Lecture Notes of
Seminario Interdisciplinare di Matematica
Vol. 16(2023), pp. 57 – 70.

Biconservative hypersurfaces with three distinct principal curvatures in a space form

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Abstract. Let M^n $(n > 4)$ be a biconservative hypersurface with constant scalar curvature and with three distinct principal curvatures in a space form $N^{n+1}(c)$. We prove that such a hypersurface M^n is either constant mean curvature (CMC) or contained in a certain non-CMC generalized cylinder $\Sigma^4 \times \mathbb{R}^{n-4}$, where Σ^4 is a non-CMC rotational hypersurface in a Euclidean 5-space $\mathbb{R}^5$.

1. Introduction

An isometric immersion $\phi : (M^n, g) \longrightarrow (N^m, \widetilde{g})$ is said to be *biconservative* if its associated divergence of the stress-bienergy tensor S_2 is zero. The concept of biconservative immersions were proposed by Caddeo-Montaldo-Oniciuc-Piu in 2014 from the principle of a stress-energy tensor for the bienergy originated from biharmonic maps [2]. Recall that a biharmonic map $\phi : (M^n, g) \longrightarrow (N^m, \widetilde{g})$ is a critical point of the bienergy functional $E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 \, dv_g$, where $\tau(\phi)$ is the tension field associated to ϕ and ϕ is characterized by the vanishing of the associated bitension field:

$$\tau_2(\phi) := -\Delta\tau(\phi) - \operatorname{trace} R^N(d\phi, \tau(\phi))d\phi = 0\ . \tag{1.1}$$

In recent years, the study of biharmonic submanifolds and biconservative submanifolds has attracted great attention, and geometers have obtained many interesting properties and classification results (cf. [21, 7, 10, 12, 8, 6, 14, 15, 13, 16, 18, 17, 20]).

Recently Fetcu and Oniciuc [7] proposed an interesting problem on biconservative hypersurfaces:

Problem: Classify all biconservative hypersurfaces with constant scalar curvature in a space form $N^{n+1}(c)$.

In the case of $n = 3$ and $n = 4$, the authors [11] have proved

Theorem 1.1 ([11])**.** *Any biconservative hypersurface with constant scalar curvature in $N^4(c)$ has constant mean curvature.*

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Keywords. Biconservative hypersurfaces, biharmonic hypersurfaces, constant mean curvature, constant scalar curvature.

AMS Subject Classification. 53C40, 58E20, 53C42.

Theorem 1.2 ([11])**.** *Any biconservative hypersurface with constant scalar curvature in $N^5(c)$ is either CMC or contained in a certain non-CMC rotational hypersurface, where the rotational hypersurface has two distinct principal curvatures with $-\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ and the scalar curvature R of this rotational hypersurface is $12c$.*

Remark 1.1. Let M^n be a non-CMC biconservative hypersurface in $N^{n+1}(c)$ with constant scalar curvature. If M^n has two distinct principal curvatures, then $n = 4$ and M^n is given by Theorem 1.2 (see Remark 3.5 in [14]).

In the current paper, we continue our study on biconservative hypersurfaces with constant scalar curvature in a space form $N^{n+1}(c)$. We will prove that

Theorem 1.3. *Let M^n $(n > 4)$ be a biconservative hypersurface with constant scalar curvature which has three distinct principal curvatures in a space form $N^{n+1}(c)$. Then either M^n is CMC in $N^{n+1}(c)$, or $c = 0$ and M^n is an open part of the extrinsic product $\Sigma^4 \times \mathbb{R}^{n-4}$, where Σ^4 is a non-CMC rotational biconservative hypersurface in $\mathbb{R}^5$.*

It is well-known that Chern's conjecture (see, for example, [3, 4, 5, 22, 23]) says that: *any closed minimal hypersurface in the unit sphere $\mathbb{S}^{n+1}$ with constant scalar curvature is isoparametric.* Since the class of biconservative hypersurfaces is a generalization of minimal hypersurfaces, it is very interesting to investigate whether Chern's conjecture holds for biconservative hypersurfaces instead of minimal hypersurfaces in a sphere.

We recall a recent result due to S. C. De Almeida et al. [1]:

Theorem 1.4 ([1])**.** *Let M^n $(n > 3)$ be a hypersurface with constant mean curvature and constant scalar curvature which has three distinct principal curvatures everywhere in a unit sphere $\mathbb{S}^{n+1}$. Then M^n is isoparametric.*

Consequently, combining Theorem 1.3 with the above result gives

Corollary 1.5. *Let M^n $(n > 4)$ be a biconservative hypersurface with constant scalar curvature which has three distinct principal curvatures in a unit sphere $\mathbb{S}^{n+1}$. Then M^n is isoparametric.*

The paper is organized as follows. In Section 2, we recall some background on the theory of biconservative hypersurfaces. In Section 3, we consider the case of biconservative hypersurfaces with constant scalar curvature and with three distinct principal curvatures in a space form $N^{n+1}(c)$, and give a complete characterization.

2. Preliminaries

Let M^n be an n-dimensional immersed hypersurface in an $(n+1)$-dimensional Riemannian space form $N^{n+1}(c)$ with constant sectional curvature c. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connection of M^n and the Levi-Civita connection of $N^{n+1}(c)$, respectively. For any $X, Y, Z \in TM$, the Gauss and Codazzi equations are respectively given by

$$R(X,Y)Z = c\big(\langle Y,Z\rangle X - \langle X,Z\rangle Y\big) + \langle AY,Z\rangle AX - \langle AX,Z\rangle AY \ , \tag{2.1}$$

$$(\nabla_X A)Y = (\nabla_Y A)X \ . \tag{2.2}$$

Note that here A is the shape operator satisfying

$$(\nabla_X A)Y = \nabla_X(AY) - A(\nabla_X Y) ,$$

and the curvature tensor of M^n is defined to be

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} . \tag{2.3}$$

Let us recall the definition of biconservative immersions (cf.[2, 19]). The stress-energy tensor S_2 of the bienergy is defined to be

$$\begin{aligned} S_2(X,Y) = {} & \frac{1}{2}|\tau(\phi)|^2\langle X,Y\rangle + \langle d\phi, \nabla\tau(\phi)\rangle\langle X,Y\rangle \\ & -\langle d\phi(X), \nabla_Y\tau(\phi)\rangle - \langle d\phi(Y), \nabla_X\tau(\phi)\rangle , \end{aligned} \tag{2.4}$$

and it satisfies

$$div\, S_2 = -\langle \tau_2(\phi), d\phi\rangle = -\tau_2(\phi)^\top . \tag{2.5}$$

