Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 16(2023), pp. 35 – 55.

Banach manifolds of weights and quantization of mechanical systems whose phase space is a complex manifold

Sorin DRAGOMIR[1]

Abstract. Let $\Omega \subset \mathbb{C}^n$ be an open set. A Lebesgue measurable function $\gamma : \Omega \to (0, +\infty)$ is a *weight* on Ω. The set $W(\Omega)$ of all weights on Ω is an infinite dimensional Banach manifold modeled on $L^\infty(\Omega)$. Let $L^2H(\Omega, \gamma)$ be the space of all holomorphic functions in $L^2(\Omega, \gamma)$. A weight $\gamma \in W(\Omega)$ is *admissible* if i) the evaluation functional $\delta_z : L^2H(\Omega, \gamma) \to \mathbb{C}$, $\delta_z(f) = f(z)$, is continuous for any $z \in \Omega$, and ii) $L^2H(\Omega, \gamma)$ is a closed subspace of $L^2(\Omega, \gamma)$. The set $AW(\Omega)$ of admissible weights on Ω is an open subset in $W(\Omega)$. To every admissible weight $\gamma \in AW(\Omega)$ one associates a kernel function $K_\gamma(z, \zeta)$ organizing $L^2H(\Omega, \gamma)$ as a RKH space (cf. [2]). The interest in weighted kernels comes from quantization theory, for given a mechanical system whose phase space is Ω (or more generally a complex manifold admitting globally defined Kähler metrics) one may quantize classical states $z \in \Omega$ (besides from quantizing observables) by building an embedding

$$\Omega \hookrightarrow \mathbb{CP}(\mathcal{M}) \;, \tag{1}$$

$$\mathcal{M} = \Big\{ \varphi \in H^0\big(\Omega, \mathcal{O}\big(T^{*(n,0)}(\Omega) \otimes E\big)\big) : \langle s, s\rangle < \infty \Big\} \;,$$

$$\langle \varphi, \psi \rangle = i^{n^2} \int_\Omega H^*(\varphi, \psi) \quad , \quad \varphi, \psi \in \mathcal{M} \;.$$

Here $E = \Omega \times \mathbb{C}$ (the trivial complex line bundle). Using the embedding (1) one can (cf. [35]) calculate the transition probability amplitude from one point of Ω to another, and actually provide the interpretation of the normalized reproducing kernel function as the transition probability amplitude between two points of the complex phase space Ω. The above interpretation is possible when the holomorphic and metric structures of the line bundle $E \to \Omega$ are tied by the requirement that the weight $\gamma \in AW(\Omega)$ satisfies the complex Monge-Ampère equation

$$\det\Big[\frac{\partial^2 \gamma}{\partial z_j \partial \overline{z}_k}(z)\Big] = (-1)^{n(n+1)/2} C \, \frac{1}{n!} \, \gamma(z) K_\gamma(z, z) \;.$$

Let $\Omega = \{\varphi < 0\} \subset \mathbb{C}^n$ be a smoothly bounded strictly pseudoconvex domain. A notable class of admissible weights is $\gamma_m(z) = |\varphi|^m$, $m \in \{0, 1, 2, \cdots\}$. Let $K_{\gamma_m}(\zeta, z)$ be the reproducing kernel for $L^2H(\Omega, \gamma_m)$. By a result of M.M. Peloso (cf. [38])

$$K_{\gamma_m}(\zeta, z) = C_\Omega \big|\nabla\varphi(z)\big|^2 \cdot \det L_\varphi(z) \cdot \Psi(\zeta, z)^{-(n+1+m)} + E(\zeta, z) \;, \tag{2}$$

$$E \in C^\infty\big(\overline{\Omega} \times \overline{\Omega} \setminus \Delta\big) \;,$$

[1]S. Dragomir, Università degli Studi della Basilicata, Dipartimento di Matematica, Informatica ed Economia, Via dell'Ateneo Lucano 10, 85100 Potenza, Italy; `sorin.dragomir@unibas.it`

Lecture at the Conference *Real and complex manifolds. The mathematical heritage of Edoardo Vesentini*, Pisa, June 29 to July 2, 2021.

Keywords. Weighted Bergman kernel, quantization, Monge-Ampère equation.

AMS Subject Classification. 53C21, 53C55, 32Q40.

$$\big|E(\zeta,z)\big| \le C'_\Omega \big|\Psi(\zeta,z)\big|^{-(n+1+m)+1/2}\big|\log|\Psi(\zeta,z)|\big| \; .$$

For $m = 0$ this is Fefferman's asymptotic expansion formula for the ordinary Bergman kernel, and Peloso recovers that for the points of the curve

$$C : (-1, +\infty\big) \to W(\Omega) \; ,$$

$$C(\alpha) = \big|\varphi\big|^\alpha \in AW(\Omega) \quad , \quad \alpha > -1 \; , \tag{3}$$

corresponding to the integer values of the parameter. Extending (2) to all weights $\gamma \in AW(\Omega)$ is so far an open problem. By a result in [5] the curve (3) is discontinuous and every point of C is an isolated point in $W(\Omega)$. The result may be looked at as a measure of the amount of job [deriving an asymptotic expansion formula for $K_\gamma(z,\,\zeta)$] left unsolved. We report on results extending (2) to ampler classes of weights (cf. [5]). There are significant classes of admissible weights going back as far as the more romantic times of the work by G. Cimmino (cf. [13]) on the Dirichlet problem with L^2 boundary data, and the classical work by A. Andreotti & E. Vesentini (cf. [1]) who proved Carleman type estimates [to the purpose of establishing vanishing results for the cohomology with compact supports $H^q_k\big(\Omega, \Omega^p(E)\big) = 0$] in which admissible weights spring from the (many possible) choices of Hermitian metrics on E.

Acknowledgements.

The Author expresses his gratitude to the organizers of the Conference *Real and complex manifolds. The mathematical heritage of Edoardo Vesentini*, Pisa, June 29 to July 2, 2021, for the invitation to give a lecture. The Author was very much influenced by the scientific creation of Edoardo Vesentini, although he didn't have the privilege of directly working with the great Italian scientist.

1. Admissible weights, weighted Bergman kernels

Let $\Omega \subset \mathbb{C}^n$ be a domain. Let $W(\Omega)$ denote the set of all Lebesgue measurable functions $\gamma : \Omega \to (0, +\infty)$. An element $\gamma \in W(\Omega)$ is a *weight* on Ω. Two weights equal almost everywhere are identified. Let $L^2(\Omega, \gamma)$ be the space of all Lebesgue measurable functions $f : \Omega \to \mathbb{C}$ such that

$$\int_\Omega |f(\zeta)|^2\, \gamma(\zeta)\; d\mu(\zeta) < \infty$$

where μ is the Lebesgue measure on $\mathbb{R}^{2n}$. Then $L^2(\Omega, \gamma)$ is a separable Hilbert space with the L^2 inner product

$$(f, g)_\gamma = \int_\Omega f(\zeta)\,\overline{g(\zeta)}\gamma(\zeta)\; d\mu(\zeta) \; .$$

Let $L^2H(\Omega, \gamma)$ be the space of all holomorphic functions in $L^2(\Omega)$. A weight $\gamma \in W(\Omega)$ is *admissible* if

i) For any $z \in \Omega$ the evaluation functional

$$\delta_z : L^2H(\Omega, \gamma) \to \mathbb{C} \quad , \quad \delta_z(f) = f(z) \; ,$$

is continuous, and

ii) $L^2H(\Omega, \gamma)$ is a closed subspace of $L^2(\Omega, \gamma)$.

Let $AW(\Omega)$ be the set of all admissible weights on Ω. We shall shortly see that $W(\Omega)$ may be organized as a Banach manifold modeled on $L^\infty(\Omega)$ and then $AW(\Omega)$ is an open subset. If $\gamma \in AW(\Omega)$ then $L^2H(\Omega,\gamma)$ is a Hilbert space and $\delta_z : L^2H(\Omega,\gamma) \to \mathbb{C}$ is continuous, hence (by the Riesz representation theorem) there is a unique $k_{z,\gamma} \in L^2H(\Omega,\gamma)$ such that

$$\delta_z(f) = \left(f, k_{z,\gamma}\right)_\gamma \quad , \quad \forall\, f \in L^2H(\Omega,\gamma)\ ,\ \ \forall\, z \in \Omega\ ,$$

or

$$f(z) = \int_\Omega K_\gamma(z,\zeta)\, f(\zeta)\, \gamma(\zeta)\, d\mu(\zeta)$$

where

$$K_\gamma : \Omega \times \Omega \to \mathbb{C} \quad , \quad K_\gamma(z,\zeta) = \overline{k_{z,\gamma}} \quad , \quad z,\zeta \in \Omega\ ,$$

the *weighted Bergman kernel* of weight γ, or the γ-*Bergman kernel*, of Ω. If $\gamma(\zeta) \equiv 1$ then $K(z,\zeta) = K_1(z,\zeta)$ is the ordinary Bergman kernel of Ω, as discovered by S. Bergman (cf. [9]).

Let E be a set and let $\mathcal{F}(E)$ be the set of all complex-valued functions $f : E \to \mathbb{C}$. A complex Hilbert space H is a *reproducing kernel Hilbert space* (a RKH space) if $H \subset \mathcal{F}(E)$ for some $E \neq \emptyset$ and the evaluation functional $\delta_x : H \to \mathbb{C}$, $\delta_x(f) = f(z)$, is continuous for every $x \in E$. Once again by the Riesz representation theorem there is a unique $k_x \in H$ such that $\delta_x(f) = (f, k_x)_H$ for any $f \in H$ and the function $k : E \times E \to \mathbb{C}$, $k(x,y) = \overline{k_x(y)}$ is the *reproducing kernel* of H. Hence $L^2H(\Omega,\gamma)$ is a RKH space and K_γ is its reproducing kernel. N. Aronszajn developed (cf. [2]) a general theory of RKH spaces. The work [2] was published in 1950 yet it appears to have been written previously, in the rather difficult second world war conditions[2]. However, the notion of a reproducing kernel is much older and was perhaps first introduced by the famous Polish mathematician S. Zaremba in connection with his work (cf. [42]) on boundary value problems for harmonic and biharmonic functions.

