{i=1}^m [\nabla_{e_i} S_{F,u}^{GL}(e_i, X)] - S_{F,u}^{GL}(e_i, \nabla_{e_i} X) \\ &= \sum_{i=1}^m e_i \left\{ \left[\frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1-|u|^2)^2 \right] \langle e_i, X \rangle \right\} \\ &\quad - \sum_{i=1}^m e_i \{ |du|^{p-2} \langle du(e_i), du(X) \rangle \} \\ &\quad - \left[\frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1-|u|^2)^2 \right] \sum_{i=1}^m \langle e_i, \nabla_{e_i} X \rangle \\ &\quad + |du|^{p-2} \sum_{i=1}^m \langle du(e_i), du(\nabla_{e_i} X) \rangle \\ &= X \left[\frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1-|u|^2)^2 \right] - \sum_{i=1}^m e_i (|du|^{p-2}) \langle du(e_i), du(X) \rangle \\ &\quad - \sum_{i=1}^m |du|^{p-2} \langle \nabla_{e_i} du(e_i), du(X) \rangle - \sum_{i=1}^m |du|^{p-2} \langle du(e_i), \nabla_{e_i} du(X) \rangle \\ &\quad + |du|^{p-2} \sum_{i=1}^m \langle du(e_i), du(\nabla_{e_i} X) \rangle \\ &= -\langle \operatorname{div}(|du|^{p-2}du) + \frac{1}{\epsilon^n}(1-|u|^2)u, du(X) \rangle. \end{aligned}$$

□

Definition 2.3. We say that u satisfies the *conservation law* if $\operatorname{div} S_{u,p}^{GL} = 0$.

Corollary 2.1. If $u : (M^m, g) \rightarrow (R^n, h)$ is a critical point of p -Ginzburg-Landau energy functional, then u satisfies the conservation law, i.e., $\operatorname{div} S_{u,p}^{GL} = 0$.

For any vector field $X \in \Gamma(TM)$, let θ_X denote the dual one form of X , that is,

$$(2.2) \quad \theta_X(Y) = g(X, Y) \quad , \quad \forall Y \in \Gamma(TM) .$$

The covariant derivative of θ_X is given by

$$(\nabla^M \theta_X)(Y, Z) = (\nabla_Y^M \theta_X)(Z) = g(\nabla_Y^M X, Z) ,$$

for any $X, Y, Z \in \Gamma(TM)$. If $X = \nabla^M \psi$ is the gradient of some smooth function ψ on M , then $\theta_X = d\psi$ and $\nabla^M \theta_X = \operatorname{Hess}_g(\psi)$.

Let $W \in \Gamma(T^*M \otimes T^*M)$ be any symmetric 2-tensor. By a direct computation, we have

$$\begin{aligned} \operatorname{div}(i_X W) &= (\operatorname{div} W)(X) + \langle W, \nabla^M \theta_X \rangle \\ &= (\operatorname{div} W)(X) + \frac{1}{2} \langle W, L_X g \rangle, \end{aligned}$$

where $i_X W \in A^1(M)$ denotes the interior product by any $X \in \Gamma(TM)$.

In terms of the Stoke's formula we get

Lemma 2.2. *Let D be any bounded domain of M with C^1 boundary. Denote by ν the unit outward normal vector field along ∂D . For any symmetric 2-tensor $W \in \Gamma(T^*M \otimes T^*M)$ and any vector field $X \in \Gamma(TM)$, we have*

$$\int_{\partial D} (i_X W)(\nu) ds_g = \int_D \langle W, \nabla^M \theta_X \rangle + (\operatorname{div} W)(X) dv_g$$

and

$$\int_{\partial D} (i_X W)(\nu) ds_g = \int_D \frac{1}{2} \langle W, L_X g \rangle + (\operatorname{div} W)(X) dv_g.$$

Applying Lemma 2.2 to $S_{u,p}^{GL}$, we immediately obtain the following integral formulae:

$$\int_{\partial D} S_{u,p}^{GL}(X, \nu) ds_g = \int_D \langle S_{u,p}^{GL}, \nabla \theta_X \rangle + (\operatorname{div} S_{u,p}^{GL})(X) dv_g$$

and

$$\int_{\partial D} S_{u,p}^{GL}(X, \nu) ds_g = \int_D \frac{1}{2} \langle S_{u,p}^{GL}, L_X g \rangle + (\operatorname{div} S_{u,p}^{GL})(X) dv_g.$$

If u is a critical point of p -Ginzburg-Landau energy functional, by Corollary 2.1 we obtain

$$\begin{aligned} \int_{\partial D} S_{u,p}^{GL}(X, \nu) ds_g &= \int_D \langle S_{u,p}^{GL}, \nabla \theta_X \rangle dv_g \\ (2.3) \qquad \qquad \qquad &= \int_D \frac{1}{2} \langle S_{u,p}^{GL}, L_X g \rangle dv_g. \end{aligned}$$

For applications of the stress-energy tensor, the readers may refer to [5, 6, 7, 23]. In next section, we will use the similar method to establish the monotonicity formulae.

3. MONOTONICITY FORMULAE AND LIOUVILLE TYPE RESULTS UNDER GROWTH CONDITIONS.

From now on, we always assume that (M^m, g) is a complete Riemannian manifold with a pole x_0 . A pole $x_0 \in M$ is a point such that the exponential map from the tangent space to M at x_0 into M is a diffeomorphism. We will establish monotonicity formulae on these manifolds.

Denote by $r(x)$ the distance function relative to the pole x_0 . For any $x \in M$, let $\lambda_1(x) \leq \lambda_2(x) \leq \dots \leq \lambda_m(x)$ be the eigenvalues of $Hess_g(r^2)$ at x .

Theorem 3.1. *Assume that $u : (M^m, g) \rightarrow (R^n, h)$ is the critical point of p -Ginzburg-Landau energy functional. If there exists a constant $\sigma > 0$ such that*

$$(P_1) \qquad \qquad \qquad \frac{1}{2} \left(\sum_{i=1}^m \lambda_i - p \lambda_m \right) \geq \sigma,$$

then

$$\frac{\int_{B_{\rho_1}(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g}{\rho_1^\sigma} \leq \frac{\int_{B_{\rho_2}(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g}{\rho_2^\sigma},$$

for any $0 < \rho_1 \leq \rho_2$.

Proof. Set $D = B_R(x_0) = \{x \in M \mid r(x) \leq R\}$ and $X = \frac{1}{2} \nabla r^2 = r \frac{\partial}{\partial r}$. Since u is a critical point, use (2.3) we have

$$(3.1) \quad \int_{\partial B_R(x_0)} S_{u,p}^{GL}(X, \nu) ds_g = \int_{B_R(x_0)} \frac{1}{2} \langle S_{u,p}^{GL}, L_X g \rangle dv_g,$$

where $\nu = \frac{\partial}{\partial r}$ is the unit outward normal vector field of $B_R(x_0)$.