The isometric immersion $\phi : (M^n, g) \longrightarrow (N^m, \widetilde{g})$ is said to be *biconservative* if $div\, S_2 = 0$. Taking into account (1.1) and (2.5), we have

Proposition 2.1 (cf. [7]). *A hypersurface $\phi : M^n \to N^{n+1}(c)$ is biconservative if the mean curvature H and the shape operator A on M^n satisfy*

$$A\,\mathrm{grad}\, H = -\frac{n}{2}\, H\, \mathrm{grad}\, H . \tag{2.6}$$

According to (2.6), it is straightforward to see that CMC hypersurfaces are automatically biconservative in $N^{n+1}(c)$. We call non-CMC biconservative hypesurfaces as proper biconservative hypesurfaces in $N^{n+1}(c)$.

From now on, we consider a proper biconservative hypersurface M^n in a space form $N^{n+1}(c)$, that is, H is not identically constant on a connected component of M^n, where the number of distinct principal curvatures is constant, and all of the principal curvature functions of the shape operator A are always smooth. Fix $p \in M^n$, there exists a neighborhood U_p of p such that $\mathrm{grad}\, H \neq 0$ on U_p.

From (2.6), we see that $\mathrm{grad}\, H$ is an eigenvector of A with the corresponding principal curvature $-nH/2$. Then we can choose a suitable orthonormal frame $\{e_1, \cdots, e_n\}$ such that $e_n = \mathrm{grad}\, H/|\mathrm{grad}\, H|$ and the shape operator A has the form

$$A = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) , \tag{2.7}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_{n-1}$ and $\lambda_n = -nH/2$ are the principal curvatures of M^n. According to the definition of the mean curvature, it follows that $\sum_{i=1}^{n} \lambda_i = nH$, and hence

$$\lambda_1 + \lambda_2 + \cdots + \lambda_{n-1} = -3\lambda_n . \tag{2.8}$$

The squared length of the second fundamental form, denoted by S, is defined as

$$S = \mathrm{trace}\, A^2 = \sum_{i=1}^{n-1} \lambda_i^2 + \lambda_n^2 . \tag{2.9}$$

By Gauss equation (2.1), it is easy to check that the scalar curvature R of M^n is given by

$$R = n(n-1)c + 4\lambda_n^2 - S . \tag{2.10}$$

Combining (2.9) with (2.10), we find that

$$\sum_{i=1}^{n-1} \lambda_i^2 = n(n-1)c + 3\lambda_n^2 - R \,. \tag{2.11}$$

Since e_n is parallel to $\operatorname{grad} H$, it follows that $e_n(H) \neq 0$ and $e_i(H) = 0$ for $1 \leq i \leq n-1$, and hence

$$e_n(\lambda_n) \neq 0, \quad e_i(\lambda_n) = 0, \quad 1 \leq i \leq n-1 \,. \tag{2.12}$$

Let $\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_{ij}^k e_k$ $(1 \leq i, j \leq n)$. Note that ω_{ij}^k is the connection coefficients of M^n.

By the compatibility conditions $\nabla_{e_k}\langle e_i, e_i\rangle = 0$ and $\nabla_{e_k}\langle e_i, e_j\rangle = 0$ $(i \neq j)$, we have

$$\omega_{ki}^i = 0 \quad , \quad \omega_{ki}^j + \omega_{kj}^i = 0 \quad , \quad i \neq j \,. \tag{2.13}$$

Using Codazzi equation (2.2), it yields

$$e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j \,, \tag{2.14}$$

$$(\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j \tag{2.15}$$

for distinct i, j, k. Using (2.13) with (2.14), we then actually have

$$e_n(\lambda_i) = (\lambda_i - \lambda_n)\omega_{ii}^n, \; i = 1, 2, \cdots, n-1 \,. \tag{2.16}$$

According to the formula (3-13) in Lemma 3.2 of [9], it follows that

$$e_n(\omega_{ii}^n) = (\omega_{ii}^n)^2 + \lambda_n\lambda_i + c, \quad i = 1, 2, \cdots, n-1 \,. \tag{2.17}$$

3. The proof of the main theorem

In this section, we assume that the biharmonic hypersurface M^n with constant scalar curvature in a space form $\mathbb{M}^{n+1}(c)$, which has three distinct principal curvatures. For the sake of convenience, the principal curvature $\lambda_n = -\frac{nH}{2}$ is denoted by λ. We will use the notations μ and ν to denote the other two principal curvatures with the corresponding multiplicities p and q, respectively. Without loss of generality we can assume that

$$\mu := \lambda_1 = \lambda_2 = \cdots = \lambda_p \,,$$

and

$$\nu := \lambda_{p+1} = \lambda_{p+1} = \cdots = \lambda_{p+q} \,,$$

where $p + q = n - 1$.