The main properties of weighted Bergman kernels, *versus* the ordinary Bergman kernel $K(z,\zeta)$, were investigated by Z. Pasternak-Winiarski (cf. [36] and [37]):

i) For any complete orthonormal system
$$\{\phi_\nu\}_{\nu \geq 0} \subset L^2H(\Omega,\gamma)$$
the series $\sum_{\nu \geq 0} \phi_\nu(z)\, \overline{\phi_\nu(\zeta)}$ converges uniformly on any compact subset of $\Omega \times \Omega$ and its sum is
$$\sum_{\nu=0}^{\infty} \phi_\nu(z)\, \overline{\phi(\zeta)} = K_\gamma(z,\zeta)\ .$$

ii) $K_\gamma(\zeta,z) = \overline{K_\gamma(z,\zeta)}$ for any $z,\zeta \in \Omega$.

iii) $K_\gamma(z,\zeta)$ is holomorphic in z and anti-holomorphic in ζ.

iv) K_γ is real analytic.

[2]Nachman Aronszajn (26 July 1907 - 5 February 1980) was a Polish American mathematician, and an Ashkenazi Jew. **A.** got a degree in mathematics in 1930, from the University of Warsaw, under the supervision of Stefan Mazurkiewicz, and a Ph.D. in mathematics in 1935, from Paris University, with Maurice Frchet as an advisor. The mentioned work [2] appeared while **A.** was on the Oklahoma A & M faculty. The civil views of **A.** were not amended by his religious background, for **A.** moved to the University of Kansas in 1951 with his colleague Ainsley Diamond after Diamond, a Quaker, was fired for refusing to sign a newly instituted loyalty oath.

v) If $P_\gamma : L^2(\Omega,\gamma) \to L^2 H(\Omega,\gamma)$ is the L^2 orthogonal projection then

$$(P_\gamma f)(z) = \int_\Omega K_\gamma(z,\zeta)\, f(\zeta)\, d\mu(\zeta)$$

for any $f \in L^2(\Omega,\gamma)$ and $z \in \Omega$.

Example 1. (*Admissible weights on* $\mathbb{B}^1$) Let $\Omega = \mathbb{B}^1 = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc and let us set

$$f_t(z) = \big|\mathrm{Im}(z)\big|^t \quad , \quad t \in (0,+\infty)\ .$$

$$g(z) = \big|\mathrm{Im}(z)\big|^{1/(1-|z|)} \quad , \quad h(z) = \begin{cases} \exp\left[|z|^{-1/2}\right] & \text{for } \ z \neq 0\ , \\ 0 & \text{for } \ z = 0\ . \end{cases}$$

Then (cf. [36])

$$f_t,\, g,\, h \in AW\big(\mathbb{B}^1\big) \quad , \quad t \in (0,+\infty)\ .$$

In his pioneering work [13], G. Cimmino studied the Dirichlet problem for the ordinary Laplacian on domains in $\mathbb{R}^2$ with L^2 boundary data. We restate Cimmino's approach (accredited by him to R. Caccioppoli, [11]) in a slightly generalized form, on domains $\Omega \subset \mathbb{C}^n$.

Example 2. (*Cimmino's admissible weights*) Let $\gamma \in W(\Omega)$ and let $f \in L^2(\partial\Omega)$. Let $\mathcal{F}$ be a foliation by real hypersurfaces in $\mathbb{C}^n$, of a one sided neighborhood $V \subset \Omega$ of the boundary $\partial\Omega$. Next, let $\{\Phi_L\}_{L\in V/\mathcal{F}}$ be a family of C^∞ diffeomorphisms $\Phi_L : \partial\Omega \setminus F \to L$, for some subset $F \subset \partial\Omega$ of "surface" measure zero. Given a function $u : \Omega \to \mathbb{C}$ we say that $u = f$ on $\partial\Omega$ if

$$\lim_{\epsilon\to 0^+} \int_{\partial\Omega} \big|u \circ \Phi_{L_\epsilon} - f\big|^2 (\gamma \circ \Phi_{L_\epsilon})\, d\sigma = 0$$

for any generalized sequence of leaves $\{L_\epsilon\}_{\epsilon>0} \subset V/\mathcal{F}$ tending to $\partial\Omega$ in the Gromov-Hausdorff distance as $\epsilon \to 0^+$. As emphasized by G. Cimmino (cf. *op. cit.*, or [14], p. 266) the choice of data

$$\big(\gamma,\, f,\, \mathcal{F},\, \{\Phi_L\}_{L\in V/\mathcal{F}}\big)$$

is eventually responsible for the loss of uniqueness in the Dirichlet problem

$$\Delta u = 0 \quad \text{in } \Omega \quad , \quad u = f \quad \text{on } \partial\Omega\ .$$

For instance, let $\Omega = \mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$ (with $n = 1$ in [13]) and let $\zeta_0 \in \partial\mathbb{B}^n$ and let us set

$$\gamma(\zeta) = \big|\zeta - \zeta_0\big|^2 \quad , \quad \zeta \in \Omega\ .$$

Then $\gamma \in AW(\mathbb{B}^n)$. Next, let $\mathcal{F}$ be the foliation of $V = \mathbb{B}^n \setminus \{0\}$ whose leaf space is

$$V/\mathcal{F} = \big\{S^{2n-1}(0, 1-t) :\ 0 < t < 1\big\}\ .$$

Here $S^{2n-1}(z,r) = \partial B_r(z)$ is the sphere of radius $r > 0$ and center $z \in \mathbb{C}^n$. If $L_\epsilon = S^{2n-1}(0, 1-\epsilon)$, $0 < \epsilon < 1$, then

$$\Phi_\epsilon : \partial\mathbb{B}^n \to L_\epsilon \quad , \quad \Phi_\epsilon(\zeta) = (1-\epsilon)\zeta\ ,$$

are C^∞ diffeomorphisms. Let us consider the function

$$u(\zeta) = \frac{1 - |\zeta|^2}{|\zeta - \zeta_0|^2} \quad , \quad \zeta \in \mathbb{B}^n\ .$$

Then

$$\lim_{\epsilon\to 0^+}\int_{\partial\mathbb{B}^n}\left|u\big(\Phi_\epsilon(z)\big)\right|^2\gamma\big(\Phi_\epsilon(z)\big)\,d\sigma(z)=0$$

so that $u=0$ on $\partial\mathbb{B}^n$ with respect to $\big(\gamma,\mathcal{F},\{\Phi_\epsilon\}_{\epsilon>0}\big)$. On the other hand, if $n=1$ and $\zeta_0=1$ then

$$u=\mathrm{Re}(h)\quad,\quad h(\zeta)=\frac{1+\zeta}{1-\zeta}\,.$$

Here h is holomorphic, hence u is a nonzero harmonic function on $\mathbb{B}^1$ having zero boundary data in the L^2 sense adopted by C. Cimmino, accounting for non uniqueness in the Dirichlet problem. Non uniqueness is produced by the vanishing of the weight on a subset of the boundary (a point, in Cimmino's example). To be entirely fair to the crowd, we should add that the phenomenon is not governed by the weight alone. Indeed, let $\gamma\equiv 1$ and let $\mathcal{F}$ be the foliation of $\mathbb{B}^n$ whose leaf space is

$$\mathbb{B}^n/\mathcal{F}=\{L_t:0<t<1\}\quad,\quad L_t=S^{2n-1}\big(t\,\zeta_0,1-t\big)\,,$$

and let $\Phi_t:\partial\mathbb{B}^n\setminus\{\zeta_0\}\to L_t$ be defined by

$$\big\{\Phi_t(\zeta)\big\}=\ell_\zeta\cap L_t\quad,\quad \zeta\in\partial\mathbb{B}^n\setminus\{\zeta_0\}\,,$$
$$\ell_\zeta=\big\{(1-s)\zeta_0+s\,\zeta:0<s<1\big\}\,.$$

Once again

$$\lim_{t\to 0^+}\int_{\partial\mathbb{B}^n}\left|u\circ\Phi_t\right|^2 d\sigma=0$$

[and if $n=1$ and $\zeta_0=1$ then u is a nonzero harmonic function on $\mathbb{B}^1$].

The next example is due to F. Forelli & W. Rudin (cf. [22]). The fact that their construction fits into the theory of weighted Bergman kernels was observed by E. Ligocka (cf. [32]).

Example 3. (*Forelli & Rudin's admissible weights*) Let $\sigma=s+it\in\mathbb{C}$ be a complex number with $\sigma>-1$ and $t\in\mathbb{R}$ and let us set

$$\gamma_s(\zeta)=\left(1-|\zeta|^2\right)^s\quad,\quad\zeta\in\mathbb{B}^n\,.$$

Then $\gamma_s\in AW(\mathbb{B}^n)$ and the γ_s-Bergman kernel is

$$K_{\gamma_s}(z,\zeta)=\frac{\binom{n+s}{n}}{(1-z\cdot\overline{\zeta})^{n+1+s}}\quad,\quad\zeta,z\in\mathbb{B}^n\,,$$

$$\binom{n+s}{n}=\frac{\Gamma(n+s+1)}{\Gamma(n+1)\,\Gamma(s+1)}\,.$$

$L^2H(\mathbb{B}^n,\ \gamma_s)$ is a RKH space with the inner product

$$(f,g)_{\gamma_s}=\int_{\mathbb{B}^n}f(\zeta)\,\overline{g(\zeta)}\,(1-|\zeta|^2)^s\,d\mu(\zeta)\,.$$

The following example builds on work by M.M. Djrbashian & A.H. Karapetyan (cf. [15]) and is due to E. Barletta et al. (cf. [6]).

