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Gradient estimates for Dirac-Harmonic maps with curvature term

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Abstract. In this paper, we derive a gradient estimate and Liouville theorem for Dirac-harmonic maps with curvature term.

1. Introduction

Let (M,g) be a Riemannian spin manifold, denote ΣM the spin bundle on M. Let (N,h) be another Riemannian manifold, $\Phi : M \to N$ a map. On the twisted bundle $\Sigma M \otimes \Phi^{-1}TN$, there is a Hermitian product $\langle \cdot, \cdot \rangle$ and a connection ∇ induced from those on ΣM and $\Phi^{-1}TN$. Denote $\not{D}\Psi := e_i \cdot \nabla_{e_i}\Psi$ the *Dirac operator along the map* Φ, $\forall \Psi \in \Gamma(\Sigma M \otimes \Phi^{-1}TN)$, where $\{e_i, i = 1, 2, \cdots, m\}$ is a local orthonormal frame on M. *Dirac-harmonic maps* ([2, 3]) are critical ponints (Φ, Ψ) of the following functional:

$$L(\Phi, \Psi) = \frac{1}{2}\int_M [\|d\Phi\|^2 + \langle \Psi, \not{D}\Psi \rangle] \,. \tag{1.1}$$

This functional comes from the nonlinear supersymmetric sigma model of quantum field theory (see [9]), the only difference is that in physics, the components of the field Ψ take values in the Grassmannian algebra of infinitely dimension while in (1.1) the components of Ψ are just the usual spinors in spin geometry, leaving the model in the category of geometric variational problems.

The Euler-Lagrange equations for L are given by

$$\begin{aligned} \tau(\Phi) &= \frac{1}{2}\langle \Psi^\alpha, e_i \cdot \Psi^\beta \rangle R^N(\theta_\alpha, \theta_\beta)\Phi_*(e_i) \,, \\ \not{D}\Psi &= 0 \,, \end{aligned} \tag{1.2}$$

where $R^N(X,Y) := [\nabla^N_X, \nabla^N_Y] - \nabla^N_{[X,Y]}$, $\forall X, Y \in \Gamma(TN)$ is the curvature operator of N, $\{\theta_\alpha\}$ a local frame of TN, $\Psi = \Psi^\alpha \otimes \theta_\alpha$ and $\tau(\Phi) := (\nabla^{T^*M \otimes \Phi^{-1}TN}_{e_i} d\Phi)(e_i)$ is the tension field of Φ.

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In [5], the authors derived Liouville theorems for Dirac-harmonic maps, namely, all solutions (Φ, Ψ) of (1.2) from the Euclidean space $\mathbb{R}^n$ $(n \geq 3)$, the hyperbolic space $\mathbb{H}^n$ $(n \geq 3)$ and the Riemannian Schwarzchild space $\mathfrak{S}^n$ $(n \geq 3)$ to any Riemannian manifolds with finite energy must be $(Constant, 0)$.

In fact, the supersymmetric sigma model in superstring theory includes an additional curvature term in (1.1) (see [9]). In [5], the authors introduced the following functional:

$$L_c(\Phi, \Psi) = \frac{1}{2}\int_M [\|d\Phi\|^2 + \langle \Psi, \not{D}\Psi\rangle - \frac{1}{6}R_{\alpha\gamma\beta\zeta}\langle \Psi^\alpha, \Psi^\beta\rangle\langle \Psi^\gamma, \Psi^\zeta\rangle] \ . \tag{1.3}$$

Critical points (Φ, Ψ) of L_c are called *Dirac harmonic maps with curvature term.*

The Euler-Lagrange equations for the action functional L_c are given by

$$\begin{aligned} \tau(\Phi) \ &= \frac{1}{2}\langle \Psi^\alpha, e_i \cdot \Psi^\beta\rangle R^N(\theta_\alpha, \theta_\beta)\Phi_*(e_i) \\ &\quad -\frac{1}{12}\Phi^*(\nabla^{TN}R^N)_{\alpha\beta\gamma\zeta}\langle \Psi^\alpha, \Psi^\gamma\rangle\langle \Psi^\beta, \Psi^\zeta\rangle \ , \\ \not{D}\Psi &= \frac{1}{3}R(\Psi, \Psi)\Psi \ . \end{aligned} \tag{1.4}$$

Due to the new nonlinear terms in the equations, (1.4) is more subtle than (1.2). A Liouville theorem for these kind of maps was also established in [5].

On the other hand, gradient estimates for harmonic functions was developed by S.T. Yau in his seminal paper [14], later, it was extended to the case of harmonic maps by Cheng [6] and Choi [8]. The method of gradient estimates has become a fundamental tool for the existence and uniqueness problems of differential equations on manifolds. In [4], extending Cheng and Choi's results, the authors obtained a gradient estimate and Liouville theorem for Dirac-harmonic maps. Later, Branding [1] proved a gradient estimate for the nonlinear supersymmetric sigma model.

The aim of the present paper is to extending results in [4] to the case of Dirac-harmonic maps with curvature term, we will prove a gradient estimate (Theorem 2.1) with concise and clear dependence on the curvatures of both domain and target manifolds, which enables us to derive a Liouville theorem (Theorem 3.1) as an application.

2. Gradient estimates

In the sequel, we assume that (M, g) is a complete non-compact Riemannian spin manifold. In this section, we derive a gradient estimate for Dirac-harmonnic maps with curvature term from M into a regular ball on arbitrary Riemannian manifold (N, h).