Let $\{e_i\}_{i=1}^m$ be an orthonormal frame of (M, g) . Moreover, we can assume that $Hess_g(r^2)$ becomes a diagonal matrix with respect to $\{e_i\}_{i=1}^m$.

$$\begin{aligned} \langle S_{u,p}^{GL}, \frac{1}{2} L_X g \rangle &= \frac{1}{2} \langle S_{u,p}^{GL}, Hess_g(r^2) \rangle \\ &= \frac{1}{2} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) \langle g, Hess_g(r^2) \rangle - \frac{1}{2} |du|^{p-2} \langle u^* h, Hess_g(r^2) \rangle \\ &= \frac{1}{2} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) \sum_{i,j=1}^m g(e_i, e_j) \cdot Hess_g(r^2)(e_i, e_j) \\ &\quad - \frac{1}{2} |du|^{p-2} \sum_{i,j=1}^m u^* h(e_i, e_j) \cdot Hess_g(r^2)(e_i, e_j) \\ &\geq \frac{1}{2} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) \sum_{i=1}^m \lambda_i - \frac{1}{2} |du|^p \lambda_m \\ &\geq \frac{1}{2} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) \sum_{i=1}^m \lambda_i - \frac{1}{2} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) p \lambda_m \\ (3.2) \quad &= \frac{1}{2} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) \left(\sum_{i=1}^m \lambda_i - p \lambda_m \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} \int_{\partial B_R(x_0)} S_{u,p}^{GL}(r \frac{\partial}{\partial r}, \nu) ds_g &\leq \int_{\partial B_R(x_0)} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) g(r \frac{\partial}{\partial r}, \nu) ds_g \\ &= R \int_{\partial B_R(x_0)} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) ds_g \\ &= R \frac{d}{dR} \int_0^R \left(\int_{\partial B_r(x_0)} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) ds_g \right) dr \\ (3.3) \quad &= R \frac{d}{dR} \int_{B_R(x_0)} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g. \end{aligned}$$

It follows from (3.1), (3.2) and (3.3) that

$$(3.4) \quad \begin{aligned} & R \frac{d}{dR} \int_{B_R(x_0)} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g \\ & \geq \int_{B_R(x_0)} \frac{1}{2} \left(\sum_i^m \lambda_i - p\lambda_m \right) \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g . \end{aligned}$$

By condition (P_1) , we obtain

$$R \frac{d}{dR} \int_{B_R(x_0)} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g \geq \sigma \int_{B_R(x_0)} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g .$$

Integrating the above formula on $[\rho_1, \rho_2]$, finally, we can get

$$\frac{\int_{B_{\rho_1}(x_0)} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g}{\rho_1^\sigma} \leq \frac{\int_{B_{\rho_2}(x_0)} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g}{\rho_2^\sigma} .$$

□

Next we list some vanishing results which are immediate applications of the monotonicity formulae.

Theorem 3.2. *Under the same condition of Theorem 3.1 and*

$$\int_{B_r(x_0)} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g = o(r^\sigma) ,$$

then $du = 0$, that is, u is a constant.

Proof. By Theorem 3.1, for any $0 < \rho < r$, we have the following inequality

$$\frac{1}{\rho^\sigma} \int_{B_\rho(x_0)} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g \leq \frac{1}{r^\sigma} \int_{B_r(x_0)} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g .$$

Letting $r \rightarrow +\infty$, under the the assumption, we may conclude

$$\int_{B_\rho(x_0)} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g = 0 .$$

Since ρ is arbitrary, then $du = 0$.

□

Next we will introduce some comparison theorems in Riemannian geometry.

Lemma 3.1 (cf.[7, 8, 19]). *Let (M, g) be a complete Riemannian manifold with a pole x_0 and let r be the distance function relative to x_0 . Denote by K_r the radial curvature of M .*

(i) *If $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha > 0$, $\beta > 0$, then*

$$\beta \coth(\beta r) [g - dr \otimes dr] \leq \text{Hess}_g(r) \leq \alpha \coth(\alpha r) [g - dr \otimes dr] .$$

(ii) *If $-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\varepsilon}}$ with $\varepsilon > 0$, $A \geq 0$, $0 \leq B < 2\varepsilon$, then*

$$\frac{1 - \frac{B}{2\varepsilon}}{r} \leq \text{Hess}_g(r) \leq \frac{e^{A/2\varepsilon}}{r} [g - dr \otimes dr] .$$

(iii) If $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ with $a \geq 0, b^2 \in [0, \frac{1}{4}]$, then

$$\frac{1 + \sqrt{1 - 4b^2}}{2r} [g - dr \otimes dr] \leq Hess_g(r) \leq \frac{1 + \sqrt{1 + 4a^2}}{2r} [g - dr \otimes dr].$$

Proof. The case (i) and (iii) are standard results in Riemannian geometry. Indeed by Example 2.25 in [8] and Theorem 27 in [18], we can obtain (i). The readers can also see [14] and [19] for details. In [12], the authors have generalized the work of [8] and [19]. Using Theorem 1.1 in [12], we can get case (iii). The case (ii) is discussed in [7]. \square

Lemma 3.2. *Let (M, g) be a complete Riemannian manifold with a pole x_0 and let r be the distance function relative to x_0 . Assume that there exist two positive functions $h_1(r)$ and $h_2(r)$ such that*

$$h_1(r)[g - dr \otimes dr] \leq Hess_g(r) \leq h_2(r)[g - dr \otimes dr] \quad \text{and} \quad rh_2(r) \geq 1,$$

then

$$(3.5) \quad \sum_{i=1}^m \lambda_i - p\lambda_m \geq 2\{1 + (m-1)rh_1(r) - ph_2(r)\}.$$

Combing Lemma 3.1 and Lemma 3.2, we can obtain the following.

Lemma 3.3. *Let (M, g) be a complete Riemannian manifold with a pole x_0 and let r be the distance function relative to x_0 . Denote by K_r the radial curvature of M .*

(i) *If $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha > 0, \beta > 0$ and $(m-1)\beta - p\alpha > 0$, then*

$$\sum_i^m \lambda_i - p\lambda_m \geq 2(m - p\frac{\alpha}{\beta}).$$

(ii) *If $-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\varepsilon}}$ with $\varepsilon > 0, A \geq 0, 0 \leq B < 2\varepsilon$ and $1 + (m-1)(1 - \frac{B}{2\varepsilon}) - pe^{A/2\varepsilon} > 0$, then*

$$\sum_i^m \lambda_i - p\lambda_m \geq 2[1 + (m-1)(1 - \frac{B}{2\varepsilon}) - e^{A/2\varepsilon}p].$$