Due to (2.8), we get that

$$p\mu + q\nu = -3\lambda \,. \tag{3.1}$$

From (2.11) one has

$$p\mu^2 + q\nu^2 = n(n-1)c + 3\lambda^2 - R \,. \tag{3.2}$$

Eliminating ν between (3.1) and (3.2), it follows that

$$3(3-q)\lambda^2 + 6p\lambda\mu + (n-1)p\mu^2 + qR - n(n-1)qc = 0 \,,$$

i.e.

$$3(p+4-n)\lambda^2 + 6p\lambda\mu + (n-1)p\mu^2 \tag{3.3}$$

$$+(n-p-1)R+n(n-1)(p+1-n)c=0\ .$$

Likewise, after eliminating μ between (3.1) and (3.2), it is easily seen that

$$3(3-p)\lambda^2+6q\lambda\nu+(n-1)q\nu^2+pR-n(n-1)pc=0\ ,$$

that is

$$3(3-p)\lambda^2+6(n-p-1)\lambda\nu+(n-1)(n-p-1)\nu^2+pR-n(n-1)pc=0\ . \tag{3.4}$$

By differentiating (3.1) and (3.2) with respect to e_i, we find

$$pe_i(\mu)+qe_i(\nu)=-3e_i(\lambda)\ ,$$

and

$$p\mu e_i(\mu)+q\nu e_i(\nu)=3\lambda e_i(\lambda)$$

for $i=1,2,\cdots,n$.
By solving the above system of equations concerning $e_i(\mu)$ and $e_i(\nu)$, we obtain

$$p(\mu-\nu)e_i(\mu)=3(\lambda+\nu)e_i(\lambda)\ ,$$

$$q(\nu-\mu)e_i(\nu)=3(\lambda+\mu)e_i(\lambda)\ .$$

Taking into account (2.12), we derive that

$$e_i(\mu)=e_i(\nu)=0,\ i=1,2,\cdots,n-1\ .$$

To shorten the notations, we use f', f'' to stand for the first and second derivatives $e_n(f)$, $e_ne_n(f)$ of the function f with respective to e_n.

Hence, we have

$$\mu'=\frac{3(\lambda+\nu)}{p(\mu-\nu)}\lambda'=-\frac{3\big[(p+4-n)\lambda+p\mu\big]\lambda'}{p\big[3\lambda+(n-1)\mu\big]}\ , \tag{3.5}$$

and

$$\nu'=\frac{3(\lambda+\mu)}{q(\nu-\mu)}\lambda'=-\frac{3\big[(3-p)\lambda+(n-p-1)\nu\big]\lambda'}{(n-p-1)\big[3\lambda+(n-1)\nu\big]}\ . \tag{3.6}$$

By calculating the sectional curvatures $\langle R(e_i,e_j)e_i,e_j\rangle$ for $i=1,2,\cdots,p$ and $j=p+1,p+2,\cdots,p+q$, we can derive the following formulae

$$\omega_{ii}^n\omega_{jj}^n=-(c+\lambda_i\lambda_j)$$

for $i=1,2,\cdots,p$ and $j=p+1,p+2,\cdots,p+q$. Then, from (2.16) we have

$$\frac{\mu'}{\mu-\lambda}\cdot\frac{\nu'}{\nu-\lambda}=-(c+\mu\nu)\ . \tag{3.7}$$

Inserting (3.5) and (3.6) into (3.7) leads to

$$\frac{9\big[(p+4-n)\lambda+p\mu\big]\big[(3-p)\lambda+(n-p-1)\nu\big]\lambda'^2}{p(n-p-1)\big[3\lambda+(n-1)\mu\big]\big[3\lambda+(n-1)\nu\big]}\ . \tag{3.8}$$

$$+(c+\mu\nu)(\mu-\lambda)(\nu-\lambda)=0\ .$$

Moreover, differentiating (3.5) and (3.6) with respect to e_n, we obtain

$$\mu''=-\frac{3\big[(p+4-n)\lambda+p\mu\big]\lambda''}{p\big[3\lambda+(n-1)\mu\big]} \tag{3.9}$$

$$-\frac{3(n-4)(p+1-n)\big[3(p+4-n)\lambda^2+6p\lambda\mu+(n-1)p\mu^2\big]\lambda'^2}{p^2\big[3\lambda+(n-1)\mu\big]^3}\ ,$$

and

$$\nu'' = -\frac{3\big[(3-p)\lambda + (n-p-1)\nu\big]\lambda''}{(n-p-1)\big[3\lambda + (n-1)\nu\big]} \tag{3.10}$$

$$-\frac{3(n-4)p\big[3(p-3)\lambda^2 + 6(p+1-n)\lambda\nu + (n-1)(p+1-n)\nu^2\big]\lambda'^2}{(p+1-n)^2\big[3\lambda + (n-1)\nu\big]^3} .$$

By applying (2.17) we derive that

$$\Big(\frac{\mu'}{\mu-\lambda}\Big)' = \Big(\frac{\mu'}{\mu-\lambda}\Big)^2 + \lambda\mu + c ,$$
$$\Big(\frac{\nu'}{\nu-\lambda}\Big)' = \Big(\frac{\nu'}{\nu-\lambda}\Big)^2 + \lambda\nu + c ,$$

and hence

$$\frac{\mu''}{\mu-\lambda} + \frac{\lambda'\mu' - 2\mu'^2}{(\mu-\lambda)^2} - (\lambda\mu + c) = 0 , \tag{3.11}$$

$$\frac{\nu''}{\nu-\lambda} + \frac{\lambda'\nu' - 2\nu'^2}{(\nu-\lambda)^2} - (\lambda\nu + c) = 0 . \tag{3.12}$$