Example 4. (*Djrbashian kernels*) Let $\Omega_n = \{\zeta \in \mathbb{C}^n : \mathrm{Im}(\zeta_1) > |\zeta'|^2\}$ be the Siegel domain. Here $\zeta' = (\zeta_2, \cdots, \zeta_n)$ so that $\zeta = (\zeta_1, \zeta')$. For every $\alpha > -1$ and let $\gamma_\alpha \in W(\Omega_n)$ be the weight

$$\gamma_\alpha(\zeta) = \left[\mathrm{Im}(\zeta_1) - |\zeta'|^2\right]^\alpha .$$

By a result in [6] $\gamma_\alpha \in AW(\Omega_n)$ and the γ_α-Bergman kernel is

$$K_{\gamma_\alpha}(z, \zeta) = \frac{2^{n-1+\alpha} c_{n,\alpha}}{\left[i\left(\overline{\zeta}_1 - z_1\right) - 2\langle z', \zeta'\rangle\right]^{n+1+\alpha}} ,$$

$$c_{n,\alpha} = \pi^{-n}(\alpha+1)\cdots(\alpha+n) .$$

2. Analyticity of $\gamma \in AW(\Omega) \mapsto K_\gamma \in HA(\Omega)$

Let $HA(\Omega)$ denote the space of functions $F : \Omega \times \Omega \to \mathbb{C}$ such that F is holomorphic in the first n variables and anti-holomorphic in the last n variables. Then $HA(\Omega)$ is a complex Fréchet space whose topology as a locally convex space is determined by the family of semi-norms

$$\left\{\|\cdot\|_A : A \subset \Omega,\ A \text{ compact}\right\} ,$$

$$\|F\|_A = \sup_{(z,\zeta)\in A\times A} \left|F(z,\zeta)\right| \quad , \quad F \in HA(\Omega) .$$

The regularity properties (continuity, differentiability, analyticity) of the map

$$\gamma \in AW(\Omega) \longmapsto K_\gamma \in HA(\Omega)$$

were studied by Z. Pasternak-Winiarski (cf. [37]). To make sense of those properties one needs to organize $W(\Omega)$ as a manifold of sorts. One will also need a concept of analyticity for functions $F : U \subset \mathfrak{X} \to \mathfrak{Y}$ where $\mathfrak{X}$ is a normed space, while $\mathfrak{Y}$ is a topological space, and $U \subset \mathfrak{X}$ is an open set. We start with that and call F *analytic* on U if for any $x \in U$ there is a ball $B \subset \mathfrak{X}$ of center ζ with $x + B \subset U$, and there is a sequence $\{a_m\}_{m\in\mathbb{N}}$ of continuous multi-linear (m-linear) maps $a_m : \mathfrak{X}^m \to \mathfrak{Y}$ such that

$$F(x+h) = F(x) + \sum_{m=1}^{\infty} a_m\left(h, \cdots, h\right)$$

for any $h \in B$ and the series $\sum_{m\geq 1} a_m\left(h, \cdots, h\right)$ converges uniformly in $h \in B$.

Next, let $L^\infty(\Omega)$ be the Banach algebra of all real-valued essentially bounded functions $g : \Omega \to \mathbb{R}$, with the norm

$$\|g\|_\infty = \operatorname*{ess\,sup}_{z\in\Omega} |g(z)|$$

$$= \inf\left\{K > 0 : |g(z)| \leq K \text{ for a.e. } z \in \Omega\right\} .$$

Let us also set

$$\operatorname*{ess\,inf}_{z\in\Omega} g(z) = \sup\left\{L \in \mathbb{R} : L \leq g(z) \text{ for a.e. } z \in \Omega\right\} ,$$

$$U(\Omega) = \left\{g \in L^\infty(\Omega) : \operatorname*{ess\,inf}_{z\in\Omega} g(z) > 0\right\} ,$$

so that $U(\Omega)$ is an open subset in $L^\infty(\Omega)$. For every $\gamma \in W(\Omega)$ we consider the map

$$\Phi_\gamma : U(\Omega) \to W(\Omega) ,$$

$$\left[\Phi_\gamma(g)\right](z) = g(z)\,\gamma(z) \quad , \quad g \in U(\Omega) ,\ \ z \in \Omega ,$$

and let us set

$$U(\Omega,\gamma) = \Phi_\gamma\big[U(\Omega)\big] \ .$$

By a result in [37]

i) $\Phi_\gamma : U(\Omega) \to W(\Omega)$ is injective.

ii) For every $\varphi \in W(\Omega)$

$$U(\Omega,\varphi) \cap U(\Omega,\gamma) \neq \emptyset \implies U(\Omega,\varphi) = U(\Omega,\gamma) \ .$$

iii) There is a topology τ on $W(\Omega)$ such that the family $B = \{\Phi_\gamma(X) : \gamma \in W(\Omega),\ X \subset U(\Omega),\ X \text{ open}\}$ is a base for τ.

iv) Let $\Phi_\gamma^{-1} : U(\Omega,\gamma) \to U(\Omega)$ be the inverse of $\Phi_\gamma : U(\Omega) \to U(\Omega,\gamma)$. Then $\{\Phi_\gamma^{-1} : \gamma \in W(\Omega)\}$ is an analytic atlas on $W(\Omega)$ organizing $\big(W(\Omega),\tau\big)$ as a Banach manifold.

v) If $\varphi_1, \varphi_2 \in U(\Omega,\gamma)$ then $L^2(\Omega,\varphi_1)$ and $L^2(\Omega,\varphi_2)$ coincide as vector spaces and the norms $\|\cdot\|_{\varphi_1}$ and $\|\cdot\|_{\varphi_2}$ are equivalent.

vi) If $\gamma \in AW(\Omega)$ then $U(\Omega,\gamma) \subset AW(\Omega)$. In particular $AW(\Omega)$ is an open subset of $(W(\Omega),\tau)$.

Once again by a result in [37] the map

$$AW(\Omega) \ni \gamma \mapsto K_\gamma \in HA(\Omega)$$

is analytic and one may write down an explicit development

$$K_{(g+h)\,\gamma} = K_{g\,\gamma} + \sum_{k=1}^{\infty} (-1)^k K_{g,\gamma}^{(k)}\, h^{(k)} \ , \tag{4}$$

$$\gamma \in AW(\Omega) \quad , \quad g \in U(\Omega) \quad , \quad h \in B_g \ ,$$

$$B_g = B_{i(g)/2}(0) = \Big\{ h \in L^\infty(\Omega) : \|h\|_\infty < \frac{i(g)}{2} \Big\} \ ,$$

$$i(g) = \operatorname*{ess\,inf}_{z\in\Omega}\ g(z) \ .$$

Details about the construction of the k-linear maps

$$K_{g,\gamma}^{(k)} : L^\infty(\Omega)^k \to HA(\Omega)$$

will be given as we shall indicate applications of (4).

3. Quantization of states and reproducing kernels

3.1. Hilbert spaces of holomorphic L^2 sections in $\Lambda^{n,0}(M)\otimes E$. Let M be a complex n-dimensional manifold, and let E be a holomorphic line bundle over M, with projection $\pi : E \to M$. Let us fix a local trivialization atlas of E

$$\Big\{ T_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C} \Big\}_{\alpha\in I}$$

such that each U_α is the domain of a local complex coordinate system $(U_\alpha, z_\alpha^1, \cdots, z_\alpha^n)$ on M. The transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}\setminus\{0\}$ are

$$T_{\beta,z} \circ T_{\alpha,z}^{-1}(\xi) = g_{\alpha\beta}(z)\,\xi \quad , \quad \xi \in \mathbb{C} \quad , \quad z \in U_\alpha \cap U_\beta \ ,$$

$$T_{\alpha,z} : E_z \to \mathbb{C} \quad , \quad z \in U_\alpha \ ,$$

$$T_{\alpha,z} = \Pi \circ \big(T_\alpha\big)\Big|_{E_z} \quad , \quad E_z = \pi^{-1}(z) \ ,$$

$$\Pi : M \times \mathbb{C} \to \mathbb{C} \quad , \quad \Pi(x,\xi) = \xi \ ,$$

and $g_{\alpha\beta} \in \mathcal{O}\big(U_\alpha \cap U_\beta\big)$, describing the holomorphic structure of the given complex line bundle E, which manifests into the first order differential operator

$$\overline{\partial}_E : C^1(E) \to C\big(T^{0,1}(M)^* \otimes E\big) \, ,$$
$$\big(\overline{\partial}_E s\big)\Big|_{U_\alpha} = \big(\overline{\partial} f_\alpha\big) \otimes s_\alpha \, ,$$
$$s\big|_{U_\alpha} = f_\alpha \, s_\alpha \quad , \quad f_\alpha \in C^1(U_\alpha, \mathbb{C}) \, ,$$
$$s_\alpha : U_\alpha \to E \quad , \quad s_\alpha(z) = T_\alpha^{-1}\big(z, 1\big) \quad , \quad z \in U_\alpha \, .$$

A section $s \in C^1(E)$ is holomorphic if $\overline{\partial}_E s = 0$. Let $\mathcal{O}(E)$ be the space of all holomorphic sections in E. Then $s_\alpha \in \mathcal{O}\big(E\big|_{U_\alpha}\big)$. Let $\Lambda^{n,0}(M) \to M$ be the bundle of all complex valued differential n-forms of type $(n, 0)$. Then $\Lambda^{n,0}(M) \otimes E$ is a complex line bundle, and a section $\varphi \in \Gamma\big(\Lambda^{n,0}(M) \otimes E\big)$ is an E valued $(n, 0)$-form on M locally expressed as

$$\varphi\Big|_{U_\alpha} = \Psi_\alpha \, s_\alpha \otimes \big(dz_\alpha^1 \wedge \cdots \wedge dz_\alpha^n\big) \, ,$$
$$\Psi_\alpha : U_\alpha \to \mathbb{C} \, .$$