2.1. **Preliminaries.** We first recall the following Weitzenböck formula for a map Φ (see e.g. [11]).

Proposition 2.1. *For a smooth map* $\Phi : M \to N$*,*

$$\begin{aligned} \frac{1}{2}\Delta\|d\Phi\|^2 &= \|\nabla d\Phi\|^2 + \langle \nabla_{e_i}\tau(\Phi), \Phi_*(e_i)\rangle + \langle \Phi_*(Ric^M(e_i)), \Phi_*(e_i)\rangle \\ &\quad -R^N(\Phi_*(e_i), \Phi_*(e_j), \Phi_*(e_i), \Phi_*(e_j)) \ , \end{aligned} \tag{2.1}$$

where $\{e_i\}$ *is a local orthonormal frame on* M*.*

For a harmonic spinor field along a map Φ, we also have the similar Weitzenböck formula (cf. [3], Proposition 3.4).

Proposition 2.2. *Let M and N be Riemannian manifolds, $\Phi : M \to N$, and $\Psi = \Psi^\alpha \otimes \theta_\alpha \in \Gamma(\Sigma M \otimes \Phi^{-1}TN)$. Then*

$$\begin{aligned}\frac{1}{2}\Delta\|\Psi\|^2 = \|\nabla\Psi\|^2 - \langle \not{D}^2\Psi, \Psi\rangle + \frac{1}{4}S_M\|\Psi\|^2 \\ -\frac{1}{2}R_{\alpha\beta\gamma\zeta}\langle e_i \cdot \Psi^\alpha, e_j \cdot \Psi^\beta\rangle \phi_i^\gamma \phi_j^\zeta \,,\end{aligned} \tag{2.2}$$

here $\{e_i\}$ and $\{\theta_\alpha\}$ are the local orthonormal frame on M, N, respectively. S_M is the scalar curvature of M.

Now we recall the following Kato-Yau inequalities in [4], which generalize both the result for harmonic maps in [13] and the result for harmonic spinors in [10].

Proposition 2.3 (Kato-Yau inequality)**.** *Let E be any Dirac bundle on M with dimension m. Then for any cross-section $\Psi \in \Gamma(E)$ and $\delta > 0$, we have*

$$\|\nabla\Psi\|^2 \geq \left(1 + \frac{1}{m-1+\delta}\right)\|\nabla\|\Psi\|\|^2 - \frac{1}{\delta}\|\not{D}\Psi\|^2 \,, \tag{2.3}$$

provided that $\Psi \neq 0$.

In particular, for any map $\Phi : M \to N$ and $\delta > 0$, we have

$$\|\nabla d\Phi\|^2 \geq (1 + \frac{1}{m-1+\delta})\|\nabla\|d\Phi\|\|^2 - \frac{1}{\delta}\|\tau(\Phi)\|^2 \,. \tag{2.4}$$

Lemma 2.1. *Assume that the pair (Φ, Ψ) is a smooth solution of (1.4). Suppose that the Ricci curvature of M satisfies $Ric_M \geq -\kappa$ for some nonnegative constant κ, the sectional curvature sec_N and the curvature tensor R^N of N satisfy $-b_2 \leq sec_N \leq b_1$, $\|\nabla R^N\| \leq b_3$ and $\|\nabla^2 R^N\| \leq b_4$ respectively, where b_i are constants with $b_2 \geq b_1 > 0, b_3 \geq 0, b_4 \geq 0$. Then*

$$\begin{aligned}&|\langle \nabla_{e_i}\tau(\Phi), \Phi_*(e_i)\rangle| \leq b_3\, C(m,n)\|\Psi\|^2\|d\Phi\|^3 \\ &+b_2\, C(m,n)\|\Psi\|^2\|d\Phi\|\|\nabla d\Phi\| + b_2\, C(m,n)\|\Psi\|\|\nabla\Psi\|\|d\Phi\|^2 \\ &+b_4\, C(m,n)\|\Psi\|^4\|d\Phi\|^2 + b_3\, C(m,n)\|\Psi\|^3\|\nabla\Psi\|\|d\Phi\|\end{aligned} \tag{2.5}$$

for some constant $C(m,n) > 0$ depending only on m and n.

Proof. From the first equation in (1.4), we have

$$\begin{aligned}&\langle \nabla_{e_i}\tau(\Phi), \Phi_*(e_i)\rangle = \frac{1}{2}\left\langle \langle \Psi^\alpha, e_k \cdot \Psi^\beta\rangle (\nabla_{\Phi_*(e_i)}R^N)(\theta_\alpha, \theta_\beta)\Phi_*(e_k), \Phi_*(e_i)\right\rangle \\ &+\frac{1}{2}\left\langle \langle \Psi^\alpha, e_k \cdot \Psi^\beta\rangle R^N(\theta_\alpha, \theta_\beta)(\nabla_{e_i}d\Phi)(e_k), \Phi_*(e_i)\right\rangle \\ &+\frac{1}{2}\left\langle \langle \nabla_{e_i}\Psi^\alpha, e_k \cdot \Psi^\beta\rangle R^N(\theta_\alpha, \theta_\beta)\Phi_*(e_k), \Phi_*(e_i)\right\rangle \\ &+\frac{1}{2}\left\langle \langle \Psi^\alpha, \nabla_{e_i}(e_k \cdot \Psi^\beta)\rangle R^N(\theta_\alpha, \theta_\beta)\Phi_*(e_k), \Phi_*(e_i)\right\rangle \\ &-\frac{1}{12}\langle (\nabla_{\Phi_*(e_p)}(\nabla R^N)_{\alpha\beta\gamma\zeta})\langle \Psi^\alpha, \Psi^\gamma\rangle\langle \Psi^\beta, \Psi^\zeta\rangle, \Phi_*(e_p)\rangle \\ &-\frac{1}{3}\langle (\nabla R^N)_{\alpha\beta\gamma\zeta}\langle \Psi^\alpha, \Psi^\gamma\rangle\langle \Psi^\beta, \nabla_{e_p}(\Psi^\zeta)\rangle, \Phi_*(e_p)\rangle \,,\end{aligned}$$