Substituting (3.5) and (3.9) into (3.11) gives

$$\begin{aligned} 0 = &-\frac{3\big[(p+4-n)\lambda + p\mu\big]\lambda''}{p(\mu-\lambda)\big[3\lambda + (n-1)\mu\big]} \\ &-\frac{3\lambda'^2}{p^2(\lambda-\mu)^2\big[3\lambda + (n-1)\mu\big]^3} \times \Big\{ -3\lambda^3 n^3 + 6\lambda^3 n^2 p \\ &+45\lambda^3 n^2 - 3\lambda^3 np^2 - 84\lambda^3 np - 216\lambda^3 n + 39\lambda^3 p^2 \\ &+240\lambda^3 p + 336\lambda^3 + 9\lambda^2\mu n^3 - 18\lambda^2\mu n^2 p - 81\lambda^2\mu n^2 \\ &+9\lambda^2\mu np^2 + 63\lambda^2\mu np + 216\lambda^2\mu n + 45\lambda^2\mu p^2 + 36\lambda^2\mu p \\ &-144\lambda^2\mu - 18\lambda\mu^2 n^2 p + 27\lambda\mu^2 np^2 + 90\lambda\mu^2 np \\ &-27\lambda\mu^2 p^2 - 72\lambda\mu^2 p - \mu^3 n^3 p + 2\mu^3 n^2 p^2 + 6\mu^3 n^2 p \\ &-\mu^3 np^2 - 9\mu^3 np - \mu^3 p^2 + 4\mu^3 p\Big\} - (\lambda\mu + c) . \end{aligned} \tag{3.13}$$

Furthermore, substituting (3.6) and (3.10) into (3.12) gives

$$-\frac{3\big[(3-p)\lambda + (n-p-1)\nu\big]\lambda''}{(n-p-1)(\nu-\lambda)\big[3\lambda + (n-1)\nu\big]} \tag{3.14}$$

$$
-\frac{3\lambda'^2}{(p+1-n)^2(\lambda-\nu)^2\big[3\lambda+(n-1)\nu\big]^3}\times\Big\{-3\lambda^3np^2+27\lambda^3n
+39\lambda^3p^2-162\lambda^3p+135\lambda^3+27\lambda^2\nu n^2+9\lambda^2\nu np^2-135\lambda^2\nu np
+108\lambda^2\nu n+45\lambda^2\nu p^2+54\lambda^2\nu p-135\lambda^2\nu+9\lambda\nu^2n^3-36\lambda\nu^2n^2p
+27\lambda\nu^2n^2+27\lambda\nu^2np^2+18\lambda\nu^2np-81\lambda\nu^2n-27\lambda\nu^2p^2+18\lambda\nu^2p
+45\lambda\nu^2+\nu^3n^4-3\nu^3n^3p+2\nu^3n^3+2\nu^3n^2p^2-12\nu^3n^2-\nu^3np^2
+9\nu^3np+14\nu^3n-\nu^3p^2-6\nu^3p-5\nu^3\Big\}-(\lambda\nu+c)=0\ .
$$

If $q\neq 3$, we claim that $\mu'\neq 0$. Otherwise, we assume that $\mu'=0$, and hence μ is a constant. From (3.5) one has $\lambda+\nu=0$. In addition, by using (3.1) we get

$$
p\mu=-3\lambda-q\nu=(q-3)\lambda\ ,
$$

This implies that λ is also a constant, contradiction. Similarly, if $p\neq 3$, then $\nu'\neq 0$. We need to consider the following three cases.

Case A. Assume that $p\neq 3$ and $q\neq 3$. Then $\mu'\neq 0$ and $\nu'\neq 0$.

By using (3.1), (3.8), (3.13) and (3.14), we can eliminate the terms of λ'',λ'^2 and ν, and finally obtain the following

$$
\begin{aligned}
(3.15)\quad & 9p^3(n-1)\mu^6+2p^2\big\{-n^3+2n^2p+2np+30n+23p-29\big\}\lambda\mu^5\\
&+3p\big\{-\lambda^2n^3-6\lambda^2n^2p-3\lambda^2n^2+16\lambda^2np^2+15\lambda^2np+36\lambda^2n\\
&+11\lambda^2p^2+126\lambda^2p-32\lambda^2-4cn^2p+4cnp^2+8cnp-4cp^2-4cp\big\}\mu^4\\
&+2p\big\{6\lambda^2n^3-18\lambda^2n^2p-63\lambda^2n^2+12\lambda^2np^2+99\lambda^2np+27\lambda^2n\\
&+42\lambda^2p^2+162\lambda^2p+516\lambda^2+cn^4-3cn^3p-cn^3+2cn^2p^2\\
&-21cn^2+2cnp^2-9cnp+41cn+32cp^2+12cp-20c\big\}\lambda\mu^3\\
&+3\big\{\lambda^2n^3p+9\lambda^2n^3-2\lambda^2n^2p^2-36\lambda^2n^2p-54\lambda^2n^2\\
&+\lambda^2np^3+16\lambda^2np^2+60\lambda^2np+11\lambda^2p^3+148\lambda^2p^2\\
&+272\lambda^2p+288\lambda^2+8cn^3p-22cn^2p^2-12cn^2p+14cnp^3\\
&+2cnp^2-108cnp+16cp^3+128cp^2+112cp\big\}\lambda^2\mu^2\\
&+6\big\{3\lambda^2n^3-9\lambda^2n^2p-18\lambda^2n^2+9\lambda^2np^2+9\lambda^2np-3\lambda^2p^3\\
&+9\lambda^2p^2+108\lambda^2p+96\lambda^2-2cn^4+8cn^3p+14cn^3-10cn^2p^2\\
&-33cn^2p-12cn^2+4cnp^3+11cnp^2-21cnp-64cn+8cp^3\\
&+44cp^2+100cp+64c\big\}\lambda^3\mu+3(-n+p+4)^2\big\{3\lambda^2n-3\lambda^2p\\
&+6\lambda^2-2cn^2+2cnp-2cn+4cp+4c\big\}\lambda^4=0\ .
\end{aligned}
$$