Let H be a Hermitian bundle metric on E. For every $\varphi \in \mathcal{O}\big(\Lambda^{n,0}(M) \otimes E\big)$ let us consider the scalar (n, n)-form

$$H^*(\varphi, \varphi)\Big|_{U_\alpha} = \big|\Psi_\alpha\big|^2 \gamma_\alpha dz_\alpha^1 \wedge \cdots \wedge dz_\alpha^n \wedge d\overline{z}_\alpha^1 \wedge \cdots \wedge d\overline{z}_\alpha^n \, ,$$
$$\Psi_\alpha \in \mathcal{O}\big(U_\alpha\big) \quad , \quad \gamma_\alpha = H\big(s_\alpha, s_\alpha\big) \, .$$

Let $\mathcal{M}$ be the space

$$\mathcal{M} = \Big\{\varphi \in \mathcal{O}\big(\Lambda^{n,0}(M) \otimes E\big) : \int_M H^*\big(\varphi, \varphi\big) < \infty\Big\} \, .$$

By a result of K. Gawędzki (cf. [23]) $\mathcal{M}$ is a complex Hilbert space with the inner product

$$\langle \varphi, \psi \rangle = i^{n^2} \int_M H^*\big(\varphi, \psi\big) \, ,$$
$$H^*\big(\varphi, \psi\big)\Big|_{U_\alpha} = \Psi_\alpha \, \overline{\Phi}_\alpha \, \gamma_\alpha dz_\alpha^{1\cdots n} \wedge d\overline{z}^{1\cdots n} \, ,$$
$$\psi\big|_{U_\alpha} = \Phi_\alpha \, s_\alpha \otimes dz_\alpha^{1\cdots n} \quad , \quad \Phi_\alpha \in \mathcal{O}(U_\alpha) \, .$$

From a physical viewpoint one may think of M as the *classical phase space* (the phase space of a classical physical system). The complex projective space $\mathbb{CP}(\mathcal{M})$ is then the *quantum phase space* and quantization of classical states amounts to building an embedding

$$\mathcal{K} : M \to \mathbb{CP}(\mathcal{M}) \, .$$

The better known approach to quantization of classical states relies upon the space of holomorphic sections in E which are square integrable with respect to the Liouville measure yet the present approach (due to A. Odzijewicz, [35]) brings into the picture weighted Bergman kernels, which can then be used to compute the transition probability amplitude from $z \in M$ to ζ [after identifying the classical states $z, \zeta \in M$ with the coherent states $\mathcal{K}(z), \mathcal{K}(\zeta) \in \mathbb{CP}(\mathcal{M})$].

To explain the ideas of A. Odzijewicz (cf. *op. cit.*) we start with his assumption

$$\forall \, z, \zeta \in M \quad , \quad \exists \, \varphi, \psi \in \mathcal{M} \, :$$

(5)
$$\det \begin{bmatrix} \Psi_\alpha(z) & \Psi_\beta(\zeta) \\ \Phi_\alpha(z) & \Phi_\beta(\zeta) \end{bmatrix} \neq 0 \, ,$$
$$\forall \, \alpha, \beta \in I \ \text{ such that } \ z \in U_\alpha \, , \ \ \zeta \in U_\beta \, ,$$

a requirement that guarantees that $\mathcal{M}$ is "sufficiently ample". Let $z \in M$ and let $\alpha \in I$ such that $z \in U_\alpha$ and let us consider the evaluation functional

$$\delta_z^\alpha : \mathcal{M} \to \mathbb{C} \quad , \quad \delta_z^\alpha(\varphi) = \Psi_\alpha(z) \quad , \quad \varphi \in \mathcal{M} \, .$$

Again by a result of K. Gawędzki (cf. *op. cit.*)

(6)
$$\big|\Psi_\alpha(z)\big| \le C_\alpha \, \|\varphi\|$$

for some constant $C_\alpha > 0$ and any $\varphi \in \mathcal{M}$. By the Riesz representation theorem there is a unique $k_{z,\overline{\alpha}} \in \mathcal{M}$ such that

$$\delta_z^\alpha(\varphi) = \big\langle \varphi, k_{z,\overline{\alpha}} \big\rangle = i^{n^2} \int_M H^*\big(\varphi, k_{z,\overline{\alpha}}\big) \, ,$$
$$k_{z,\overline{\alpha}}\Big|_{U_\beta} = \overline{K_{\alpha\overline{\beta}}(z, \cdot \,)} \, s_\beta \otimes d\zeta_\beta^1 \wedge \cdots \wedge d\zeta_\beta^n \, ,$$
$$H^*\big(\varphi, k_{z,\overline{\alpha}}\big)\Big|_{U_\beta} = K_{\alpha\overline{\beta}}(z, \cdot \,) \, \Psi_\beta \, \gamma_\beta \, d\zeta_\beta^{1\cdots n} \wedge d\overline{\zeta}_\beta^{1\cdots n} \, .$$

We shall see that $K_{\alpha\overline{\beta}}(z,\zeta)$ are reproducing kernels enjoying properties similar to those of the Bergman kernel. As a consequence of Odzijewicz's assumption (5) above $k_{z,\overline{\alpha}}$ is never the zero section. One has

$$k_{z,\overline{\beta}} = \overline{g_{\alpha\beta}(z)} \; \overline{\frac{\partial \zeta_\alpha}{\partial \zeta_\beta}(z)} \, k_{z,\overline{\alpha}}$$

for any $z \in U_\alpha \cap U_\beta$ so that the map

$$\mathcal{K} : M \to \mathbb{CP}(\mathcal{M}) \quad , \quad \mathcal{K}(z) = \big[k_{z,\overline{\alpha}}\big] \quad , \quad z \in U_\alpha \, ,$$

is well defined. Here $\big[\varphi\big] = \{\lambda\varphi : \lambda \in \mathbb{C} \setminus \{0\}\}$ is the (projective) ray represented by $\varphi \in \mathcal{M} \setminus \{0\}$. The physical meaning of $\mathbb{CP}(\mathcal{M})$ is that in quantum theory the wave functions $\varphi \in \mathcal{M}$ and $\lambda\varphi \in \mathcal{M}$ [or Ψ_α and $\lambda\Psi_\alpha$ with respect to the local description $\varphi\big|_{U_\alpha} = \Psi_\alpha \, s_\alpha \otimes dz_\alpha^{1\cdots n}$] represent the same physical state for any $\lambda \in \mathbb{C} \setminus \{0\}$. A normalized wave function [i.e. $\varphi \in \mathcal{M}$ with $\langle \varphi, \varphi \rangle = 1$] may be chosen in a ray, yet the normalization procedure determines φ only up to a factor $\lambda = e^{i\phi} \in \mathrm{U}(1)$. Here ϕ is the *global phase*. The phase of a ray is not observable.

3.2. **A Bergman-like metric.** There is a globally defined $(0,2)$-tensor field g on M such that

(7)
$$g\big|_{U_\alpha} = \sum_{j,k=1}^n \frac{\partial^2 \log K_{\alpha\overline{\alpha}}(z,z)}{\partial z_\alpha^j \partial \overline{z}_\alpha^k} \, dz_\alpha^j \odot d\overline{z}_\alpha^k \quad , \quad \alpha \in I \, .$$

By a result of A. Odzijewicz (cf. *op. cit.*) the following statements are equivalent

i) $\mathcal{K} : M \to \mathbb{CP}(\mathcal{M})$ is injective.

ii) $\forall \, z_1, z_2 \in M$, $\exists \, \zeta_1, \zeta_2 \in M$ such that

(8)
$$\det \begin{bmatrix} K_{\gamma\overline{\alpha}}(\zeta_1, z_1) & K_{\gamma\overline{\beta}}(\zeta_1, z_2) \\ K_{\delta\overline{\alpha}}(\zeta_2, z_1) & K_{\delta\overline{\beta}}(\zeta_2, z_2) \end{bmatrix} \neq 0$$

provided that $\zeta_1 \in U_\gamma$, $\zeta_2 \in U_\delta$, $z_1 \in U_\alpha$ and $z_2 \in U_\beta$.

iii) The ampleness condition (5) is fulfilled.

Again by a result of A. Odzijewicz (cf. *op. cit.*) the following statements are equivalent

a) $\mathcal{K} : M \to \mathbb{CP}(\mathcal{M})$ is a holomorphic embedding.

b) The condition (8) is fulfilled and the $(0, 2)$-tensor field g is positive definite.

According to the result just stated, if $\mathcal{K} : M \to \mathbb{CP}(\mathcal{M})$ is a holomorphic embedding then g is a Riemannian metric on M (referred to as the *Bergman metric*). Of course metrics derived from a potential [such as g, cf. (7)] are Kählerian. Therefore the quantization procedure devised by A. Odzijewicz will not work when M is a complex manifold admitting no globally defined Kählerian metric.

The proof of "(a) $\Longleftrightarrow$ (b)" is an easy adaptation of arguments in [26]. At the time when Kobayashi's paper [26] was written locally conformal Kähler (lc.K.) metrics[3] were practically unknown to the community devoted to the study of complex analysis and geometry. To set matters in a correct historical perspective, we should mention that l.c.K. metrics were introduced by P. Libermann (cf. [31]) in 1954-1955 (while [26] was published in 1959). Also, the first example of a Hermitian metric (on a compact complex manifold) which is l.c.K. but not Kähler was discovered in 1954 by W.M. Boothby (cf. [10]) yet the example was not recognized as such until 1976, with the publication of I. Vaisman's work [40].

The reason one should be concerned with l.c.K. metrics (among the many known classes of Hermitian metrics) comes from A. Andreotti & E. Vesentini's work[4] [1] (that we shall examine in the next section of the present talk): they do start with a Kählerian metric g yet their theory requires completeness and they therefore modify g by a conformal factor i.e. consider $f \in C^\infty(M, \mathbb{R})$ such that $e^f g$ is complete. Such a conformal transformation is feasible by a result of K. Nomizu & H. Ozeki (cf. [34]) yet $e^f\, g$ is only globally conformal Kähler (for any two conformally related Kähler metrics are homothetic).

Are there mechanical systems whose classical phase space is a complex manifold not satisfying the topological constraints of a Kählerian manifold? The (perhaps purely mathematical) problem of the quantization of classical states when the phase space is a non-Kähler l.c.K. manifold is open.