thus (2.5) follows.

□

Lemma 2.2. *If* (Φ, Ψ) *is a smooth solutions of* (1.4)*, then*

$$\|\not{D}^2\Psi\| \leq b_3 C(m,n)\|\Psi\|^3\|d\Phi\| + b_2 C(m,n)\|\Psi\|^2\|\nabla\Psi\| \ , \tag{2.6}$$

for some constant $C(m,n) > 0$ *depending only on* m *and* n*.*

Proof. By the second equation in (1.4), it is easy to see that

$$\|\not{D}^2\Psi\| \leq C(m,n)\left(\|\nabla R\|\|\Psi\|^3\|d\Phi\| + \|R^N\|\|\Psi\|^2\|\nabla\Psi\|\right) \ ,$$

so (2.6) holds.

□

2.2. **Gradient estimates.** Combined with (2.5) and Weitzenböck formula (2.1), one can get that

$$\begin{aligned}\frac{1}{2}\Delta\|d\Phi\|^2 \geq \ & \|\nabla d\Phi\|^2 - b_3 C(m,n)\|\Psi\|^2\|d\Phi\|^3 - b_2 C(m,n)\|\Psi\|^2\|d\Phi\|\|\nabla d\Phi\| \\ & - b_2 C(m,n)\|\Psi\|\|\nabla\Psi\|\|d\Phi\|^2 - b_4 C(m,n)\|\Psi\|^4\|d\Phi\|^2 \\ & - b_3 C(m,n)\|\Psi\|^3\|\nabla\Psi\|\|d\Phi\| - \kappa\|d\Phi\|^2 - (1-\frac{1}{p})b_1\|d\Phi\|^4 \ .\end{aligned}$$

Applying Cauchy-Schwarz inequality, it follows that, for any constant $\delta_1 > 0$ depending only on m, n,

$$\begin{aligned}\frac{1}{2}\Delta\|d\Phi\|^2 \geq \ & (1-\delta_1)\|\nabla d\Phi\|^2 - \frac{b_2^2}{\delta_1}C(m,n)\|\Psi\|^4\|d\Phi\|^2 \\ & - \frac{b_3^2}{b_1}C(m,n)2p(p+1)\|\Psi\|^4\|d\Phi\|^2 \\ & - \frac{b_1}{2p(p+1)}\|d\Phi\|^4 - \frac{b_2^2}{b_1}C(m,n)2p(p+1)\|\Psi\|^2\|\nabla\Psi\|^2 \\ & - \frac{b_1}{2p(p+1)}\|d\Phi\|^4 \\ & - b_4 C(m,n)\|\Psi\|^4\|d\Phi\|^2 - \frac{b_3^2}{b_1}C(m,n)\|\Psi\|^4\|d\Phi\|^2 \\ & - b_1 C(m,n)\|\Psi\|^2\|\nabla\Psi\|^2 \\ & - \kappa\|d\Phi\|^2 - (1-\frac{1}{p})b_1\|d\Phi\|^4 \ .\end{aligned}$$

Since $b_2 \geq b_1 > 0$ and δ_1 depending only on m, n, we obtain

$$\begin{aligned}\frac{1}{2}\Delta\|d\Phi\|^2 \geq \ & (1-\delta_1)\|\nabla d\Phi\|^2 - (b_2^2 + \frac{b_3^2}{b_1} + b_4)C(m,n)\|\Psi\|^4\|d\Phi\|^2 \\ & - \frac{b_2^2}{b_1}C(m,n)\|\Psi\|^2\|\nabla\Psi\|^2 - \kappa\|d\Phi\|^2 - (1-\frac{1}{p+1})b_1\|d\Phi\|^4 \ .\end{aligned} \tag{2.7}$$

By using (2.6) and Weitzenböck formula (2.2), we estimate $\frac{1}{2}\|\Psi\|^4$ as follows:

$$\begin{aligned}\frac{1}{2}\Delta\|\Psi\|^4 &= \|\Psi\|^2\Delta\|\Psi\|^2 + \|\nabla\|\Psi\|^2\|^2\\ &\geq 2\|\Psi\|^2\|\nabla\Psi\|^2 + \|\nabla\|\Psi\|^2\|^2 + \frac{S_M}{2}\|\Psi\|^4 - b_2^2\, C(m,n)\|\Psi\|^4\|d\Phi\|^2\\ &\quad -2\langle \not{D}^2\Psi, \Psi\rangle\|\Psi\|^2\\ &\geq 2\|\Psi\|^2\|\nabla\Psi\|^2 + \|\nabla\|\Psi\|^2\|^2 - \frac{m}{2}\kappa\|\Psi\|^4 - b_2^2\, C(m,n)\|\Psi\|^4\|d\Phi\|^2\\ &\quad -b_3C(m,n)\|\Psi\|^6\|d\Phi\| - b_2C(m,n)\|\Psi\|^5\|\nabla\Psi\| \ .\end{aligned}$$