We apply (3.3) to (3.15) to eliminate the term of $\mu^6,\mu^5,\mu^4,\mu^3,\mu^2$. Consequently, a tedious calculation shows that

$$
(3.16)\qquad L_1\mu+L_2=0\ ,
$$

where

$$\begin{aligned}L_1 = & -2\lambda p(2p+1-n)\Big[-18\lambda^4(n-7)(n-4)^2\\ & +3\lambda^2(n-4)(-9Rn+63R+10cn^3-78cn^2+72cn-4c)\\ & +(n-1)(cn^2-cn-R)(-Rn^2-Rn+29R\\ & +cn^4-cn^3-30cn^2+50cn-20c)\Big]\,,\end{aligned}$$

$$\begin{aligned}L_2 = \quad & 3\Big[6\lambda^6(n-4)^2(p+4-n)(np+3n-13p-3)\\ & +\lambda^4(n-4)(Rn^3p+3Rn^3-Rn^2p^2-36Rn^2p-45Rn^2\\ & +35Rnp^2+303Rnp+162Rn-214Rp^2-592Rp\\ & -120R-cn^5p-3cn^5+cn^4p^2+39cn^4p+48cn^4\\ & -38cn^3p^2-359cn^3p-207cn^3+261cn^2p^2+961cn^2p\\ & +282cn^2-232cnp^2-672cnp-120cn+8cp^2+32cp)\\ & +\lambda^2(cn^2-cn-R)(-Rn^4+2Rn^3p-2Rn^3+48Rn^2p\\ & +39Rn^2-48Rnp^2-264Rnp-68Rn+156Rp^2\\ & +214Rp+32R+cn^6-2cn^5p+cn^5-50cn^4p-41cn^4\\ & +54cn^3p^2+352cn^3p+107cn^3-240cn^2p^2-610cn^2p\\ & -100cn^2+210cnp^2+374cnp+32cn-24cp^2-64cp)\\ & +p(n-1)(cn^2-cn-R)^2(p+1-n)(3cn^2-7cn+4c-3R)\Big]\,.\end{aligned}$$

Suppose that $L_1 = 0$, which indicates that $n = 2p+1$ or $n = 7$. Therefore, the coefficient of λ^6 is $-36p(p-3)^2(2p-3)^2$ or $-972(p-3)^2$ in L_2. That is to say, the coefficient of λ^6 is nonzero. Hence, λ is a constant, contradiction. We thus get $L_1 \neq 0$. Finally, combining (3.3) with (3.16) gives rise to the following equation

$$a_{12}\lambda^{12}+a_{10}\lambda^{10}+a_8\lambda^8+a_6\lambda^6+a_4\lambda^4+a_2\lambda^2+a_0=0\,, \tag{3.17}$$

where

$$\begin{aligned}a_{12} = & \ 324(n-4)^4(p-3)^2(p+4-n)^2\,,\\ a_{10} = & \ 108(n-4)^3(p-3)(p+4-n)(Rn^2p-3Rn^2-Rnp^2-11Rnp+42Rn\\ & +10Rp^2+10Rp-120R-cn^4p+3cn^4+cn^3p^2+14cn^3p-45cn^3\\ & -13cn^2p^2-45cn^2p+162cn^2+32cnp^2+88cnp-120cn-56cp^2-56cp)\,,\\ a_8 = & \ 9(n-4)^2(R^2n^4p^2+42R^2n^4p-27R^2n^4-2R^2n^3p^3-256R^2n^3p^2\\ & +66R^2n^3p-216R^2n^3+R^2n^2p^4+426R^2n^2p^3+1185R^2n^2p^2\\ & -2448R^2n^2p+4212R^2n^2-212R^2np^4-2160R^2np^3+392R^2np^2\\ & +7284R^2np-16416R^2n+868R^2p^4+1736R^2p^3-4076R^2p^2-4944R^2p\\ & +19008R^2-2Rcn^6p^2-84Rcn^6p+54Rcn^6+4Rcn^5p^3+518Rcn^5p^2\\ & -36Rcn^5p+378Rcn^5-2Rcn^4p^4-864Rcn^4p^3-3106Rcn^4p^2\\ & +5148Rcn^4p-8856Rcn^4+430Rcn^3p^4+5588Rcn^3p^3+3594Rcn^3p^2\\ & -22980Rcn^3p+41256Rcn^3-2364Rcn^2p^4-11656Rcn^2p^3+4756Rcn^2p^2\end{aligned}$$