(iii) *If $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ with $a \geq 0, b^2 \in [0, \frac{1}{4}]$ and $2 + (m-1)(1 + \sqrt{1 - 4b^2}) - p(1 + \sqrt{1 + 4a^2}) > 0$, then*

$$\sum_i^m \lambda_i - p\lambda_m \geq 2[1 - \frac{p}{2} + (m-1)\frac{1 + \sqrt{1 - 4b^2}}{2} - \frac{p}{2}\sqrt{1 + 4a^2}].$$

Corollary 3.1. *Let $u : (M, g) \rightarrow (R^n, h)$ be a critical point of p -Ginzburg-Landau energy functional from a Riemannian manifold with a pole x_0 to a standard Euclidean space. Assume that the radial curvature K_r of M satisfies one of the following three conditions:*

(i) $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha > 0, \beta > 0$ and $(m-1)\beta - p\alpha > 0$;

$$\begin{aligned}
(ii) \quad & -\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\varepsilon}} > 0 \quad \text{with } \varepsilon, A \geq 0, 0 \leq B < 2\varepsilon \text{ and} \\
& 1 + (m-1)\left(1 - \frac{B}{2\varepsilon}\right) - pe^{A/2\varepsilon} > 0; \\
(iii) \quad & -\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2} \quad \text{with } a \geq 0, b^2 \in \left[0, \frac{1}{4}\right] \text{ and } 2 + (m-1)(1+ \\
& \sqrt{1-4b^2}) - p(1 + \sqrt{1+4a^2}) > 0.
\end{aligned}$$

Then

$$\frac{\int_{B_{\rho_1}(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1-|u|^2)^2 dv_g}{\rho_1^\sigma} \leq \frac{\int_{B_{\rho_2}(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1-|u|^2)^2 dv_g}{\rho_2^\sigma}$$

for any $0 < \rho_1 \leq \rho_2$, where

$$\sigma = \begin{cases} \left(m - p\frac{\alpha}{\beta}\right) & \text{for } K_r \text{ satisfies (i)} \\ 1 + (m-1)\left(1 - \frac{B}{2\varepsilon}\right) - pe^{A/2\varepsilon} & \text{for } K_r \text{ satisfies (ii)} \\ \frac{2 - p + (m-1)(1 + \sqrt{1-4b^2}) - p\sqrt{1+4a^2}}{2} & \text{for } K_r \text{ satisfies (iii)} \end{cases}$$

Corollary 3.1 yields immediately the following vanishing result.

Theorem 3.3. *Suppose that $u : (M, g) \rightarrow (R^n, h)$ is a critical point of p -Ginzburg-Landau energy functional. Let r be the distance function relative to x_0 . If the radial curvature K_r of M satisfies one of the three conditions in Corollary 3.1 and*

$$\int_{B_r(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1-|u|^2)^2 dv_g = o(r^\sigma),$$

where σ is given by Corollary 3.1. Then $du = 0$, that is, u is constant.

Definition 3.1. $E_p^{GL}(u)$ is said to have *slowly divergent energy* if there exists a positive continuous function $\psi(r)$ such that

$$\int_{R_1}^{+\infty} \frac{dr}{r\psi(r)} = +\infty$$

for some $R_1 > 0$, and

$$(3.6) \quad \lim_{R \rightarrow \infty} \int_{B_R(x_0)} \frac{\frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1-|u|^2)^2}{\psi(r(x))} dv_g < \infty.$$

Theorem 3.4. *Let $u : (M, g) \rightarrow (R^n, h)$ be the critical point of p -Ginzburg-Landau energy functional. If $r(x)$ satisfies the condition (P_1) and $E_p^{GL}(u)$ has slowly divergent energy, then u is a constant map and $u(M) \subseteq S^{n-1}$.*

Proof. From Theorem 3.1, we obtain

$$\begin{aligned}
& R \frac{d}{dR} \int_{B_R(x_0)} \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1-|u|^2)^2 \right) dv_g \\
& \geq \int_{B_R(x_0)} \frac{1}{2} \left(\sum_i^m \lambda_i - p\lambda_{max} \right) \left(\frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1-|u|^2)^2 \right) dv_g.
\end{aligned}$$

If u is not a constant map contained in S^{n-1} , there exists constants $R_0 > 0$ and $C_0 > 0$ such that

$$\int_{B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g \geq C_0$$

for any $R \geq R_0$. Thus

$$\int_{\partial B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 ds_g \geq \frac{\sigma C_0}{R} \quad , \quad \forall R \geq R_0 .$$

Since $E_p^{GL}(u)$ has slowly divergent energy, then

$$\begin{aligned} & \lim_{R \rightarrow \infty} \int_{B_R(x_0)} \frac{|du|^p + \frac{1}{4\epsilon^n} (1 - |u|^2)^2}{\psi(r(x))} dv_g \\ &= \int_0^\infty \frac{dR}{\psi(R)} \int_{\partial B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 ds_g \\ &\geq \int_{R_0}^\infty \frac{dR}{\psi(R)} \int_{\partial B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 ds_g \\ &\geq \sigma C_0 \int_{R_0}^\infty \frac{dr}{r\psi(r)} = \infty . \end{aligned}$$

It is in contradiction to (3.6). □

4. LIOUVILLE THEOREM UNDER THE ASYMPTOTIC CONDITIONS

In [13], Jin established several Liouville theorems for harmonic maps between some Riemannian manifold under some asymptotic condition of the maps at infinity. In particular, he proved that for any harmonic map $f : (R^m, g_0) \rightarrow (N^n, h)$ ($m \geq 3$), if $f(x) \rightarrow P_0 \in N^n$ as $|x| \rightarrow +\infty$, then f must be a constant map. In this section, using a similar technique or idea, we can derive a Liouville theorem for the critical points of the p -Ginzburg-Landau energy. To generalize this result to our case it is necessary to give more strictly asymptotic condition at infinity. We begin with evaluating the lower bound of the energy.

Proposition 4.1. *Assume that $u : (M^m, g) \rightarrow (R^n, h)$ is a critical point of the p -Ginzburg-Landau energy functional from a Riemannian manifold with a pole x_0 to a standard Euclidean space. $r(x)$ is the distance function relative to the pole x_0 . If it satisfies the condition (P_1) and $u(M)$ is not contained in S^{n-1} , then*

$$\int_{B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g \geq C(u)R^\sigma \quad , \quad R \rightarrow \infty$$

where $C(u)$ is a positive constant only depending on u .

Proof. Since u satisfies the conditions in Theorem 3.1, we obtain

$$\frac{\int_{B_\rho(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g}{\rho^\sigma} \leq \frac{\int_{B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g}{R^\sigma}$$

for any $0 < \rho < R$. Note that $u(M)$ is not contained in S^{n-1} , there exist some $\rho > 0$ such that

$$\int_{B_\rho(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g > 0.$$

Denoted by

$$C(u) = \frac{\int_{B_\rho(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g}{\rho^\sigma},$$

then

$$\int_{B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g \geq C(u)R^\sigma.$$

□

By Corollary 3.1, and using the above proposition, we easily obtain the following corollary.