Applying Cauchy-Schwarz inequality,

$$\begin{aligned}\frac{1}{2}\Delta\|\Psi\|^4 &\geq 2\|\Psi\|^2\|\nabla\Psi\|^2 + \|\nabla\|\Psi\|^2\|^2 - \frac{m}{2}\kappa\|\Psi\|^4\\ &\quad -b_2^2\, C(m,n)\|\Psi\|^4\|d\Phi\|^2\\ &\quad -b_3C(m,n)\|\Psi\|^4\|d\Phi\|^2 - b_3C(m,n)\|\Psi\|^8\\ &\quad -\frac{1}{2}\|\Psi\|^2\|\nabla\Psi\|^2 - b_2^2\, C(m,n)\|\Psi\|^8\\ &= \frac{3}{2}\|\Psi\|^2\|\nabla\Psi\|^2 + \|\nabla\|\Psi\|^2\|^2 - \frac{m}{2}\kappa\|\Psi\|^4\\ &\quad -(b_2^2+b_3)C(m,n)\|\Psi\|^4\|d\Phi\|^2 - (b_3+b_2^2)C(m,n)\|\Psi\|^8 \ .\end{aligned} \tag{2.8}$$

Putting Kato-Yau inequality (2.4) into (2.7), when $\delta = 1$. Set $\delta_1 = \frac{3}{4+4m}$ such that $(1-\delta_1)(1+\frac{1}{m}) = 1 + \frac{1}{4m}$, then

$$\begin{aligned}\frac{1}{2}\Delta\|d\Phi\|^2 &\geq (1+\frac{1}{4m})\|\nabla\|d\Phi\|\|^2 - (b_2^2 + \frac{b_3^2}{b_1} + b_4)C(m,n)\|\Psi\|^4\|d\Phi\|^2\\ &\quad -b_3^2C(m,n)\|\Psi\|^8 - \frac{b_2^2}{b_1}\, C(m,n)\|\Psi\|^2\|\nabla\Psi\|^2\\ &\quad -\kappa\|d\Phi\|^2 - (1-\frac{1}{p+1})b_1\|d\Phi\|^4 \ .\end{aligned} \tag{2.9}$$

As for $\frac{1}{2}\Delta\|\Psi\|^4$, because of $\|\nabla\Psi\|^2 \geq \frac{1}{m}\|\nabla\|\Psi\|\|^2$,

$$\begin{aligned}\frac{1}{2}\Delta\|\Psi\|^4 &\geq (1+\frac{1}{4m})\|\nabla\|\Psi\|^2\|^2 + \frac{1}{2}\|\Psi\|^2\|\nabla\Psi\|^2 - \frac{m}{2}\kappa\|\Psi\|^4\\ &\quad -(b_2^2+b_3)C(m,n)\|\Psi\|^4\|d\Phi\|^2 - (b_3+b_2^2)C(m,n)\|\Psi\|^8 \ .\end{aligned} \tag{2.10}$$

Now we fix the constant $C_0 = C(m,n)$ in the above inequalities (2.9) (2.10). Set

$$C_1 = \sqrt{\frac{2b_2^2}{b_1}\, C_0} \quad , \quad e_1 = C_1\|\Psi\|^2 \quad , \quad e := \sqrt{\|d\Phi\|^2 + e_1^2} = \sqrt{\|d\Phi\|^2 + C_1^2\|\Psi\|^4} \ ,$$

then

$$\begin{aligned}\frac{1}{2}\Delta e^2 &= \frac{1}{2}\Delta\|d\Phi\|^2 + \frac{1}{2}\Delta C_1^2\|\Psi\|^4 \\ &\geq \big(1+\frac{1}{4m}\big)\|\nabla\|d\Phi\|\|^2 + \big(1+\frac{1}{4m}\big)C_1^2\|\nabla\|\Psi\|^2\|^2 \\ &\quad -\big(b_3^2 + \frac{b_2^2 b_3}{b_1} + \frac{b_2^4}{b_1}\big)C(m,n)\|\Psi\|^8 \\ &\quad -\big(b_2^2 + \frac{b_3^2}{b_1} + b_4 + \frac{b_2^4}{b_1} + \frac{b_2^2 b_3}{b_1}\big)C(m,n)\|\Psi\|^4\|d\Phi\|^2 \\ &\quad -\kappa\|d\Phi\|^2 - \frac{m}{2}\kappa\|\Psi\|^4 - \big(1-\frac{1}{p+1}\big)b_1\|d\Phi\|^4 \\ &\geq \big(1+\frac{1}{4m}\big)(\|\nabla\|d\Phi\|\|^2 + C_1^2\|\nabla\|\Psi\|^2\|^2) \\ &\quad -\big(b_2^2 + \frac{b_3^2}{b_1} + b_4 + \frac{b_2^4}{b_1} + \frac{b_2^2 b_3}{b_1} + b_3^2\big)C(m,n)\|\Psi\|^4 e^2 \\ &\quad -\big(1+\frac{m}{2}\big)\kappa e^2 - \big(1-\frac{1}{p+1}\big)b_1\|d\Phi\|^2 e^2 \ .\end{aligned}$$