$$
\begin{aligned}
&+45504Rcn^2p - 70848Rcn^2 + 3464Rcnp^4 + 14736Rcnp^3 - 16280Rcnp^2 \\
&-55584Rcnp + 38016Rcn - 3904Rcp^4 - 7808Rcp^3 + 24128Rcp^2 \\
&+28032Rcp + c^2n^8p^2 + 42c^2n^8p - 27c^2n^8 - 2c^2n^7p^3 - 262c^2n^7p^2 \\
&-30c^2n^7p - 162c^2n^7 + c^2n^6p^4 + 438c^2n^6p^3 + 1930c^2n^6p^2 - 2658c^2n^6p \\
&+4617c^2n^6 - 218c^2n^5p^4 - 3446c^2n^5p^3 - 4502c^2n^5p^2 + 16062c^2n^5p \\
&-25056c^2n^5 + 1505c^2n^4p^4 + 10850c^2n^4p^3 + 2561c^2n^4p^2 - 47568c^2n^4p \\
&+56052c^2n^4 - 3920c^2n^3p^4 - 21104c^2n^3p^3 + 12904c^2n^3p^2 + 86856c^2n^3p \\
&-54432c^2n^3 + 6632c^2n^2p^4 + 21392c^2n^2p^3 - 37880c^2n^2p^2 - 82848c^2n^2p \\
&+19008c^2n^2 - 4064c^2np^4 - 8256c^2np^3 + 25952c^2np^2 + 30912c^2np \\
&+64c^2p^4 + 128c^2p^3 - 704c^2p^2 - 768c^2p)\ , \\
a_6 = &-18(n-4)^2(-R + cn^2 - cn)(-7R^2n^4p + 3R^2n^4 + 35R^2n^3p^2 \\
&+52R^2n^3p - 21R^2n^3 - 56R^2n^2p^3 - 327R^2n^2p^2 + 165R^2n^2p \\
&-198R^2n^2 + 28R^2np^4 + 508R^2np^3 + 270R^2np^2 - 944R^2np \\
&+1176R^2n - 226R^2p^4 - 452R^2p^3 + 508R^2p^2 + 734R^2p \\
&-960R^2 + 14Rcn^6p - 6Rcn^6 - 70Rcn^5p^2 - 128Rcn^5p + 48Rcn^5 \\
&+112Rcn^4p^3 + 776Rcn^4p^2 - 146Rcn^4p + 354Rcn^4 - 56Rcn^3p^4 \\
&-1212Rcn^3p^3 - 1714Rcn^3p^2 + 2304Rcn^3p - 2748Rcn^3 + 550Rcn^2p^4 \\
&+2736Rcn^2p^3 + 970Rcn^2p^2 - 5560Rcn^2p + 4272Rcn^2 - 818Rcnp^4 \\
&-3292Rcnp^3 + 1042Rcnp^2 + 7292Rcnp - 1920Rcn + 828Rcp^4 \\
&+1656Rcp^3 - 2948Rcp^2 - 3776Rcp - 7c^2n^8p + 3c^2n^8 + 35c^2n^7p^2 \\
&+76c^2n^7p - 27c^2n^7 - 56c^2n^6p^3 - 449c^2n^6p^2 - 38c^2n^6p - 153c^2n^6 \\
&+28c^2n^5p^4 + 704c^2n^5p^3 + 1545c^2n^5p^2 - 1228c^2n^5p + 1551c^2n^5 \\
&-324c^2n^4p^4 - 2448c^2n^4p^3 - 2419c^2n^4p^2 + 5233c^2n^4p - 3510c^2n^4 \\
&+900c^2n^3p^4 + 4784c^2n^3p^3 + 784c^2n^3p^2 - 11240c^2n^3p + 3096c^2n^3 \\
&-1492c^2n^2p^4 - 4856c^2n^2p^3 + 3792c^2n^2p^2 + 11828c^2n^2p - 960c^2n^2 \\
&+936c^2np^4 + 1968c^2np^3 - 3592c^2np^2 - 4976c^2np - 48c^2p^4 \\
&-96c^2p^3 + 304c^2p^2 + 352c^2p)\ , \\
a_4 = &\ 3(n-4)(-R + cn^2 - cn)^2(-4R^2n^5p + 3R^2n^5 + 20R^2n^4p^2 \\
&-100R^2n^4p + 30R^2n^4 - 32R^2n^3p^3 + 490R^2n^3p^2 + 314R^2n^3p \\
&-69R^2n^3 + 16R^2n^2p^4 - 804R^2n^2p^3 - 2058R^2n^2p^2 + 994R^2n^2p \\
&-732R^2n^2 + 418R^2np^4 + 2892R^2np^3 + 1270R^2np^2 - 3968R^2np \\
&+1536R^2n - 1028R^2p^4 - 2056R^2p^3 + 1736R^2p^2 + 2764R^2p \\
&-768R^2 + 8Rcn^7p - 6Rcn^7 - 40Rcn^6p^2 + 192Rcn^6p - 54Rcn^6 \\
&+64Rcn^5p^3 - 940Rcn^5p^2 - 936Rcn^5p + 198Rcn^5 - 32Rcn^4p^4 \\
&+1544Rcn^4p^3 + 5696Rcn^4p^2 - 604Rcn^4p + 1326Rcn^4
\end{aligned}
$$

$$
\begin{aligned}
&-804Rcn^3p^4 - 8376Rcn^3p^3 - 11396Rcn^3p^2 + 10248Rcn^3p\\
&-4536Rcn^3 + 3384Rcn^2p^4 + 17096Rcn^2p^3 + 8932Rcn^2p^2\\
&-23724Rcn^2p + 4608Rcn^2 - 5164Rcnp^4 - 18584Rcnp^3\\
&+1396Rcnp^2 + 22592Rcnp - 1536Rcn + 4128Rcp^4 + 8256Rcp^3\\
&-3648Rcp^2 - 7776Rcp - 4c^2n^9p + 3c^2n^9 + 20c^2n^8p^2 - 92c^2n^8p\\
&+24c^2n^8 - 32c^2n^7p^3 + 450c^2n^7p^2 + 622c^2n^7p - 126c^2n^7\\
&+16c^2n^6p^4 - 740c^2n^6p^3 - 3638c^2n^6p^2 - 650c^2n^6p - 564c^2n^6\\
&+386c^2n^5p^4 + 5484c^2n^5p^3 + 11516c^2n^5p^2 - 5090c^2n^5p + 2931c^2n^5\\
&-2356c^2n^4p^4 - 17300c^2n^4p^3 - 18882c^2n^4p^2 + 23182c^2n^4p - 4572c^2n^4\\
&+6294c^2n^3p^4 + 30892c^2n^3p^3 + 11938c^2n^3p^2 - 42032c^2n^3p\\
&+3072c^2n^3 - 9152c^2n^2p^4 - 28920c^2n^2p^3 + 3800c^2n^2p^2 + 37312c^2n^2p\\
&-768c^2n^2 + 5308c^2np^4 + 11608c^2np^3 - 6948c^2np^2 - 15488c^2np\\
&-496c^2p^4 - 992c^2p^3 + 1744c^2p^2 + 2240c^2p),\\
a_2 = {}&2p(-R + cn^2 - cn)^3(-n + p + 1)(2R^2n^5 - 8R^2n^4p + 2R^2n^4\\
&+8R^2n^3p^2 - 16R^2n^3p - 91R^2n^3 + 24R^2n^2p^2 + 348R^2n^2p\\
&+79R^2n^2 - 324R^2np^2 - 940R^2np + 826R^2n + 616R^2p^2\\
&+616R^2p - 818R^2 - 4Rcn^7 + 16Rcn^6p + 4Rcn^6 - 16Rcn^5p^2\\
&+186Rcn^5 - 16Rcn^4p^2 - 744Rcn^4p - 508Rcn^4 + 728Rcn^3p^2\\
&+3286Rcn^3p - 1226Rcn^3 - 2558Rcn^2p^2 - 5796Rcn^2p + 4404Rcn^2\\
&+3238Rcnp^2 + 6558Rcnp - 4024Rcn - 3320Rcp^2 - 3320Rcp\\
&+1168Rc + 2c^2n^9 - 8c^2n^8p - 6c^2n^8 + 8c^2n^7p^2 + 16c^2n^7p\\
&-87c^2n^7 - 8c^2n^6p^2 + 364c^2n^6p + 429c^2n^6 - 372c^2n^5p^2 - 2378c^2n^5p\\
&+57c^2n^5 + 2006c^2n^4p^2 + 6646c^2n^4p - 3075c^2n^4 - 4640c^2n^3p^2\\
&-11870c^2n^3p + 6768c^2n^3 + 7230c^2n^2p^2 + 11998c^2n^2p - 6936c^2n^2\\
&-4768c^2np^2 - 5312c^2np + 3648c^2n + 544c^2p^2 + 544c^2p - 800c^2)\ ,\\
a_0 = {}&9p^2(-R + cn^2 - cn)^4(-n + p + 1)^2(-3R + 3cn^2 - 7cn + 4c)^2\ .
\end{aligned}
$$