Corollary 4.1. *Let $u : (M, g) \rightarrow (R^n, h)$ be a critical map of p -Ginzburg-Landau energy functional from a Riemannian manifold with a pole x_0 to a standard Euclidean space. Assume that the radial curvature K_r of M satisfies one of the following three conditions:*

- (i) $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha > 0$, $\beta > 0$ and $(m-1)\beta - p\alpha > 0$;
- (ii) $-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\varepsilon}} > 0$ with $\varepsilon, A \geq 0$, $0 \leq B < 2\varepsilon$ and $1 + (m-1)(1 - \frac{B}{2\varepsilon}) - pe^{A/2\varepsilon} > 0$;
- (iii) $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ with $a \geq 0$, $b^2 \in [0, \frac{1}{4}]$ and $2 + (m-1)(1 + \sqrt{1-4b^2}) - p(1 + \sqrt{1+4a^2}) > 0$.

If $u(M)$ is not contained in S^{n-1} , then

$$\int_{B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g \geq C(u)R^\sigma, \quad \text{as } R \rightarrow \infty$$

where $C(u)$ is a positive constant only depending on u , and

$$\sigma = \begin{cases} (m - p\frac{\alpha}{\beta}) & \text{for } K_r \text{ satisfies (i)} \\ 1 + (m-1)(1 - \frac{B}{2\varepsilon}) - pe^{A/2\varepsilon} & \text{for } K_r \text{ satisfies (ii)} \\ \frac{2 - p + (m-1)(1 + \sqrt{1-4b^2}) - p\sqrt{1+4a^2}}{2} & \text{for } K_r \text{ satisfies (iii)} \end{cases}$$

Next, we will show that if u is the critical map of the 2-Ginzburg-Landau energy functional and it is uniformly bounded, the condition (P_1) may be replaced by

(\tilde{P}_1) : The left hand side of the inequality in (P_1) is nonnegative on the whole M and there exists an $R_0 > 0$ such that (P_1) holds for $r(x) \geq R_0$.

To prove this assertion, we start with the following lemmas.

Lemma 4.1. *Suppose that $u : (M^m, g) \rightarrow (R^n, h)$ is a critical map of the 2-Ginzburg-Landau energy functional and u is uniformly bounded. If u is constant in an open set of M , then u is constant on M .*

Proof. From the Euler-Largrange equation, we have

$$\Delta u + \frac{1}{\epsilon^n}(1 - |u|^2)u = 0.$$

Since u is bounded, using unique continuation theorem in [1], one can deduce that u is constant on M . \square

Lemma 4.2. *Assume that u satisfies (\tilde{P}_1) . If du is not identically zero, then $E_2^{GL}(u) = +\infty$.*

Proof. By co-area formula, we have

$$\begin{aligned} & \int_M \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g \\ &= \int_0^{+\infty} \frac{dr}{r} \cdot r \int_{\partial B_r(x_0)} \left(\frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 \right) \frac{1}{|\nabla r|} ds_g \\ &= \int_0^{+\infty} \frac{dr}{r} \cdot r \int_{\partial B_r(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 ds_g. \end{aligned}$$

If $E_2^{GL}(u) < +\infty$, we can choose a sequence $\{r_i\}$ such that

$$(4.1) \quad \lim_{r_i \rightarrow +\infty} r_i \int_{\partial B_{r_i}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 ds_g = 0.$$

Since $r(x)$ satisfies (\tilde{P}_1) , by (3.4) we obtain

$$\begin{aligned} & r \frac{d}{dr} \int_{B_r(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g \\ & \geq \sigma \int_{B_r(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g, \end{aligned}$$

that is

$$r \int_{\partial B_r(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g \geq \sigma \int_{B_r(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g.$$

Let $r = r_i$ tend to infinity in above inequality, it follows from (4.1) that

$$\int_{M \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n}(1 - |u|^2)^2 dv_g = 0.$$

It follows from Lemma 4.1 that $du = 0$ on M which contradicts with the condition. Therefore $E_2^{GL}(u) = +\infty$. \square

Proposition 4.2. *Assume that $u : (M^m, g) \rightarrow (R^n, h)$ is a critical point of the 2-Ginzburg-Landau energy functional and u is uniformly bounded. If $r(x)$ satisfies*

(\tilde{P}_1) and $u(M)$ is not contained in S^{n-1} , then

$$\int_{B_R(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g \geq C(u) R^\sigma ,$$

where R is sufficiently large and $C(u)$ is a positive constant only depending on u .

Proof. Taking $D = B_R(x_0) \setminus B_{R_0}(x_0)$ and $X = r \frac{\partial}{\partial r} = \frac{1}{2} \nabla r^2$, by (2.3) we get

$$\begin{aligned} & \int_{\partial B_R(x_0)} S_{2,u}^{GL}(X, \nu) ds_g - \int_{\partial B_{R_0}(x_0)} S_{2,u}^{GL}(X, \nu) ds_g \\ &= \int_{B_R(x_0) \setminus B_{R_0}(x_0)} \langle S_{2,u}^{GL}, \nabla \theta_X \rangle dv_g \\ &\geq \int_{B_R(x_0) \setminus B_{R_0}(x_0)} \left[\frac{1}{2} \left(\sum_i^m \lambda_i - 2\lambda_{max} \right) \right] \left(\frac{|du|^2}{2} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right) dv_g . \end{aligned}$$

Denoting $\int_{\partial B_{R_0}(x_0)} S_{p=2,u}^H(X, \nu) ds_g$ by $H(R_0)$, then (3.3) and condition (\tilde{P}_1) yield

$$\begin{aligned} & R \int_{\partial B_R(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 ds_g - 2H(R_0) \\ &\geq \sigma \int_{B_R(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g . \end{aligned}$$

It also can be written as

$$\begin{aligned} & R \frac{d}{dR} \left\{ \int_{B_R(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g + \frac{2H(R_0)}{\sigma} \right\} \\ &\geq \sigma \left\{ \int_{B_R(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g + \frac{2H(R_0)}{\sigma} \right\} . \end{aligned}$$

From Lemma 4.2, we know that $E_2^{GL}(u) = +\infty$. Therefore, when R is sufficiently large, we get

$$\int_{B_R(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g + \frac{2H(R_0)}{\sigma} > 0 .$$

Then

$$\frac{\frac{d}{dR} \left\{ \int_{B_R(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g + \frac{2H(R_0)}{\sigma} \right\}}{\int_{B_R(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g + \frac{2H(R_0)}{\sigma}} \geq \frac{\sigma}{R} .$$

Fixing some $R_0 < \bar{R} < R$ and integrating the above formula on $[\bar{R}, R]$, we get

$$\int_{B_R(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g \geq \left\{ H(R_0, \bar{R}) - \frac{2H(R_0)}{\sigma R^\sigma} \right\} R^\sigma ,$$

where

$$H(R_0, \bar{R}) = \frac{\int_{B_{\bar{R}}(x_0) \setminus B_{R_0}(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g + \frac{2H(R_0)}{\sigma}}{\bar{R}^\sigma}.$$