Since $\|\nabla e\|^2 \leq \|\nabla\|d\Phi\|\|^2 + \|\nabla e_1\|^2$, we have

$$\begin{aligned}\Delta e &= e^{-1}\big(\frac{1}{2}\Delta e^2 - \|\nabla e\|^2\big) \\ &\geq \frac{1}{4m}\frac{\|\nabla e\|^2}{e} - \big(b_2^2 + \frac{b_3^2}{b_1} + b_4 + \frac{b_2^4}{b_1} + \frac{b_2^2 b_3}{b_1} + b_3^2\big)C(m,n)\|\Psi\|^4 e \\ &\quad -\big(1+\frac{m}{2}\big)\kappa e - \big(1-\frac{1}{p+1}\big)b_1\|d\Phi\|^2 e \ .\end{aligned} \tag{2.11}$$

Let $r(x)$ be the distance function from the point x_0 in M. Define a function on the geodesic ball $B_a(x_0)$ by

$$F = (a^2 - r^2)f = (a^2 - r^2)\frac{e}{B\circ\Phi} \ ,$$

where $B : N \to \mathbb{R}^+$ is a function which we will be defined later. Clearly, the function F vanishes on the boundary $B_a(x_0)$, then F must achieve its maximum at an interior point x^* when $e \neq 0$. In addition, the distance function r is a smooth near the point x^*, therefore we may assume that the distance function r is twice differentiable near the point x^* (cf. [7]). By the maximum principle, we have

$$\nabla F(x^*) = 0 \ , \tag{2.12}$$

$$\Delta F(x^*) \leq 0 \ . \tag{2.13}$$

Now we recall the Laplacian Comparison Theorem [12], for some constant $C(m) >$ depending only on m,

$$\Delta r^2 \leq C(m)(1+\sqrt{\kappa}\, r) \ . \tag{2.14}$$

Lemma 2.3. *At the point x^*, we have the following estimate*

$$
\begin{aligned}
&\frac{1}{4m}\frac{\|\nabla(B\circ\Phi)\|^2}{(B\circ\Phi)^2}-\frac{\Delta(B\circ\Phi)}{B\circ\Phi}\\
&-4r\frac{\|\nabla(B\circ\Phi)\|}{(a^2-r^2)(B\circ\Phi)}-\tilde{b}\,C(m,n)\|\Psi\|^4\\
&-\big(1-\frac{1}{p+1}\big)b_1\|d\Phi\|^2-\big(1+\frac{m}{2}\big)\kappa\\
&-\frac{C(m)(1+\sqrt{\kappa}\,r)}{a^2-r^2}-\frac{8r^2}{(a^2-r^2)^2}\le 0\,.
\end{aligned}
\tag{2.15}
$$

Here $\tilde{b}=b_2^2+\dfrac{b_3^2}{b_1}+b_4+\dfrac{b_2^4}{b_1}+\dfrac{b_2^2b_3}{b_1}+b_3^2$.

Proof. We first have

$$
\nabla f=\frac{\nabla e}{B\circ\Phi}-\frac{e\nabla(B\circ\Phi)}{(B\circ\Phi)^2}\,, \tag{2.16}
$$

and

$$
\Delta f=\frac{\Delta e}{B\circ\Phi}-\frac{f\Delta(B\circ\Phi)}{B\circ\Phi}-\frac{2\langle\nabla(B\circ\Phi),\nabla f\rangle}{B\circ\Phi}\,. \tag{2.17}
$$

By using (2.11) and setting $\mu=\dfrac{1}{4m}$, $\tilde{b}=b_2^2+\dfrac{b_3^2}{b_1}+b_4+\dfrac{b_2^4}{b_1}+\dfrac{b_2^2b_3}{b_1}+b_3^2$, we get

$$
\begin{aligned}
\Delta f\ge{}&\mu\frac{\|\nabla e\|^2}{e(B\circ\Phi)}-\tilde{b}\,C(m,n)\|\Psi\|^4f-\big(1+\frac{m}{2}\big)\kappa f\\
&-\big(1-\frac{1}{p+1}\big)b_1\|d\Phi\|^2f-\frac{f\Delta(B\circ\Phi)}{B\circ\Phi}-\frac{2\langle\nabla(B\circ\Phi),\nabla f\rangle}{B\circ\Phi}\,.
\end{aligned}
\tag{2.18}
$$

Applying Cauchy-Schwarz inequality, we also have that

$$
\begin{aligned}
-\frac{2\langle\nabla(B\circ\Phi),\nabla f\rangle}{B\circ\Phi}&=-(2-2\mu)\frac{\langle\nabla(B\circ\Phi),\nabla f\rangle}{B\circ\Phi}\\
&\quad-2\mu\frac{\langle\nabla(B\circ\Phi),\nabla f\rangle}{B\circ\Phi}\\
&=-(2-2\mu)\frac{\langle\nabla(B\circ\Phi),\nabla f\rangle}{B\circ\Phi}-2\mu\frac{\langle\nabla(B\circ\Phi),\nabla e\rangle}{(B\circ\Phi)^2}\\
&\quad+2\mu\frac{f\|\nabla(B\circ\Phi)\|^2}{(B\circ\Phi)^2}\\
&\ge-(2-2\mu)\frac{\langle\nabla(B\circ\Phi),\nabla f\rangle}{B\circ\Phi}-\mu\frac{\|\nabla e\|^2}{e(B\circ\Phi)}\\
&\quad+\mu\frac{f\|\nabla(B\circ\Phi)\|^2}{(B\circ\Phi)^2}\,.
\end{aligned}
\tag{2.19}
$$