Example 5. (*Classical phase space is a domain* $\Omega \subset \mathbb{C}^n$) Let $M = \Omega$ be a domain in $\mathbb{C}^n$ and let $E \to \Omega$ be the trivial complex line bundle with the Hermitian bundle metric H. Let $s_0 \in \mathcal{O}(E)$ be given by $s_0(z) = (z, 1)$ for any $z \in \Omega$ and let us set $\gamma = H(s_0, s_0)$ so that $\gamma \in W(\Omega)$. Every $\varphi \in \mathcal{M}$ may be (globally) represented as $\varphi = \Psi\, s_0 \otimes d\zeta^1 \wedge \cdots \wedge d\zeta^n$ for some $\Psi \in \mathcal{O}(\Omega)$ and

$$\langle \varphi, \varphi \rangle = i^{n^2} \int_\Omega H^*(\varphi, \varphi)$$

[3] Cf. e.g. L. Ornea et al., [16], for an account on the geometry of l.c.K. metrics at the level of the year 1998.

[4] The fundamental paper by A. Andreotti & E. Vesentini (cf. [1]) has much in common with the famous work by G.B. Folland & J.J. Kohn (cf. [21]) and [1] precedes [21] by seven years. However much of the material on which the presentation in [21] relies was published earlier [e.g. J.J. Kohn, [27], J.J. Kohn & L. Nirenberg, [28].

$$= i^{n^2} \int_\Omega \big|\Psi\big|^2 \gamma \, d\zeta^1 \wedge d\zeta^n \wedge d\overline{\zeta}^1 \wedge \cdots \wedge d\overline{\zeta}^n = 2^n \|\Psi\|_\gamma^2$$

so that $\Psi \in L^2 H(\Omega, \gamma)$ and the map

$$\mathcal{M} \to L^2 H(\Omega, \gamma) \quad , \quad \varphi \longmapsto 2^{-n/2}\, \Psi \ ,$$

is an isometry. By Gawędzki's estimate (6) the weight $\gamma = H(s, s)$ is admissible i.e. $\gamma \in AW(\Omega)$ and

$$K_{0\overline{0}}(z, \zeta) = 2^{-n} K_\gamma(z, \zeta) \quad , \quad z, \zeta \in \Omega \ .$$

Here we have set $I = \{0\}$ and $T_0 = 1_E$.

Example 6. (*Segal-Bargmann space*) Let $\alpha > 0$ and let $\gamma_\alpha \in W\big(\mathbb{C}^n\big)$ be given by $\gamma_\alpha(z) = \exp\big(-\alpha|z|^2\big)$ for any $z \in \mathbb{C}^n$. Then $L^2 H(\mathbb{C}^n, \gamma_\alpha)$ is the Segal-Bargmann-Fock space of quantum mechanics (with parameter α) cf. G. B. Folland, [20]. Then $\gamma_\alpha \in AW(\mathbb{C}^n)$ and the γ_α-Bergman kernel is

$$K_{\gamma_\alpha}(z, \zeta) = \Big(\frac{\alpha}{\pi}\Big)^n \exp(z \cdot \overline{\zeta}) \quad , \quad z, \zeta \in \mathbb{C}^n \ .$$

Cf. also V. Bargmann, [4]. A unit vector in $L^2 H(\mathbb{C}^n, \gamma_\alpha)$ is thought of as the wave function of a quantum particle moving in configuration space $\mathbb{R}^n$ [while $\mathbb{C}^n$ is the classical phase space].

3.3. **Andreotti-Vesentini external fields.** One of the fundamental ideas in R. Penrose's twistor theory is that electromagnetic and gravitational interactions may be described by the deformation of the holomorphic structure of certain complex vector bundles over twistor flag spaces, cf. [39]. The relevance of that approach was exhibited for instance with the Atiyah-Drinfeld-Hitchin-Manin classification (cf. [3]) of instanton solutions to Yang-Mills equations.

By building on Penrose's ideas one fixes the differential structure of the quantum bundle $E \to M$ and varies its holomorphic and metric structures, interpreting them as external fields interacting with the physical system described by M. The famous work by A. Andreotti & E. Vesentini, [1] (that we shall start describing in a moment) does precisely that i.e. deforms the Hermitian structure on E in the quest for Carleman type estimates for the complex Laplacian $\square = \overline{\partial}^{\,*}\, \overline{\partial} + \overline{\partial}\, \overline{\partial}^{\,*}$, allowing one to speak about *Andreotti-Vesentini external fields*.

Though being (and remaining) a beautiful piece of differential geometry and complex analysis, the work [1] contains a number of misleading phrases, starting from its very title *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*. As previously recalled the authors start with a Kählerian metric g but need to work with the conformally related metric $\hat{g} = e^f g$ where the conformal factor is chosen[5] such that $\hat{g}$ be complete. Then the Laplace-Beltrami operator $\Delta = d\, d^* + d^* d$ of $(M, \hat{g})$ doesn't agree with $\square$ (one has $\Delta = 2\,\square$ if and only if the background metric[6] is Kählerian).

[5]The matter is briefly relegated to a footnote on p. 107 of [1] and the paper by K. Nomizu & H. Ozeki (cf. [34]) isn't quoted. It should be observed that the arguments in [34] aren't trivial at all.

[6]In the case at hand $\Delta = 2\,\square - A$ for some first order operator A (vanishing if f is a constant) whose precise form was determined by I. Vaisman (cf. (2.13) in [41], p. 292).

Moreover the Carleman estimate in [1] is but an analog to the ordinary Carleman estimate (cf. T. Carleman, [12]) serving distinct purposes in the long run.

Also the (rather "mysterious"[7]) notion of W-ellipticity of the given Hermitian vector bundle (E, H) over the complete Hermitian manifold (M, g) turns out to be postulating an analog to the Poincaré inequality [with the Euclidean gradient replaced by the pair $\big(\overline{\partial}, \overline{\partial}^{\,*}\big)$, and an ingredient in solving the generalized Dirichlet problem for the complex Laplacian, very much the way Poincaré inequality is useful in solving the generalized Dirichlet problem for a second order elliptic equation in divergence form].

Let $E \to M$ be a holomorphic line bundle, and let (H, g) be a pair consisting of a Hermitian bundle metric on E and a complete Hermitian metric on M. Let $p, q \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ such that $p + q = n$.

The complex line bundle E is said to be $W^{p,q}$-*elliptic* with respect to the pair (H, g) if there is a constant $C > 0$ such that

$$\int_M A(\varphi, \varphi) * 1 \le C\, \mathbb{D}(\varphi, \varphi) \tag{9}$$

for any $\varphi \in C_0^\infty\big(\Lambda^{p,q}(M) \otimes E\big)$. Here

$$A(\varphi, \psi) * 1 = \varphi \wedge * \sharp\, \psi\ ,$$

$$\mathbb{D}(\varphi, \psi) = \int_M A\big(\overline{\partial}_E \varphi, \overline{\partial}_E \psi\big) * 1 + \int_M A\big(\overline{\partial}^{\,*}_E \varphi, \overline{\partial}^{\,*}_E \psi\big) * 1\ .$$

Also $\sharp : E \to E^*$ is the canonical anti-isomorphism determined by the Hermitian metric H [extended to E-valued (p, q)-forms as a map $\sharp : \Gamma\big(\Lambda^{p,q}(M) \otimes E\big) \to \Gamma\big(\Lambda^{q,p}(M) \otimes E^*\big)$].

Let $\overset{\circ}{W}{}^{1,2}\big(\Lambda^{p,q}(M) \otimes E\big)$ be the completion of $C_0^\infty\big(\Lambda^{p,q}(M) \otimes E\big)$ with respect to the norm $N(\varphi) = a(\varphi, \varphi)^{1/2}$ where

$$a(\varphi, \psi) = \int_M A(\varphi, \psi) * 1 + \mathbb{D}(\varphi, \psi)\ .$$

By the W-ellipticity condition (9)

$$\|\varphi\|_{\overset{\circ}{W}{}^{1,2}} = \mathbb{D}(\varphi, \varphi)^{1/2}$$

is a norm on $\overset{\circ}{W}{}^{1,2}\big(\Lambda^{p,q}(M) \otimes E\big)$ and

$$\|\varphi\|_{\overset{\circ}{W}{}^{1,2}} \le N(\varphi) \le (1 + C)^{1/2} \|\varphi\|_{\overset{\circ}{W}{}^{1,2}}$$

so that the norms $\|\cdot\|_{\overset{\circ}{W}{}^{1,2}}$ and N induce on $\overset{\circ}{W}{}^{1,2}\big(\Lambda^{p,q}(M) \otimes E\big)$ the same topology (and viceversa).

Let $\mathcal{C}$ be the set of all convex, non decreasing, C^∞ functions $\lambda(t)$, $0 \le t < +\infty$. Together with [1], p. 327, we assume there is a C^∞ function $\Phi : M \to [0, +\infty)$ such that for any $p, q \in \mathbb{Z}_+$ with $p + q = n$ and any $\lambda \in \mathcal{C}$ the holomorphic line bundle

[7] That is appearing not to be fully understood, cf. e.g. G. Gentili, F. Ricci & G. Tomassini, [24], p. 102.