When $R \rightarrow +\infty$, $\frac{2H(R_0)}{\sigma R^\sigma}$ can be controlled by $H(R_0, \bar{R})$. Consequently

$$\int_{B_R(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g \geq C(u)R^\sigma,$$

where $C(u)$ is a constant depending on the map u . □

Next we will use the assumption for the map at infinity to derivative an upper bound for the growth rate. The condition that we will assume for u is as follow:

(P_2) There exists a positive constant $\tilde{\sigma}$ less than σ in (P_1) such that

$$\max_{r(x)=r} h^2(u(x), P_0) \leq r^{\tilde{\sigma}} \int_r^{+\infty} \frac{ds}{\text{vol}(\partial B_s(x_0))} \quad \text{for } r(x) \gg 1.$$

Theorem 4.1. *Let $u : (M, g) \rightarrow (R^n, h)$ be a critical point of p -Ginzburg-Landau energy functional. Suppose that $|du|^{p-2}$ is uniformly bounded and $r(x)$ satisfies the condition (P_1). If $u(x) \rightarrow P_0 \in S^{n-1}$ and u satisfies the condition (P_2), the u must be a constant map.*

Proof. Suppose the critical point u is not constant, then by Proposition 4.1, the energy of u must be infinite. That is, $\int_{B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g \rightarrow +\infty$ as $r(x) \rightarrow +\infty$.

Since $P_0 = (c_1, c_2, \dots, c_\alpha, \dots, c_n) \in S^{n-1}$, then $\sum_{\alpha=1}^n c_\alpha^2 = 1$. It is clear that we can choose a orthogonal matrix A such that $AP_0 = \tilde{P}_0 = (\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_\alpha, \dots, \tilde{c}_n)$, $\tilde{c}_\alpha \neq 0$, for each $\alpha = 1, 2, \dots, n$. Clearly if u is the critical point of p -Ginzburg-Landau energy functional, then Au is also the critical point. Hence without loss of generality we may assume that $u(x) \rightarrow p_0 \in S^{n-1}$, where $p_0 = (c_1, c_2, \dots, c_\alpha, \dots, c_n)$, $c_\alpha \neq 0$, $\alpha = 1, 2, \dots, n$.

Now the assumption that $u(x) \rightarrow P_0$ as $r(x) \rightarrow +\infty$ implies that there exists an $R_1 > 0$ and a neighbourhood U of P_0 such that for $r(x) > R_1$, $u(x) \in U$ and $u_\alpha \neq 0$ for any $\alpha = 1, 2, \dots, n$.

For $\omega \in C_0^2(M \setminus B_{R_1}(x_0), U)$, we consider the variation $u + t\omega : M \rightarrow R^n$ defined as follows:

$$(u + t\omega)(q) = \begin{cases} u(q) & q \in B_{R_1}(x_0), \\ (u + t\omega)(q) & q \in M \setminus B_{R_1}(x_0) \end{cases}$$

for sufficiently small t . Since u is the critical point of p -Ginzburg-Landau energy functional, we have

$$\left. \frac{d}{dt} \right|_{t=0} E_p^{GL}(u + t\omega) = 0$$

that is,

$$(4.2) \quad \int_{M^m \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^n g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \omega_k}{\partial x_j} - \frac{1}{\epsilon^n} \left(1 - \sum_{k=1}^n u_k^2\right) \sum_{k=1}^n u_k \omega_k \, dv_g = 0 .$$

Choose $\omega(x) = \phi(r(x))\tilde{u}(x)$ in (4.2) for $\phi(t) \in C_0^\infty(R_1, \infty)$, $\tilde{u}_k = \frac{u_k^2 - c_k^2}{u_k}$, we obtain

$$(4.3) \quad \begin{aligned} & \int_{M^m \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^n g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \tilde{u}_k}{\partial x_j} \phi(r(x)) - \frac{1}{\epsilon^n} \left(1 - \sum_{k=1}^n u_k^2\right) \phi(r(x)) \sum_{k=1}^n u_k \tilde{u}_k \, dv_g \\ &= \int_{M^m \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^n g^{ij} \frac{\partial u_k}{\partial x_i} \tilde{u}_k \frac{\partial \phi(r(x))}{\partial x_j} \, dv_g . \end{aligned}$$

By a standard approximation argument, (4.3) holds for Lipschitz functions ϕ with compact support.

For $0 < \theta \leq 1$, define

$$\varphi_\theta(t) = \begin{cases} 1 & t \leq 1 ; \\ 1 + \frac{1-t}{\theta} & 1 < t < 1 + \theta ; \\ 0 & t \geq 1 + \theta . \end{cases}$$

In (4.3), choose the Lipschitz function $\phi(r(x))$ to be

$$\phi(r(x)) = \varphi_\theta\left(\frac{r(x)}{R}\right) \left(1 - \varphi_1\left(\frac{r(x)}{R_1}\right)\right) , \quad R > 2R_1 \quad \text{and} \quad R_2 = 2R_1 .$$

Then the first term on left hand side of (4.3) becomes

$$(4.4) \quad \begin{aligned} & \int_{M^m \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^n g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \tilde{u}_k}{\partial x_j} \phi(r(x)) \, dv_g \\ &= \int_{B_{R_2}(x_0) \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^n g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \tilde{u}_k}{\partial x_j} [1 - \varphi_1\left(\frac{r(x)}{R_1}\right)] \, dv_g \\ &+ \int_{B_R(x_0) \setminus B_{R_2}(x_0)} |du|^{p-2} \sum_{k=1}^n g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \tilde{u}_k}{\partial x_j} \, dv_g \\ &+ \int_{B_{R(1+\theta)}(x_0) \setminus B_R(x_0)} |du|^{p-2} \sum_{k=1}^n g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \tilde{u}_k}{\partial x_j} \varphi_\theta\left(\frac{r(x)}{R}\right) \, dv_g . \end{aligned}$$

The second term on left hand side of (4.3) becomes

$$\begin{aligned} & - \int_{M^m \setminus B_{R_1}(x_0)} \frac{1}{\epsilon^n} \left(1 - \sum_{k=1}^n u_k^2\right) \phi(r(x)) \sum_{k=1}^n u_k \tilde{u}_k \, dv_g \\ &= - \int_{B_{R_2}(x_0) \setminus B_{R_1}(x_0)} \frac{1}{\epsilon^n} \left(1 - \sum_{k=1}^n u_k^2\right) [1 - \varphi_1\left(\frac{r(x)}{R_1}\right)] \sum_{k=1}^n u_k \tilde{u}_k \, dv_g \\ &- \int_{B_R(x_0) \setminus B_{R_2}(x_0)} \frac{1}{\epsilon^n} \left(1 - \sum_{k=1}^n u_k^2\right) \sum_{k=1}^n u_k \tilde{u}_k \, dv_g \end{aligned}$$

$$(4.5) \quad - \int_{B_{R(1+\theta)}(x_0) \setminus B_{R(x_0)}} \frac{1}{\epsilon^n} \left(1 - \sum_{k=1}^n u_k^2\right) \varphi_\theta \left(\frac{r(x)}{R}\right) \sum_{k=1}^n u_k \tilde{u}_k \, dv_g .$$