So we have

$$
\begin{aligned}
\frac{\Delta f}{f}\ge{}&-(2-2\mu)\frac{\langle\nabla(B\circ\Phi),\nabla f\rangle}{B\circ\Phi}+\mu\frac{\|\nabla(B\circ\Phi)\|^2}{(B\circ\Phi)^2}-\frac{\Delta(B\circ\Phi)}{B\circ\Phi}\\
&-\tilde{b}\,C(m,n)\|\Psi\|^4-\big(1+\frac{m}{2}\big)\kappa-\big(1-\frac{1}{p+1}\big)b_1\|d\Phi\|^2\,.
\end{aligned}
$$

At the point x^*, since F achieves its maximum, as a consequence $\nabla F(x^*) = 0$ and $\Delta F(x^*) \le 0$. By the definition of F, one has that

$$\frac{\nabla r^2}{a^2 - r^2} = \frac{\nabla f}{f} ,$$

and

$$-\frac{\Delta r^2}{a^2 - r^2} + \frac{\Delta f}{f} - \frac{2\langle \nabla r^2, \nabla f\rangle}{f(a^2 - r^2)} \le 0 .$$

It follows that

$$\frac{\Delta f}{f} - \frac{\Delta r^2}{a^2 - r^2} - \frac{2\|\nabla r^2\|^2}{(a^2 - r^2)^2} \le 0 .$$

Using $\|\nabla r^2\| = 2r$ and (2.14), we then have

$$0 \ge \frac{\Delta f}{f} - \frac{C(m)(1 + \sqrt{\kappa}\, r)}{a^2 - r^2} - \frac{8r^2}{(a^2 - r^2)^2} ,$$

i.e.

$$\begin{aligned} 0 \ge& -(2-2\mu)\frac{\langle \nabla(B\circ\Phi), \nabla f\rangle}{B\circ\Phi} + \mu\frac{\|\nabla(B\circ\Phi)\|^2}{(B\circ\Phi)^2} - \frac{\Delta(B\circ\Phi)}{B\circ\Phi} \\ & -\tilde{b}C(m,n)\|\Psi\|^4 - (1+\frac{m}{2})\kappa - (1-\frac{1}{p+1})b_1\|d\Phi\|^2 \\ & -\frac{C(m)(1+\sqrt{\kappa}\, r)}{a^2 - r^2} - \frac{8r^2}{(a^2-r^2)^2} . \end{aligned}$$

Since

$$\begin{aligned} -(2-2\mu)\frac{\langle \nabla(B\circ\Phi), \nabla f\rangle}{f(B\circ\Phi)} &= -(2-2\mu)2r\frac{\langle \nabla(B\circ\Phi), \nabla r\rangle}{(a^2-r^2)(B\circ\Phi)} \\ &\ge -(2-2\mu)2r\frac{\|\nabla(B\circ\Phi)\|}{(a^2-r^2)(B\circ\Phi)} , \end{aligned}$$

we conclude that

$$\begin{aligned} \mu & \frac{\|\nabla(B\circ\Phi)\|^2}{(B\circ\Phi)^2} - \frac{\Delta(B\circ\Phi)}{B\circ\Phi} - 4r\frac{\|\nabla(B\circ\Phi)\|}{(a^2-r^2)(B\circ\Phi)} - \tilde{b}\, C(m,n)\|\Psi\|^4 \\ & -(1-\frac{1}{p+1})b_1\|d\Phi\|^2 - \big(1+\frac{m}{2}\big)\kappa - \frac{C(m)(1+\sqrt{\kappa}\, r)}{a^2-r^2} - \frac{8r^2}{(a^2-r^2)^2} \le 0 . \end{aligned}$$

□

Theorem 2.1 (Gradient Estimate). *Suppose the Ricci curvature of M satisfies $Ric_M \ge -\kappa$ for some nonnegative constant κ, the sectional curvature sec_N and curvature tensor R^N satisfy $-b_2 \le sec_N \le b_1, \|\nabla R^N\| \le b_3$ and $\|\nabla^2 R^N\| \le b_4$ respectively, where b_i are constants with $b_2 \ge b_1 > 0, b_3 \ge 0, b_4 \ge 0$. If (Φ, Ψ) is a solution of (1.4), $\Phi : M \to B_{y_0}(R) \subset N^n, R < \pi/(2\sqrt{b_1})$, then for any $x_0 \in M$ and any positive constant a, we have*

$$\sup_{B_{x_0}(\frac{a}{2})} \|d\Phi\| \le \frac{C(m,n)}{\sqrt{b_1}\,\cos^2(\sqrt{b_1}\,R)} \left(\frac{1 + a\sqrt{\kappa}}{a} + \sqrt{\frac{b}{b_1}} \sup_{B_{x_0}(a)} \|\Psi\|^2 \right) . \tag{2.20}$$

Here $b = b_1 b_2^2 + b_3^2 + b_4 b_1 + b_2^4 + b_2^2 b_3 + b_3^2 b_1 + b_1^{3/2} b_3$, and $C(m,n)$ is a constant depending only on the dimensions m and n.