E is $W^{p,q}$-elliptic with respect to the pair $\big(e^{\lambda(\Phi)}H, g\big)$, such that the W-ellipticity constant $C > 0$ doesn't depend upon the choice of $\lambda \in \mathcal{C}$. The Carleman type estimate obtained in [1] is

$$\int_M A(\psi_\lambda, \psi_\lambda)\, e^{\lambda(\Phi)} \, * 1 \le 4C \int_M A(f, f)\, e^{\lambda(\Phi)} \, * 1 \ , \tag{10}$$

$$\psi_\lambda = \overline{\partial}^{\,*}_{E_\lambda} G_\lambda f \ ,$$

$$E_\lambda = \big(E, H_\lambda\big) \quad , \quad H_\lambda = e^{\lambda(\Phi)}\, H \ ,$$

$$G_\lambda : L^2\big(\Lambda^{p,q}(M) \otimes E_\lambda\big) \to \overset{\circ}{W}{}^{1,2}\ \big(\Lambda^{p,q}(M) \otimes E_\lambda\big)$$

$$\text{Green operator i.e.} \quad \Box_\lambda\big(G_\lambda f\big) = f \ .$$

An index λ indicates that relevant operators depending on the choice of Hermitian bundle metric on E are now taken with respect to $H_\lambda = e^{\lambda(\Phi)}\, H$. It helps to observe that

$$A_\lambda(\varphi, \psi) = e^{\lambda(\Phi)} A(\varphi, \psi)$$

[by looking at the anti-isomorphism $\sharp_\lambda : E \to E^*$, determined by H_λ]. To start a parallel between the constructions in [35] and [1] we also note that

$$\forall\ \varphi, \psi \in \mathcal{O}\big(\Gamma^{n,0}(M) \otimes E\big) \ ,$$

$$H^*(\varphi, \psi) = i^{-n^2} \varphi\big(*\ \sharp\psi\big) = i^{-n^2} A(\varphi, \psi) * 1$$

for any Hermitian metric g on M [where $* : \Gamma\big(\Lambda^{n,0}(M) \otimes E\big) \to \Gamma\big(\Lambda^{n,0}(M) \otimes E\big)$ is the corresponding Hodge star].

The inequality (10) does bear a certain similarity to the ordinary Carleman inequality

$$\int_\Omega |u|^2 e^{\tau\Phi} \, dx \le C \int_\Omega \big|P(x, D)u\big|^2 e^{\tau\Phi} \, dx \quad , \quad \tau > \tau_0 \ , \tag{11}$$

where $P(x, D)$ is a differential operator defined in an open set $\Omega \subset \mathbb{R}^n$, with C^∞ coefficients, u is a C^∞ function such that $P(x, D)u \in C_0^\infty(\Omega)$, and $\Phi : \Omega \to (0, +\infty)$ is a C^∞ function. By an observation of L. Hörmander (cf. [25]) the essential feature of (11) is the presence of the exponential weight factor $e^{\tau\Phi}$ allowing one to obtain information[8] on the support of u in terms of the support of $P(x, D)u$.

The similar use of (10) in [1] is to solve

$$\overline{\partial}_E \psi = f$$

getting at the same time information on the support of the solution. Precisely (by a result in [1], p. 97)

$$\forall\ \epsilon > 0 \quad , \quad \forall\ f \in C_0^\infty\big(\Lambda^{p,q}(M) \otimes E\big) \ \text{ with } \ \overline{\partial}^{p,q}_E f = 0 \ ,$$

$$\exists\ \psi \in \mathcal{D}\big(\overline{\partial}^{p,q-1}_E\big) = W^{1,2}\big(\Lambda^{p,q-1}(M) \otimes E\big) : \overline{\partial}^{p,q-1}_E \psi = f \ ,$$

$$\text{Supp}(\psi) \subset \Big\{ z \in M : \Phi(z) \le \sup_{w \in \text{Supp}(f)} \Phi(w) + \epsilon \Big\} \ .$$

Now, let us discuss Andreotti-Vesentini external fields within the Kostant-Odzijewicz quantization scheme. To simplify exposition, from now on we only look at

[8] Precisely $\Phi(x) \le \sup \Big\{ \Phi(y) : y \in \text{Supp}\big[P(x, D)u\big] \Big\}$ for any $x \in \text{Supp}(u)$.

the case where $M = \Omega$ [the classical phase space is a domain in $\mathbb{C}^n$] and $E = \Omega \times \mathbb{C}$ [the trivial complex line bundle over Ω] equipped with the Hermitian bundle metric H. Also let g_0 be the canonical flat Kähler metric on Ω. Next, let $\Phi : \Omega \to [0, +\infty)$ be a C^∞ function such that $E = \Omega \times \mathbb{C}$ is $W^{n,0}$-elliptic with respect to the pair of metrics $\big(H_\lambda, g_0\big)$, for any $\lambda \in \mathcal{C}$ [such that the $W^{n,0}$-ellipticity constant $C > 0$ doesn't depend on $\lambda \in \mathcal{C}$]. Here $H_\lambda = e^{\lambda(\Phi)} H$.

Let us set

$$\mathcal{M}_\lambda = \Big\{ \varphi \in \mathcal{O}\big(\Lambda^{p,q}(\Omega) \otimes E\big) : \int_\Omega H^*_\lambda(\varphi, \varphi) < \infty \Big\} ,$$

$$\langle \varphi, \psi \rangle_\lambda = i^{n^2} \int_\Omega H^*_\lambda\big(\varphi, \psi\big) .$$

Once again by Gawędzki's results $\mathcal{M}_\lambda$ is a complex Hilbert space with the scalar product $\langle \, , \, \rangle_\lambda$ and the functional

$$\delta_z : \mathcal{M}_\lambda \to \mathbb{C} \quad , \quad \delta_z(\varphi) = \Psi(z) \quad , \quad z \in \Omega ,$$

is continuous. Thus for every $z \in \Omega$ there is a unique

$$k^\lambda_z = 2^{-n} \, \overline{K_\lambda(z, \cdot \,)} \, s_0 \otimes d\zeta^{1 \cdots n} \in \mathcal{M}_\lambda$$

such that

$$\Psi(z) = i^{n^2} \int_\Omega K_\lambda(z, \zeta) \, \Psi(\zeta) \, e^{\lambda(\Phi(\zeta))} \, \gamma(\zeta) \; d\zeta^{1 \cdots n \, \overline{1} \cdots \overline{n}}$$

or [by $d\zeta^j \wedge d\overline{\zeta}^j = -2i \, dx^j \wedge dy^j$ with $\zeta^j = x^j + i y^j$]

$$\Psi(z) = \int_\Omega K_\lambda(z, \zeta) \Psi(\zeta) e^{\lambda(\Phi(\zeta))} \gamma(\zeta) \; d\mu(\zeta) ,$$

$$\Psi \in L^2 H\big(\Omega, e^{\lambda(\Phi)} \gamma\big) \quad , \quad z \in \Omega ,$$

where μ is the Lebesgue measure on $\mathbb{R}^{2n}$. In particular $e^{\lambda(\Phi)} \gamma \in AW(\Omega)$ and $L^2 H(\Omega, e^{\lambda(\Phi)} \gamma\big)$ is a RKH space with the reproducing kernel $K_\lambda(z, \zeta)$.

By following Penrose's ideas (cf. *op. cit.*) we now deform the holomorphic and metric structures on E. All holomorphic line bundles over the complex manifold M are described, up to a vector bundle isomorphism, by the elements of $H^1\big(M, \mathcal{O}^*\big)$. Starting from the exact sequence of sheaves

$$0 \to \mathbb{Z} \stackrel{\iota}{\longrightarrow} \mathcal{O} \stackrel{\exp}{\longrightarrow} \mathcal{O}^* \to 0$$

one may build the connecting homomorphism

$$\delta : H^1\big(M, \mathcal{O}^*\big) \to H^2\big(M, \mathbb{Z}\big) .$$

As the differential structure of E is thought of as fixed, the classes of isomorphic holomorphic structures on E are parametrized by

$$\delta^{-1}\big(c_1(E)\big)$$

where $c_1(E)$ is the first Chern class of E.

For a more explicit description let $T_0 = 1_E : E \to \Omega \times \mathbb{C}$ be the holomorphic atlas we fixed to start with, and $s_0 : \Omega \to E$ the corresponding (global) frame $s_0(z) = (z, 1)$. Let $T : E \to \Omega \times \mathbb{C}$ be another holomorphic structure on E, as a trivial complex line bundle, and $s(z) = T^{-1}(z, 1)$ the corresponding frame, so that

$$s = f \, s_0$$

for some $f \in C^\infty(\Omega, \mathbb{C})$. This accounts for a deformation of the holomorphic structure on E. Now let us deform the Hermitian structure H i.e. consider (together with [1])

$$H_\lambda = \rho H \quad , \quad \rho = e^{\lambda(\Phi)} \quad , \quad \lambda \in \mathcal{C} \; .$$

If $\gamma_\lambda = H_\lambda(s, s)$ and $\gamma = H(s_0, s_0)$ then

$$\gamma_\lambda = \rho \, |f|^2 \, \gamma$$

so that the resulting external field B (referred here as the *Andreotti-Vesentini external field*) is given by

$$e^B = \rho \, |f|^2$$

and the function

$$A : \Omega \to \mathbb{R} \quad , \quad e^{A(z)} = \frac{K_{\gamma_\lambda}(z, z)}{K_\gamma(z, z)} \quad , \quad z \in \Omega \; ,$$

describes the reproducing kernel deformation resulting from the deformation $B : \Omega \to \mathbb{R}$ of the holomorphic and metric structures on E. Here $K_{\gamma_\lambda}(z, \zeta)$ is the γ_λ-Bergman kernel of Ω.

The dependence of A on B plays a fundamental role in Odzijewicz's theory (cf. *op. cit.*). To apply said theory one needs to explicitly express the potential A in terms of B, which comes down to the calculation of the reproducing kernel $K_{\gamma_\lambda}(z, \zeta)$, and in general that is, as well known, an unsolved problem.

The general theoretic discussion in [35] ends up with two suggestions, as to the computability of the potential A, a concrete one but of formidable appearance, and another one which is speculative yet, as we shall shortly show, can be followed to a certain extent. The first suggestion is to think of

$$\det \left[\frac{\partial^2 \gamma}{\partial z^j \partial \overline{z}^k} + \frac{\partial^2 \lambda(\Phi)}{\partial z^j \partial \overline{z}^k} \right] = e^{A + \lambda(\Phi)} \det \left[\frac{\partial^2 \gamma}{\partial z^j \partial \overline{z}^k} \right] \tag{12}$$

as a field equation for the Andreotti-Vesentini external field $B = \lambda(\Phi)$ and, should a solution B be pinpointed, determine A as a function of B from (12). Equation (12) is got by identifying the measure associated with H (respectively H_λ) with the corresponding Liouville measure [tied to the assumption that (E, H) (respectively (E, H_λ)) is a quantum bundle, in the sense of B. Kostant (cf. [30])]. Details on said identification will be given in § 5 when discussing the calculation of transition probability amplitudes (in terms of weighted Bergman kernels). One may also exhibit a number of specific examples, where the calculations related to the use of (12) may be carried out explicitly[9]. The audience will agree on the rather formidable aspect of the field equation (12).