We can also compute the right hand side of (4.3) as follows

$$(4.6) \quad \begin{aligned} & \int_{M^m \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^n g^{ij} \frac{\partial u_k}{\partial x_i} \tilde{u}_k \frac{\partial \phi(r(x))}{\partial x_j} \, dv_g \\ &= \int_{B_{R_2}(x_0) \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^n g^{ij} \frac{\partial u_k}{\partial x_i} \tilde{u}_k \frac{\partial \varphi_1\left(\frac{r(x)}{R_1}\right)}{\partial x_j} \, dv_g \\ &+ \frac{1}{R\theta} \int_{B_{R(1+\theta)}(x_0) \setminus B_{R(x_0)}} |du|^{p-2} \sum_{k=1}^n g^{ij} \frac{\partial u_k}{\partial x_i} \tilde{u}_k \frac{\partial r(x)}{\partial x_j} \, dv_g . \end{aligned}$$

Set

$$(4.7) \quad \begin{aligned} D(R_1) &= \int_{B_{R_2}(x_0) \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^n g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \tilde{u}_k}{\partial x_j} \left[1 - \varphi_1\left(\frac{r(x)}{R_1}\right)\right] \, dv_g \\ &- \int_{B_{R_2}(x_0) \setminus B_{R_1}(x_0)} \frac{1}{\epsilon^n} \left(1 - \sum_{k=1}^n u_k^2\right) \left[1 - \varphi_1\left(\frac{r(x)}{R_1}\right)\right] \sum_{k=1}^n u_k \tilde{u}_k \, dv_g \\ &- \int_{B_{R_2}(x_0) \setminus B_{R_1}(x_0)} |du|^{p-2} \sum_{k=1}^n g^{ij} \frac{\partial u_k}{\partial x_i} \tilde{u}_k \frac{\partial \varphi_1\left(\frac{r(x)}{R_1}\right)}{\partial x_j} \, dv_g . \end{aligned}$$

Substitute (4.4), (4.5) and (4.6) into (4.3), then letting $\theta \rightarrow 0$, we have

$$(4.7) \quad \begin{aligned} & \int_{B_R(x_0) \setminus B_{R_2}(x_0)} |du|^{p-2} \sum_{k=1}^n g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \tilde{u}_k}{\partial x_j} - \frac{1}{\epsilon^n} \left(1 - \sum_{k=1}^n u_k^2\right) \sum_{k=1}^n u_k \tilde{u}_k \, dv_g + D(R_1) \\ &= \int_{\partial B_R(x_0)} |du|^{p-2} \sum_{k=1}^n \frac{\partial u_k}{\partial x_i} \nu^i \tilde{u}_k \, ds_g , \end{aligned}$$

where $\nu^i = \frac{\partial r}{\partial x_j}$ and $\nu = \nu^i \frac{\partial}{\partial x_i}$ is the outer normal vector field along $B_{R_0}(x_0)$.

Note that $\tilde{u}_k = \frac{u_k^2 - c_k^2}{u_k}$. Thus $\frac{\partial \tilde{u}_k}{\partial x_j} = \left(1 + \frac{c_k^2}{u_k^2}\right) \frac{\partial u_k}{\partial x_j}$. Then (4.7) becomes

$$(4.8) \quad \begin{aligned} & \int_{B_R(x_0) \setminus B_{R_2}(x_0)} |du|^{p-2} \sum_{k=1}^n g^{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \left(1 + \frac{c_k^2}{u_k^2}\right) + \frac{1}{\epsilon^n} (1 - |u|^2)^2 \, dv_g + D(R_1) \\ &= \int_{\partial B_R(x_0)} |du|^{p-2} \sum_{k=1}^n \frac{\partial u_k}{\partial x_i} \nu^i \tilde{u}_k \, ds_g . \end{aligned}$$

Indeed,

$$\sum_{k=1}^n \frac{\partial u_k}{\partial x_i} \nu^i \tilde{u}_k = \sum_{k=1}^n \left\langle \frac{\partial u_k}{\partial x_i} dx^i \otimes \frac{\partial}{\partial y_k}, \frac{\partial r}{\partial x_j} \tilde{u}_k dx^j \otimes \frac{\partial}{\partial y_k} \right\rangle .$$

Therefore

$$\begin{aligned}
& \int_{\partial B_R(x_0)} |du|^{p-2} \sum_{k=1}^n \frac{\partial u_k}{\partial x_i} \nu^i \tilde{u}_k ds_g \\
&= \int_{\partial B_R(x_0)} |du|^{p-2} \sum_{k=1}^n \left\langle \frac{\partial u_k}{\partial x_i} dx^i \otimes \frac{\partial}{\partial y_k}, \frac{\partial r}{\partial x_j} \tilde{u}_k dx^j \otimes \frac{\partial}{\partial y_k} \right\rangle ds_g \\
&\leq \int_{\partial B_R(x_0)} |du|^{p-2} \left| \frac{\partial u_k}{\partial x_i} dx^i \otimes \frac{\partial}{\partial y_k} \right| \left| \frac{\partial r}{\partial x_j} \tilde{u}_k dx^j \otimes \frac{\partial}{\partial y_k} \right| ds_g \\
&\leq \sqrt{\int_{\partial B_R(x_0)} |du|^{p-2} \left| \frac{\partial u_k}{\partial x_i} dx^i \otimes \frac{\partial}{\partial y_k} \right|^2 ds_g} \\
&\quad \times \sqrt{\int_{\partial B_R(x_0)} |du|^{p-2} \left| \frac{\partial r}{\partial x_j} \tilde{u}_k dx^j \otimes \frac{\partial}{\partial y_k} \right|^2 ds_g} \\
&= \sqrt{\int_{\partial B_R(x_0)} |du|^p ds_g} \sqrt{\int_{\partial B_R(x_0)} |du|^{p-2} \left(\sum_{k=1}^n \tilde{u}_k^2 \right) ds_g} \\
(4.9) \quad &\leq \sqrt{\int_{\partial B_R(x_0)} |du|^p + \frac{1}{\epsilon^n} (1 - |u|^2)^2 ds_g} \sqrt{\int_{\partial B_R(x_0)} |du|^{p-2} \left(\sum_{k=1}^n \tilde{u}_k^2 \right) ds_g}.
\end{aligned}$$