Since the coefficient of λ^{12} can not be zero, (3.17) must be a non-trivial algebraic polynomial equation concerning λ with constant coefficients. This shows that λ is a constant, a contradiction.

Case B. Assume that $p = 3$ and $R \neq n(n-1)c$.

Case B.1. If $n = 7$, then equation (3.16) becomes

$$486(R - 42c)\big[-6\lambda^2(R - 40c) + (R - 34c)(R - 42c)\big] = 0\ . \tag{3.18}$$

Since $R - 42c \neq 0$, we obtain a quadratic equation concerning λ as follows

$$-6\lambda^2(R - 40c) + (R - 34c)(R - 42c) = 0\ . \tag{3.19}$$

It is evident to see that λ is a constant. Otherwise, we have $R = 40c$ and $R = 34c$. Thus we conclude that $R = c = 0$, which contradicts the assumption that $R \neq n(n-1)c$.

Case B.2. If $n \neq 7$, then we see that $L_1 \neq 0$. Proceeding as in the proof of Case A, (3.17) reduces to

$$\begin{aligned}(3.20)\quad & 3(n-4)(-R+cn^2-cn)\Big[108\lambda^8(n-7)^2(n-4)^3(-3R+3cn^2-5cn+8c)\\ &+36\lambda^6(n-7)(n-4)^2(3R^2n^2-42R^2n+147R^2-6Rcn^4\\ &+95Rcn^3-441Rcn^2+576Rcn-440Rc+3c^2n^6-53c^2n^5\\ &+303c^2n^4-691c^2n^3+902c^2n^2-480c^2n+16c^2)\\ &-3\lambda^4(n-4)(-R+cn^2-cn)(3R^2n^4+42R^2n^3-1305R^2n^2\\ &+7008R^2n-9636R^2-6Rcn^6-78Rcn^5+2802Rcn^4\\ &-18750Rcn^3+47544Rcn^2-59664Rcn+41760Rc\\ &+3c^2n^8+36c^2n^7-1498c^2n^6+11988c^2n^5-42261c^2n^4\\ &+85256c^2n^3-101748c^2n^2+51936c^2n-3712c^2)\\ &-2\lambda^2(-R+cn^2-cn)^2(2R^2n^5-22R^2n^4-67R^2n^3+1339R^2n^2\\ &-4910R^2n+6574R^2-4Rcn^7+52Rcn^6+42Rcn^5-2884Rcn^4\\ &+15184Rcn^3-36006Rcn^2+44792Rcn-38672Rc+2c^2n^9\\ &-30c^2n^8+33c^2n^7+1449c^2n^6-10425c^2n^5+34917c^2n^4\\ &-70602c^2n^3+94128c^2n^2-55200c^2n+5728c^2)\\ &+27(n-4)(-R+cn^2-cn)^3(-3R+3cn^2-7cn+4c)^2\Big]=0\,.\end{aligned}$$

Since $R \neq n(n-1)c$, we claim that (3.20) must be a nontrivial polynomial equation concerning λ with constant coefficients. If not, then all the coefficients of λ^i vanish for $i = 8, 6, 4, 2, 0$. It follows immediately that

$$-3R+3cn^2-5cn+8c=0\,, \tag{3.21}$$

$$-3R+3cn^2-7cn+4c=0\,. \tag{3.22}$$

From (3.21)-(3.22) we find that $c = 0$, and hence that $R = 0$. This contradicts the assumption that $R \neq n(n-1)c$.

Case C. Assume that $p = 3$ and $R = n(n-1)c$.

Case C.1. If $q = 3$, it follows from (3.1) and (3.2) that $\mu = 0$ or $\nu = 0$. Without loss of generality we can assume $\nu = 0$, then $\mu = -\lambda$. Furthermore, substituting these terms into (3.7) leads to $c = 0$. Consequently, the shape operator takes the form $A = \mathrm{diag}(\lambda, -\lambda, -\lambda, -\lambda, 0, 0, 0)$. According to the Theorem 1 in [24], we obtain that M^7 is either CMC or contained in a certain non-CMC generalized cylinder $\Sigma^4 \times \mathbb{R}^3$, where $\mathbb{R}^3$ is the Euclidean space and $\Sigma^4 \subseteq \mathbb{R}^5$ is a rotational hypersurface with the shape operator given by the form $\mathrm{diag}(\lambda, -\lambda, -\lambda, -\lambda)$.