Proof. Choose $B(y) = \sqrt{b_1}\cos\big(\sqrt{b_1}\,\rho(y)\big)$ in the function $f = \dfrac{e}{B\circ\Phi}$, then

$$\begin{aligned}
\Delta(B\circ\phi) &\le (\operatorname{Hess} B)\circ\Phi\|d\Phi\|^2 + \|(\nabla B)\circ\Phi\|\,\|\tau(\Phi)\| \\
&\le -b_1^{3/2}\cos(\sqrt{b_1}\,\rho\circ\Phi)\,\|d\Phi\|^2 + b_1\,C(m,n)(b_2\|\Psi\|^2\|d\Phi\| + b_3\|\Psi\|^4) \\
&\le -b_1^{3/2}\cos(\sqrt{b_1}\,\rho\circ\Phi)\,\|d\Phi\|^2 + \frac{1}{(p+1)(p+2)}b_1 B\circ\Phi\|d\Phi\|^2 \\
&\quad + \frac{1}{B\circ\Phi}(p+1)(p+2)b_1b_2^2\,C(m,n)\|\Psi\|^4 + b_1b_3C(m,n)\|\Psi\|^4\ .
\end{aligned}$$

Putting this inequalities into (2.15), then we have

$$\begin{aligned}
&\frac{b_1}{p+2}\|d\Phi\|^2 - \frac{4rb_1}{(a^2-r^2)B\circ\Phi}\|d\Phi\| \\
&\le \tilde{b}\,C(m,n)\|\Psi\|^4 + \big(1+\frac{m}{2}\big)\kappa \\
&\quad + \frac{C(m)(1+\sqrt{\kappa}\,r)}{a^2-r^2} + \frac{8r^2}{(a^2-r^2)^2} \\
&\quad + \frac{b_1b_2^2}{(B\circ\Phi)^2}C(m,n)\|\Psi\|^4 + \frac{b_1b_3}{B\circ\Phi}C(m,n)\|\Psi\|^4 \\
&\le \big(1+\frac{m}{2}\big)\kappa + \frac{C(m)(1+\sqrt{\kappa}\,r)}{a^2-r^2} + \frac{8r^2}{(a^2-r^2)^2} \\
&\quad + \frac{1}{(B\circ\Phi)^2}(b_1b_2^2 + b_1^{3/2}b_3 + \tilde{b}b_1)C(m,n)\|\Psi\|^4\ .
\end{aligned}$$

Namely,

$$\begin{aligned}
\frac{b_1}{p+2}e^2 - \frac{4rb_1}{(a^2-r^2)B\circ\Phi}e &\le \big(1+\frac{m}{2}\big)\kappa + \frac{C(m)(1+\sqrt{\kappa}\,r)}{a^2-r^2} + \frac{8r^2}{(a^2-r^2)^2} \\
&\quad + \frac{1}{(B\circ\Phi)^2}(b_1b_2^2 + b_1^{3/2}b_3 + \tilde{b}b_1) \\
&\quad \times C(m,n)\|\Psi\|^4\ .
\end{aligned}$$

From the definition of F, we know that

$$\begin{aligned}
&\frac{b_1(B\circ\Phi)^2}{p+2}F^2(x^*) - 4rb_1F(x^*) \\
&\le \left(\big(1+\frac{m}{2}\big)\kappa + \frac{C(m)(1+\sqrt{\kappa}\,r)}{a^2-r^2} + \frac{8r^2}{(a^2-r^2)^2}\right. \\
&\quad \left. + \frac{1}{(B\circ\Phi)^2}(b_1b_2^2 + b_1^{3/2}b_3 + \tilde{b}b_1)C(m,n)\|\Psi\|^4\right)(a^2-r^2)^2\ .
\end{aligned} \tag{2.21}$$

For the RHS of (2.21), denote $b = b_1b_2^2 + b_1^{3/2}b_3 + \tilde{b}b_1$ and

$$\begin{aligned}
I := &\left(\big(1+\frac{m}{2}\big)\kappa + \frac{C(m)(1+\sqrt{\kappa}\,r)}{a^2-r^2} + \frac{8r^2}{(a^2-r^2)^2} + \frac{1}{(B\circ\Phi)^2}b\,C(m,n)\|\Psi\|^4\right) \\
&\times (a^2-r^2)^2\ .
\end{aligned}$$

We have the following estimate

$$I \le a^2\left(\left((1+\frac{m}{2})\kappa+\frac{1}{(B\circ\Phi)^2}C(m,n)b\|\Psi\|^4\right)a^2+C(m)(1+\sqrt{\kappa}\,a)+8\right)$$
$$\le \frac{C(m,n)\,b_1}{(B\circ\Phi)^2}a^2\left(1+\sqrt{\kappa}\,a+\sqrt{\frac{b}{b_1}}\,\|\Psi\|^2a\right)^2.$$

It is elementary that if $Ax^2-Bx-C\le 0$ with A,B,C all positive, then

$$x\le\frac{B}{A}+\sqrt{\frac{C}{A}}\,.$$

So we can conclude that

$$F(x^*)\le\frac{C(m,n)a}{b_1\cos^2(\sqrt{b_1}\,R)}\left(1+\sqrt{\kappa}\,a+\sqrt{\frac{b}{b_1}}\sup_{B_{x_0}(a)}\|\Psi\|^2a\right).$$

This proves Theorem 2.1.