Odzijewicz's second suggestion is, by taking into account the dependence of weighted Bergman kernels on the weight of integration [i.e. as functions on $AW(\Omega)$], to use perturbative methods in the mathematical analysis of $A = A(B)$, when $B = \lambda(\Phi)$ is small.

As with the ordinary Bergman kernel $K(z, \zeta)$, explicit formulas for $K_\gamma(z, \zeta)$ are known only for a handful of particular domains $\Omega \subset \mathbb{C}^n$ and admissible weights

[9] Said examples aren't considered in this talk: one may see E. Barletta & S. Dragomir & F. Esposito, [8].

$\gamma \in AW(\Omega)$. In general, one can only hope for asymptotic information on $K_\gamma(z,\zeta)$ near the boundary of Ω, to be discussed in the next section of the present talk.

4. Forelli-Rudin-Ligocka-Peloso asymptotic expansion formula

Let $\Omega = \{z \in \mathbb{C}^n : \varphi(z) < 0\}$ be a smoothly bounded strictly pseudoconvex domain, where φ is such that the Levi form L_φ satisfies

$$L_\varphi(w)\zeta \geq C_1 \big|\zeta\big|^2 \quad , \quad \zeta \in \mathbb{C}^n ,$$

for $\varphi(w) < \delta_0$, $\delta_0 > 0$, with C_1 depending only on Ω. Let us set

$$F(\zeta,z) = -\sum_{j=1}^n \frac{\partial \varphi}{\partial z_j}(z)\big(\zeta_j - z_j\big) - \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial z_k}(z)\big(\zeta_j - z_j\big)\big(\zeta_k - z_k\big) ,$$

$$\Gamma(\zeta,z) = \big[F(\zeta,z) - \varphi(z)\big]\chi\big(|\zeta - z|\big) + \big[1 - \chi\big(|\zeta - z|\big)\big]|\zeta - z|^2 ,$$

where χ is a C^∞ cut-off function of the real variable t, such that $\chi(t) = 1$ for $|t| < \epsilon_0/2$ and $\chi(t) = 0$ for $|t| \geq 3\epsilon_0/4$.

We recall that $|\varphi|^m \in AW(\Omega)$ for any $m \in \mathbb{Z}_+$. Let $K_m(z,\zeta)$ be the $|\varphi|^m$-Bergman kernel of Ω. Then

$$K_m(\zeta,z) = C_\Omega \big|\nabla\varphi(z)\big|^2 \det L_\varphi(z) \cdot \Gamma(\zeta,z)^{-(n+1+m)} + E(\zeta,z) , \tag{13}$$

$$E \in C^\infty\big(\overline{\Omega} \times \overline{\Omega} \setminus \Delta\big) \quad , \quad \Delta = \{(z,z) : z \in \partial\Omega\} ,$$

$$|E(\zeta,z)| \leq C'_\Omega \, |\Gamma(\zeta,z)|^{-(n+1+m)+1/2} \, \big|\log|\Gamma(\zeta,z)|\big| . \tag{14}$$

Formula (13) [together with the estimate (14)] was proved by M.M. Peloso (cf. Lemma 2.2 in [38], p. 229). It extends Fefferman's asymptotic expansion formula for the Bergman kernel of a smoothly bounded strictly pseudoconvex domain (cf. [18] for $m = 0$). M.M. Peloso claims (cf. *op. cit.*) that Theorem (13)-(14) is implicit in [32], while E. Ligocka does employ an older idea by F. Forelli & W. Rudin (cf. [22]). Aside from the correct credit, which certainly goes to M.M. Peloso, the history of Theorem (13)-(14) demonstrates the attention shown by the mathematical community (devoted to complex analysis) to an argument born with the celebrated work by C. Fefferman (cf. *op. cit.*) and emphasizes the recognition of the relevance of that argument.

Still, in a public talk such as this, one may legitimately ask "what does (13) do for you?". Anybody in the audience can use (13)-(14) together with l'Hôspital rule to show that

$$\varphi(z) = -K(z,z)^{-1/(n+1)}$$

is a defining function[10] for Ω, and that certainly gives an effective manner of relating the Kähler geometry of the interior of Ω (the Bergman metric

$$g_{j\overline{k}} = \frac{\partial^2 \log K(z,z)}{\partial z^j \partial \overline{z}^k}$$

[10] Of class C^n. This is fine if the number $n \geq 2$ of complex variables is sufficiently high (remember that one needs two derivatives to build the Bergman metric, and another two derivatives to consider its curvature).

of Ω) to the pseudohermitian geometry of the boundary $\partial\Omega$, for which a choice of contact form is of course

$$\theta = \mathbf{j}^*\Big[\frac{i}{2}(\overline{\partial} - \partial)\varphi\Big] \quad , \quad \mathbf{j} : \partial\Omega \hookrightarrow \mathbb{C}^n \; .$$

While the first use of that is in the proof of Fefferman's theorem (that biholomorphisms of smoothly bounded strictly pseudoconvex domains extend smoothly up to the boundary) a pleiad of other applications were successively found: let us mention only A. Korányi & H.M. Reimann's result (cf. [29]) that boundary values of symplectomorphisms of Ω which extend smoothly up to $\partial\Omega$ are contact transformations[11].

So we are strongly motivated to look for analogs to Fefferman's asymptotic expansion formula for weighted Bergman kernels $K_\gamma(z,\zeta)$, possibly for any $\gamma \in AW(\Omega)$. One such generalization is of course the Forelli-Rudin-Ligocka-Peloso formula, holding for admissible weights of the form $\gamma = |\varphi|^m$ with $m \in \mathbb{Z}_+$.

Let $\gamma \in AW(\Omega)$ and let $K_{\Omega_m}(Z,W)$ be the ordinary Bergman kernel of the domain

$$\Omega_m = \Big\{ Z = (z,\xi) \in \Omega \times \mathbb{C}^m : \gamma(z) > |\xi|^{2m} \Big\} \; .$$

Then

$$K_\gamma(z,w) = K_{\Omega_m}\big((z,0),(w,0)\big) \quad , \quad z,w \in \Omega \; ,$$

and one may use Fefferman's asymptotic expansion for K_{Ω_m}, provided Ω_m is bounded and strictly pseudoconvex, to derive the analogous expansion formula for K_γ. That is precisely the manner M.M. Peloso proved (13)-(14) [when $\gamma = |\varphi|^m$ and $\Omega_m = \{(z,\xi) \in \Omega \times \mathbb{C}^m : \varphi(z) + |\xi|^2 < 0\}$].

Next, to derive a similar expansion formula for $K_{\gamma_\lambda}(z,w)$ [with $\gamma_\lambda = e^{\lambda(\Phi)}\gamma$] we exploit the analyticity of the weighted Bergman kernel as a function of weight i.e. we set $g \equiv 1$ and $h = e^{\lambda(\Phi)} - 1$ in (4) so that to obtain

$$K_{e^{\lambda(\Phi)}\gamma} = K_\gamma + \sum_{k=1}^{\infty} (-1)^k \, K^{(k)}_{1,\gamma}\big[e^{\lambda(\Phi)} - 1\big]^{(k)} \tag{15}$$

where

$$K^{(k)}_{1,\gamma}\, h^{(k)} = K^{(k)}_{1,\gamma}\big(h_1, \cdots, h_k\big) \quad , \quad h_1 = \cdots = h_k = h \; ,$$

$$\begin{aligned} &\big[K^{(k)}_{1,\gamma}\big(h_1,\cdots,h_k\big)\big](\zeta,z) \\ &= \int_\Omega K_\gamma(w_1,z) h_1(w_1)\,\gamma(w_1)\,d\mu(w_1) \\ &\times \int_\Omega K_\gamma(w_2,w_1) h_2(w_2)\,\gamma(w_2)\,d\mu(w_2) \\ &\qquad \vdots \end{aligned}$$

[11]By Fefferman's theorem boundary values of biholomorphisms $\phi : \Omega \to \Omega$ are CR isomorphisms $f : \partial\Omega \to \partial\Omega$. Any biholomorphism ϕ is also an isometry (with respect to the Bergman metric $g_{j\overline{k}}$) so in particular a symplectomorphism (with respect to the symplectic structure $\Omega_{j\overline{k}} = -i g_{j\overline{k}}$). Also a CR isomorphism $f : \partial\Omega \to \partial\Omega$ is in particular a contact transformation. The Koranyi-Reimann result is then a genuine generalization of Fefferman's theorem (weakening the hypothesis from "ϕ = biholomorphism" to "ϕ = symplectomorphism extending smoothly up to the boundary").

$$\times \int_\Omega K_\gamma(w_k, w_{k-1}) h_k(w_k)\, K_\gamma(\zeta, w_k)\, \gamma(w_k)\, d\mu(w_k)\ .$$

Finally we substitute into (15) from the asymptotic expansion formula for K_γ.

Carrying out the relevant calculations [i.e. to study the geometry of the suspended[12] domain $\Omega_m \subset \mathbb{C}^{n+m}$, to apply Fefferman's asymptotic formula to its Bergman kernel $K_{\Omega_m}(Z, W)$, to exploit the analytic dependence on the weight (of weighted Bergman kernels), ecc.] is an open problem.