Next, for any $R \geq R_2$ we let

$$G(R) = \int_{B_R(x_0) \setminus B_{R_2}(x_0)} |du|^p + \frac{1}{\epsilon^n} (1 - |u|^2)^2 dv_g + D(R_1).$$

Then

$$G'(R) = \int_{\partial B_R(x_0)} |du|^p + \frac{1}{\epsilon^n} (1 - |u|^2)^2 ds_g.$$

Hence from (4.8), (4.9) and the fact that $1 + \frac{C_\alpha^2}{u_\alpha^2} \geq 1$ for any $\alpha = 1, 2, \dots, n$,

$$G^2(R) \leq G'(R) \int_{\partial B_R(x_0)} |du|^{p-2} \sum_{k=1}^n \tilde{u}_k^2 ds_g.$$

On the other hand, we have the following estimate.

$$\begin{aligned}
G(R) - D(R_1) &= \int_{B_R(x_0) \setminus B_{R_2}(x_0)} |du|^p + \frac{1}{\epsilon^n} (1 - |u|^2)^2 dv_g \\
&= p \int_{B_R(x_0) \setminus B_{R_2}(x_0)} \frac{|du|^p}{p} dv_g + 4 \int_{B_R(x_0) \setminus B_{R_2}(x_0)} \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g \\
&\geq \min\{p, 4\} \int_{B_R(x_0) \setminus B_{R_2}(x_0)} \left[\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right] dv_g \\
&= C_{p,4} \int_{B_R(x_0) \setminus B_{R_2}(x_0)} \left[\frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 \right] dv_g.
\end{aligned}$$

Since $E_p^{GL}(u)$ is infinity, there exists $\tilde{R} \geq R_2$, $G(R) > 0$ for any $R > \tilde{R}$.

Set $J(R) = \int_{\partial B_R(x_0)} |du|^{p-2} \sum_{k=1}^n \tilde{u}_k^2 ds_g$. Then

$$G^2(R) \leq G'(R)J(R).$$

For any $\tilde{R} \leq R < R_4$, we have

$$\begin{aligned} \int_R^{R_4} \frac{G'(r)}{G^2(r)} dr &\geq \int_R^{R_4} \frac{dr}{J(r)}, \\ \frac{1}{G(R)} - \frac{1}{G(R_4)} &\geq \int_R^{R_4} \frac{dr}{J(r)}. \end{aligned}$$

When $R_4 \rightarrow +\infty$, we get $G(R) \leq \frac{1}{\int_R^{+\infty} \frac{dr}{J(r)}}$.

Note the fact that $|du|^{p-2}$ is uniformly bounded. Using the condition (P_2) and $u(x) \rightarrow P_0$ as $r(x) \rightarrow +\infty$, we get

$$\begin{aligned} J(r) &= \int_{\partial B_r(x_0)} |du|^{p-2} \sum_{k=1}^n \tilde{u}_k^2 ds_g \\ &\leq \int_{\partial B_r(x_0)} |du|^{p-2} \tau(r) ds_g \\ &\leq \tilde{C} \tau(r) \cdot \text{vol}(\partial B_r(x_0)), \end{aligned}$$

where \tilde{C} is a constant only depending on u and $\tau(r)$ is chosen in such a way that

1. $\tau(r)$ is nonincreasing on $(\tilde{R}, +\infty)$ and $\tau(r) \rightarrow 0$ as $r \rightarrow +\infty$;
2. $\tau(r) \geq \max_{r(x)=r} \left\{ \sum_{k=1}^n \tilde{u}_k^2 \right\}$;
3. $\tau(r) \leq C_{P_0} r^{\tilde{\sigma}} \cdot \int_r^{+\infty} \frac{ds}{\text{vol}(\partial B_s(x_0))}$,

where C_{P_0} is a constant only depending on P_0 . Then we can derive

$$\int_R^{+\infty} \frac{dr}{J(r)} \geq \int_R^{+\infty} \frac{dr}{\tilde{C} \tau(r) \text{vol}(\partial B_r(x_0))} \geq \frac{1}{\tilde{C} \tau(R)} \int_R^{+\infty} \frac{dr}{\text{vol}(\partial B_r(x_0))} \geq \frac{1}{C_1 R^{\tilde{\sigma}}},$$

where $C_1 = \tilde{C} \cdot C_{P_0}$. Hence $G(R) \leq C_1 R^{\tilde{\sigma}}$ for any $R \geq \tilde{R}$. By the definition of $G(R)$, we have

$$\begin{aligned} &\int_{B_R(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g \\ &\leq \frac{C_1}{C_{p,4}} R^{\tilde{\sigma}} - \frac{D(R_1)}{C_{p,4}} + \int_{B_{R_2}(x_0)} \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g \\ &= \left\{ C R^{\tilde{\sigma}-\sigma} + \frac{C(u)}{R^\sigma} \right\} R^\sigma, \end{aligned}$$

where $C(u)$ is constant only depending on u . Since $\tilde{\sigma} < \sigma$, it contradicts with Proposition 4.1. \square

In [19], the authors give the volume growth estimates under Ricci curvature conditions. Hence, applying the results to the following cases, the right side of the inequality in condition (P_2) can be expressed as a polynomial.

Corollary 4.2. *Let $u : (M, g) \rightarrow (R^n, h)$ be a critical point of p -Ginzburg-Landau energy functional. Suppose that $|du|^{p-2}$ is uniformly bounded. Assume that the radial curvature K_r of M satisfies the following condition*

$$-\frac{A}{(1+r^2)^{1+\varepsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\varepsilon}} \quad \text{with } \varepsilon > 0,$$

where $A \geq 0$, $0 \leq B < 2\varepsilon$ and $1+(m-1)(1-\frac{B}{2\varepsilon})-pe^{A/2\varepsilon} > 0$. If $u(x) \rightarrow P_0 \in S^{n-1}$ as $r(x) \rightarrow +\infty$, and

$$\max_{r(x)=r} h^2(u(x), P_0) \leq \frac{r^{\tilde{\sigma}-(m-2)}}{(m-2)\omega_m e^{(m-1)A/2\varepsilon}},$$

then u must be a constant map. Here ω_m is the $(m-1)$ -volume of the unit sphere in R^m and $\tilde{\sigma}$ is any positive constant such that $\tilde{\sigma} < [1+(m-1)(1-\frac{B}{2\varepsilon})-e^{A/2\varepsilon}p]$.