□

3. Liouville Theorem

Now using the gradient estimate, we can get a Liouville type theorem for Dirac-harmonic maps with curvature term.

Theorem 3.1 (Liouville Theorem)**.** *Assume that M is complete with nonnegative Ricci curvature and the scalar curvature is bounded below by a positive constant ϵ, suppose the sectional curvature sec_N and curvature tensor R^N of N satisfy $-b_2\le sec_N\le b_1$ and $\|\nabla R^N\|\le b_3, \|\nabla^2R^N\|\le b_4$ respectively, where b_i are constants with $b_2\ge b_1>0, b_3, b_4\ge 0$. Then there is a constant $\delta>0$ such that for any smooth solution (Φ,Ψ) of (1.4) satisfying $\Phi(M)\subset B_{y_0}(R)\subset N, R<\pi/(2\sqrt{b_1})$ and $\|\Psi\|<\delta$, we have $\Phi\equiv$ constant and $\Psi\equiv 0$.*

Proof. By using Weitzenböck formula (2.2) and (2.6), we can get that

$$\begin{aligned}\frac{1}{2}\Delta\|\Psi\|^2 \ge{}& \|\nabla\Psi\|^2+\frac{\epsilon}{4}\|\Psi\|^2-b_2^2C(m,n)\|\Psi\|^2\|d\Phi\|^2-b_3C(m,n)\|\Psi\|^4\|d\Phi\|\\ &-b_2C(m,n)\|\Psi\|^3\|\nabla\Psi\|\\ \ge{}& \|\nabla\Psi\|^2+\frac{\epsilon}{4}\|\Psi\|^2-b_2^2C(m,n)\|\Psi\|^2\|d\Phi\|^2-b_3C(m,n)\|\Psi\|^4\|d\Phi\|\\ &-\frac{1}{2}\|\nabla\Psi\|^2-b_2^2C(m,n)\|\Psi\|^6\\ \ge{}& \Big(\frac{\epsilon}{4}-b_2^2C(m,n)\|d\Phi\|^2-b_3C(m,n)\|\Psi\|^2\|d\Phi\|\\ &-b_2^2C(m,n)\|\Psi\|^4\Big)\|\Psi\|^2.\end{aligned}$$

Here $C(m,n)$ is a positive constant depending only on m,n. It is obvious that this theorem is valid when M is compact. Now we assume that M is non-compact. Suppose $\|\Psi\|<\delta$, then according to Theorem 2.1,

$$\|d\Phi\|\le\frac{C(m,n)\sqrt{b}\,\delta^2}{b_1\cos^2(\sqrt{b_1}\,R)}\,.$$

Therefore,

$$\frac{1}{2}\Delta\|\Psi\|^2 \geq \left(\frac{\epsilon}{4} - \frac{bb_2^2C(m,n)}{b_1^2\cos^4(\sqrt{b_1}\,R)}\delta^4 - \frac{\sqrt{b}\,b_3C(m,n)}{b_1\cos^2(\sqrt{b_1}\,R)}\delta^4 - b_2^2C(m,n)\delta^4\right)\|\Psi\|^2 .$$

If we choose δ such that

$$\frac{\epsilon}{4} - \frac{bb_2^2C(m,n)}{b_1^2\cos^4(\sqrt{b_1}\,R)}\delta^4 - \frac{\sqrt{b}\,b_3C(m,n)}{b_1\cos^2(\sqrt{b_1}\,R)}\delta^4 - b_2^2C(m,n)\delta^4 = 0 ,$$

then $\|\Psi\|^2$ is a bounded subharmonic function on M.

Now for any positive number c, let $u = (\|\Psi\|^2 + c)^{-1/2}$, then

$$\begin{aligned}\Delta u &= -\frac{1}{2}(\|\Psi\|^2 + c)^{-3/2}\Delta\|\Psi\|^2 + \frac{3}{4}(\|\Psi\|^2 + c)^{-5/2}\|\nabla\|\Psi\|^2\|^2 \\ &= -\frac{1}{2}(\|\Psi\|^2 + c)^{-3/2}\Delta\|\Psi\|^2 + 3(\|\Psi\|^2 + c)^{1/2}\|\nabla u\|^2 \\ &\leq -\frac{C_0}{2}(\|\Psi\|^2 + c)^{-3/2}\|\Psi\|^2 + 3(\|\Psi\|^2 + c)^{1/2}\|\nabla u\|^2 .\end{aligned} \tag{3.1}$$

Since the Ricci curvature of M is nonnegative, the Omori-Yau maximum principle holds (cf. [7, 14]), that is, for any $\eta > 0$, there exists a point $p \in M$ such that at p,

$$u < \inf u + \eta \quad , \quad \|\nabla u\| < \eta \quad , \quad \Delta u > -\eta .$$

It follows from $\|\Psi\| \leq \delta$ and (3.1) that

$$\frac{C_0}{2}\|\Psi\|^2 < \eta\left(\inf(\|\Psi\|^2 + c)^{-\frac{1}{2}} + 4\eta\right)(\|\Psi\|^2 + c)^2 < \eta\left(\delta^{-1} + 4\eta\right)(\delta^2 + c)^2 .$$

Let $\eta \to 0$, we obtain $\sup\|\Psi\| = 0$.

Since Theorem 2.1, we have $\|d\Phi\| = 0$. Therefore Φ is a constant.

□

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