This scheme is however expected to be successful, for the same approach proved effective when $\gamma = |\varphi|^m$ [and led to the asymptotic expansion formula]

$$K_{(1+h)|\varphi|^m}(z, w) = C_\Omega \left|\nabla\varphi(w)\right|^2 \cdot \det L_\varphi(w) \cdot \Gamma(z, w)^{-(n+1+m)} + E_h(z, w)\ ,$$

$$E_h \in C^\infty(\Omega \times \Omega) \quad , \quad h \in B_{1/2}(0) \subset L^\infty(\Omega)\ ,$$

$$\big|E_h(z, w)\big| \leq C'_\Omega \Big\{ \big|\Gamma(z, w)^{-(n+1+m)+1/2}\, \big|\log|\Gamma(z, w)|\big|$$

$$+ |\varphi(z)|^{-(n+1+m)/2}\, |\varphi(w)|^{-(n+1+m)/2} \big[1 + G(z) + G(w) + G(z)G(w)\big] \Big\}\ ,$$

$$G(z) = |\varphi(z)|^{3/2} + |\varphi(z)|^{1/2}\, \big|\log|\varphi(z)|\big|\ .$$

Cf. E. Barletta et al., [5].

5. Transition probability amplitudes

Let $z \in M$ be a classical state and let

$$\mathcal{K}(z) = \big[k_{z,\overline{\alpha}}\big] \in \mathbb{CP}(\mathcal{M})$$

(with $\alpha \in I$ such that $z \in U_\alpha$) be the corresponding coherent state. Given two classical states $z \in U_\alpha$ and $\zeta \in U_\beta$, identified with the coherent states $\mathcal{K}(z)$ and $\mathcal{K}(\zeta)$, the *transition probability amplitude* from z to ζ is

$$a_{\beta\overline{\alpha}}(\zeta, z) = \left\langle \frac{k_{z,\overline{\alpha}}}{\|k_{z,\overline{\alpha}}\|}\ ,\ \frac{k_{\zeta,\overline{\beta}}}{\|k_{\zeta,\overline{\beta}}\|} \right\rangle \tag{16}$$

and $\left|a_{\beta\overline{\alpha}}(\zeta, z)\right|^2$ is the *transition probability density.* Then

$$a_{\beta\overline{\alpha}}(\zeta, z) = \frac{K_{\beta\overline{\alpha}}(\zeta, z)}{K_{\alpha\overline{\alpha}}(z, z)^{1/2}\, K_{\beta\overline{\beta}}(\zeta, \zeta)^{1/2}}$$

so that

$$a_{\delta\overline{\gamma}}(\zeta, z) = \frac{g_{\mu\delta}(\zeta)}{|g_{\mu\delta}(\zeta)|}\ \frac{\dfrac{\partial\zeta_\mu}{\partial\zeta_\delta}(\zeta)}{\left|\dfrac{\partial\zeta_\mu}{\partial\zeta_\delta}(\zeta)\right|}\ \frac{g_{\sigma\gamma}(z)}{|g_{\sigma\gamma}(z)|}\ \frac{\dfrac{\partial\zeta_\sigma}{\partial\zeta_\gamma}(z)}{\left|\dfrac{\partial\zeta_\sigma}{\partial\zeta_\gamma}(z)\right|}\ a_{\mu\overline{\sigma}}(\zeta, z)$$

[12]The idea of using suspension of domains is attributed by C. Fefferman (cf. [19]) to I. Naruki who is known to have reduced the proof of Fefferman's celebrated theorem (that biholomorphisms of strictly pseudoconvex domains extend smoothly to the boundary) to a (significantly simpler) statement about extension of isomorphisms of Cartan connections. Naruki's proof (that circulated at the time as a preprint) contained a few gaps and successively only the short version [33] has been published. Nevertheless the suspension of a variable idea (due to I. Naruki) led C. Fefferman to the discovery of the Lorentzian metric that bears his name (the *Fefferman metric*, cf. [19]).

that is, under a transformation of local frames $a_{\beta\overline{\alpha}}(\zeta, z)$ changes by a phase factor $e^{i\phi}$ of global phase

$$\phi = \arg\left[g_{\mu\delta}(\zeta)\, g_{\sigma\gamma}(z)\, \frac{\partial\zeta_\mu}{\partial\zeta_\delta}(\zeta)\, \frac{\partial\zeta_\sigma}{\partial\zeta_\gamma}(z)\right]$$

and in particular

$$\left|a_{\delta\overline{\gamma}}(\zeta, z)\right| = \left|a_{\beta\overline{\alpha}}(\zeta, z)\right|$$

so that the transition probability density doesn't depend upon the choice of local frames, both on E and on $T^{1,0}(M)$.

The transition probability amplitude from z to ζ with *simultaneous transition* through $w \in U_\gamma$ is

$$a_{\gamma\overline{\alpha}}(w, z)\, a_{\beta\overline{\gamma}}(\zeta, w)\ .$$

It doesn't depend on the choice of $\gamma \in I$ such that $w \in U_\gamma$. A natural question [*vis-a-vis* to Odzijewicz formula (16) for the calculation of the transition probability amplitudes] is whether averaging $a_{\gamma\overline{\alpha}}(w, z) a_{\beta\overline{\gamma}}(\zeta, w)$ over $w \in M$ one retrieves the transition probability amplitude from z to ζ. In other words, as the natural measure on the phase space M is the *Liouville measure*

$$d\mu_L(\zeta) = (-1)^n\, \Omega_{\alpha\overline{\alpha}}\, d\zeta_\alpha^{1\cdots n} \wedge d\overline{\zeta}_\alpha^{1\cdots n}\ ,$$

$$\Omega_{\alpha\overline{\alpha}} = \det\left[\Omega^\alpha_{j\overline{k}}\right] \quad , \quad \Omega^\alpha_{j\overline{k}} = \Omega\left(\frac{\partial}{\partial z^j_\alpha}\, ,\, \frac{\partial}{\partial \overline{z}^k_\alpha}\right)\ ,$$

$$\Omega = \operatorname{curv}(E, \nabla)\ ,$$

∇ canonical[13] Hermitian connection of (E, H), one asks whether

$$a_{\beta\overline{\alpha}}(\zeta, z) = i^{n^2} \sum_{\gamma\in I} \int_{U_\gamma} \chi_\gamma(w)\, a_{\gamma\overline{\alpha}}(w, z)\, a_{\beta\overline{\gamma}}(\zeta, w)\, d\mu_L(w) \tag{17}$$

eventually with $d\mu_L$ multiplied by some constant $C > 0$. Here $\{\chi_\alpha\}_{\alpha\in I}$ is a C^∞ partition of unity on M subordinated to the open cover $\{U_\alpha\}_{\alpha\in I}$ i.e.

$$\operatorname{Supp}\bigl(\chi_\alpha\bigr) \subset U_\alpha \quad , \quad \sum_{\alpha\in I} \chi_\alpha = 1\ .$$

Starting from

$$K_{\beta\overline{\alpha}}(\zeta, z) = \left\langle k_{z,\overline{\alpha}}, k_{\zeta,\overline{\beta}}\right\rangle$$

one gets

$$K_{\beta\overline{\gamma}}(\zeta, z) = i^{n^2} \sum_{\gamma\in I} \int_{U_\gamma} \chi_\gamma(w)\, K_{\gamma\overline{\alpha}}(w, z)\, K_{\beta\overline{\gamma}}(\zeta, w)\, H_{\gamma\overline{\gamma}}(w)\, dw^{1\cdots n} \wedge d\overline{w}^{1\cdots n}$$

hence (multiplying by $1/\bigl[K_{\alpha\overline{\alpha}}(z, z)^{1/2}\, K_{\beta\overline{\beta}}(\zeta, \zeta)^{1/2}\bigr]$)

$$a_{\beta\overline{\alpha}}(\zeta, z) = i^{n^2} \sum_{\gamma\in I} \int_{U_\gamma} a_{\gamma\overline{\alpha}}(w, z)\, a_{\beta\overline{\gamma}}(\zeta, w) \times \chi_\gamma(w)\, K_{\gamma\overline{\gamma}}(w, w)\, H_{\gamma\overline{\gamma}}(w)\, dw^{1\cdots n} \wedge d\overline{w}^{1\cdots n}\ . \tag{18}$$

[13]Here one tacitly requires that the curvature 2-form $\Omega = \operatorname{curv}(E, \nabla)$ [of the canonical Hermitian connection ∇ of (E, H)] be non degenerate, so that (E, H) is a quantum bundle, in the sense of B. Kostant, [30].

Odzijewicz's solution to the raised problem is to make a clever choice of the local weights of integration i.e. such that (for some constant $C > 0$)

$$\hat{H} = C\,H\,, \tag{19}$$

$$\hat{H}\big(s_\alpha, s_\alpha\big)_z = (-1)^n\, \frac{\Omega_{\alpha\overline{\alpha}}(z)}{K_{\alpha\overline{\alpha}}(z,z)} \quad , \quad z \in U_\alpha\ .$$

If such a choice is made the Liouville measure is locally given by

$$d\mu_L(\zeta) = C\,K_{\alpha\overline{\alpha}}(\zeta,\zeta)\,\gamma_\alpha(\zeta)\,d\zeta_\alpha^{1\cdots n} \wedge d\overline{\zeta}^{1\cdots n}$$

and formula (18) yields (17).

Both Hermitian metrics H and $\hat{H}$ are sections of the same complex line bundle $E^* \otimes E^*$ hence $\hat{H} = f\,H$ for some $f \in C^\infty(M)$, so with (19) one chooses H such that $f =$ constant, a rather innocent looking like requirement which amounts to solving for γ_α in the complex Monge-Ampère equation

$$\det\left[\frac{\partial^2 \gamma_\alpha}{\partial z_\alpha^j \partial \overline{z}_\alpha^k}(z)\right] = C\,i^{n(n+1)}\ \frac{1}{n!}\ \gamma_\alpha(z)\,K_{\alpha\overline{\alpha}}(z,z)$$

on U_α.

Despite the formidable aspect of the equation, several examples of explicit solutions may be produced (cf. A. Odzijewicz, [35], E. Barletta et al., [6], [7]).

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