Proof. For the condition on the radial curvature K_r , it follows that

$$\text{Ric}(x) \geq -\frac{(m-1)A}{(1+r^2(x))^{1+\varepsilon}},$$

for any $x \in M$. Then a direct calculation yields

$$\int_0^{+\infty} \frac{Ar}{(1+r^2)^{1+\varepsilon}} dr = \frac{A}{2\varepsilon}.$$

By the volume comparison theorem (cf. Corollary 2.17 in [PRS]), we obtain

$$\text{vol}(\partial B_r(x_0)) \leq \omega_m e^{(m-1)A/2\varepsilon} r^{m-1},$$

where ω_m is the $(m-1)$ -volume of the unit sphere in R^m , and thus

$$\left(\int_R^{+\infty} \frac{dr}{\text{vol}(\partial B_r(x_0))} \right)^{-1} \leq (m-2)\omega_m e^{(m-1)A/2\varepsilon} R^{m-2}$$

for $R \gg 1$. Using Corollary 4.1 and Theorem 4.1, we can get the result. \square

Corollary 4.3. *Let $u : (M, g) \rightarrow (R^n, h)$ be a critical point of p -Ginzburg-Landau energy functional. Suppose that $|du|^{p-2}$ is uniformly bounded. Assume that the radial curvature K_r of M satisfies the following condition*

$$-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$$

with $a \geq 0$, $b^2 \in [0, \frac{1}{4}]$ and $2+(m-1)(1+\sqrt{1-4b^2})-p(1+\sqrt{1+4a^2}) > 0$. If $u(x) \rightarrow P_0 \in S^{n-1}$ as $r(x) \rightarrow +\infty$ and

$$\max_{r(x)=r} h^2(u(x), P_0) \leq Cr^{\tilde{\sigma}-(m-1)A'+1},$$

then u must be a constant map. Here $A' = \frac{1 + \sqrt{1 + 4a^2}}{2}$ and $\tilde{\sigma}$ is any positive constant such that $\tilde{\sigma} < [1 - \frac{p}{2} + (m-1)\frac{1 + \sqrt{1 - 4b^2}}{2} - \frac{p}{2}\sqrt{1 + 4a^2}]$.

Proof. For the condition on the radial curvature K_r , it follows that

$$\text{Ric}(x) \geq -\frac{(m-1)a^2}{1+r^2(x)}$$

for any $x \in M$. We can also use the volume comparison theorem (cf. Corollary 2.17 in [19]), then

$$\text{vol}(\partial B_R(x_0)) \leq CR^{(m-1)A'},$$

where C is suitable constant and $A' = \frac{1 + \sqrt{1 + 4a^2}}{2}$. Thus

$$\left(\int_R^{+\infty} \frac{dr}{\text{vol}(\partial B_r(x_0))} \right)^{-1} \leq CR^{(m-1)A'-1}$$

for $R \gg 1$. Using Corollary 4.1 and Theorem 4.1, we can get the result. \square

If the Riemannian manifold (M, g) is the standard Euclidean space (R^m, h) , the eigenvalues of $\text{Hess}_g(r^2)$ are all 2. When $p = 2$, $\frac{1}{2}(\sum_{i=1}^m \lambda_i - 2\lambda_m) = m - 2$. Thus we have the following result.

Corollary 4.4. *Let $u : (R^m, h) \rightarrow (R^n, h)$ be a critical point of 2-Ginzburg-Landau energy functional. If $u(x) \rightarrow P_0 \in S^{n-1}$ as $r(x) \rightarrow +\infty$, u must be a constant map contained in S^{n-1} .*

Proof. Since (M, g) is the standard Euclidean space, from the proofs in Theorem 4.1, we obtain

$$\int_R^{+\infty} \frac{dr}{J(r)} \geq \frac{C_m}{\tau(R)} \frac{1}{R^{m-2}}, \quad \text{for any } R \geq \tilde{R}$$

where C_m is a positive constant only depending on m and $\tau(r)$ satisfies the following conditions

1. $\tau(r)$ is nonincreasing on $(\tilde{R}, +\infty)$ and $\tau(r) \rightarrow 0$ as $r \rightarrow +\infty$;
2. $\tau(r) \geq \max_{r(x)=r} \left\{ \sum_{k=1}^n \tilde{u}_k^2 \right\}$.

Then

$$\int_{B_R(x_0)} \frac{|du|^2}{2} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g \leq C(\tau(R) + \frac{C(u)}{R^{m-2}}) R^{m-2}.$$

\square

5. CONSTANT DIRICHLET BOUNDARY-VALUE PROBLEMS

Definition 5.1. A bounded domain $D \subseteq M$ with C^1 boundary ∂D is called *starlike* if there exists an interior point $x_0 \in D$ such that

$$\left\langle \frac{\partial}{\partial r_{x_0}}, \nu \right\rangle \Big|_{\partial D} \geq 0$$

where ν is the unit outer normal to ∂D , and the vector field $\frac{\partial}{\partial r_{x_0}}$ is the unit vector field such that for any $x \in D \setminus \{x_0\} \cup \partial D$, $\frac{\partial}{\partial r_{x_0}}$ is the unit vector tangent to the unique geodesic joining x_0 to x and pointing away from x_0 .

Theorem 5.1. *Suppose M satisfies the same condition of Theorem 3.1 and $D \subseteq M$ is a bounded starlike domain with C^1 boundary. If $u : (M, g) \rightarrow (R^n, h)$ is a critical point of The p -Ginzburg-Landau energy functional and $u|_{\partial D} \subseteq S^{n-1}$ is constant, then $u|_D$ is constant.*

Proof. Set $X = r \frac{\partial}{\partial r}$, where $r = r_{x_0}$. From the proof of Theorem 3.1, we have

$$(5.1) \quad \int_D \langle S_{u,p}^{GL}, \frac{1}{2} L_X g \rangle dv_g \geq \sigma \int_D \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g .$$

Since $u|_{\partial D} \subseteq S^{n-1}$ is constant, then $|u|^2 = 1$ and for any $\eta \in T(\partial D)$, $du(\eta) = 0$. Thus

$$\begin{aligned} \int_{\partial D} S_{u,p}^{GL}(r \frac{\partial}{\partial r}, \nu) ds_g &= \int_{\partial D} r \left\{ \frac{|du|^p}{p} \langle \frac{\partial}{\partial r}, \nu \rangle - |du|^{p-2} \langle du(\frac{\partial}{\partial r}), du(\nu) \rangle \right\} ds_g \\ &= \int_{\partial D} r \left\{ \frac{|du|^p}{p} \langle \frac{\partial}{\partial r}, \nu \rangle - |du|^{p-2} \langle \frac{\partial}{\partial r}, \nu \rangle |du|^2 \right\} ds_g \\ (5.2) \quad &= \int_{\partial D} r |du|^p \langle \frac{\partial}{\partial r}, \nu \rangle \frac{1-p}{p} ds_g . \end{aligned}$$

Note that D is starlike, by (2.3) and (5.2)

$$(5.3) \quad \int_D \langle S_{GL}^p, \frac{1}{2} L_X g \rangle dv_g \leq 0 .$$

From (5.1) and (5.3), we have

$$\int_D \frac{|du|^p}{p} + \frac{1}{4\epsilon^n} (1 - |u|^2)^2 dv_g = 0 .$$

Therefore u is constant. □

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