Case C.2. If $q \neq 3$, we have proved that $\mu \neq 0$. From (3.3) one has

$$(\lambda+\mu)\big[(3-q)\lambda-(3+q)\mu\big]=0\,. \tag{3.23}$$

If $\lambda + \mu = 0$, we find that $\mu = -\lambda$, which together (3.1) implies that $\nu = 0$. So the shape operator has the form $A = \mathrm{diag}(\lambda, -\lambda, -\lambda, -\lambda, 0, \cdots, 0)$. By using the Theorem 1 and Proposition 4.3 in [24], we obtain that M^n is either CMC or contained in a certain non-CMC generalized cylinder $\Sigma^4 \times \mathbb{R}^{n-4}$, where $\mathbb{R}^{n-4}$ is an $(n-4)$-dimensional Euclidean space and $\Sigma^4 \subseteq \mathbb{R}^5$ is a rotational hypersurface with the shape operator given by the form $\mathrm{diag}(\lambda, -\lambda, -\lambda, -\lambda)$.

If $\lambda+\mu \neq 0$, then $\nu \neq 0$, and the equation (3.23) leads to $(3-q)\lambda-(3+q)\mu = 0$. Hence, it follows that

$$\mu = \frac{q-3}{q+3}\lambda\ , \tag{3.24}$$

$$\nu = \frac{-6}{q+3}\lambda\ . \tag{3.25}$$

By substituting (3.24)-(3.25) into (3.11) and (3.12), we derive that

$$\begin{aligned} &6(q-3)(q+3)\lambda\lambda'' + (q-9)(q-3)(q+3){\lambda'}^2 \\ &+36(\lambda^2 q - 3\lambda^2 + cq + 3c)\lambda^2 = 0\ , \end{aligned} \tag{3.26}$$

$$\begin{aligned} &6(q+3)(q+9)\lambda\lambda'' - 6(q+3)(q+15){\lambda'}^2 \\ &-(q+9)^2(-6\lambda^2 + cq + 3c)\lambda^2 = 0\ . \end{aligned} \tag{3.27}$$

By (3.26) and (3.27), we may now eliminate λ'' and conclude that

$$(q-3)(q+3)^2{\lambda'}^2 + (q+9)\big[-6(q-3)\lambda^2 + (q+3)^2c\big]\lambda^2 = 0\ . \tag{3.28}$$

In view of the formula (3.7), one has

$$-(q-3)(q+3)^2{\lambda'}^2 + (q+9)\big[-6(q-3)\lambda^2 + (q+3)^2c\big]\lambda^2 = 0\ . \tag{3.29}$$

Combining (3.28) with (3.29) gives

$$2(q+9)\big[-6(q-3)\lambda^2 + (q+3)^2c\big]\lambda^2 = 0\ . \tag{3.30}$$

Obviously, equation (3.30) must be a nontrivial polynomial equation of λ. This shows that λ is a constant, which is a contradiction.

In conclusion, we have completed the proof of Theorem 1.3.

Remark 3.1. The rotational hypersurface $x : \Sigma^4 \hookrightarrow \mathbb{R}^5$ in Theorem 1.3 has zero scalar curvature and its principal curvatures satisfies $-\lambda_1 = -\lambda_2 = -\lambda_3 = \lambda_4$. By Theorem 1 of [24], the rotational hypersurface $x : \Sigma^4 \hookrightarrow \mathbb{R}^5$ can be characterized by

$$\begin{aligned} &x(s,t_2,t_3,t_4) \\ &= (\psi(s)\cos t_2, \psi(s)\sin t_2\cos t_3, \psi(s)\sin t_2\sin t_3\cos t_4, \psi(s)\sin t_2\sin t_3\sin t_4, \varphi(s)) \end{aligned} \tag{3.31}$$

with the profile curve $(\psi(s), \varphi(s))$ satisfying

$${\psi'}^2(s) + {\varphi'}^2(s) = 1\ , \tag{3.32}$$

and

$$\varphi'\psi'' - \varphi''\psi' = \frac{\varphi'}{\psi}\ . \tag{3.33}$$

Moreover, the shape operator is given by

$$A = (-\lambda_4, -\lambda_4, -\lambda_4, \lambda_4) = \mathrm{diag}\,(\frac{\varphi'}{\psi}, \frac{\varphi'}{\psi}, \frac{\varphi'}{\psi}, -\frac{\varphi'}{\psi})\ . \tag{3.34}$$

Let λ stand for λ_4 as in the proof of Theorem 1.3. By taking into account (3.11), it is easy to check that

$$2\lambda\lambda'' - 3{\lambda'}^2 + 4\lambda^4 = 0\ . \tag{3.35}$$

The substitution $\omega(\lambda) = {\lambda'}^2$ leads to a first-order linear ordinary differential equation (ODE) as follows

$$\omega'(\lambda) - \frac{3}{\lambda}\omega = -4\lambda^3\ . \tag{3.36}$$

By the method of variation of parameters, the solution of ODE (3.36) is given by

$${\lambda'}^2 = \omega(\lambda) = \lambda^3(C - 4\lambda)$$

with a suitable constant C. In other word, λ satisfies

$$\lambda'(s) = \pm\sqrt{\lambda^3(C - 4\lambda)}\ , \tag{3.37}$$

which is a first-order separable equation.

With a suitable choice of a normal vector field we assume that $\lambda > 0$ in a neighborhood U_p. Taking in (3.37) $\lambda = f^{-2}$ we obtain

$$-2f'(s) = \pm\sqrt{C - 4f^{-2}}\ ,$$

which is equivalent to

$$\frac{df^2}{\sqrt{Cf^2 - 4}} = \mp ds\ . \tag{3.38}$$

Integrating Eq. (3.38) directly yields

$$\frac{2}{C}\sqrt{Cf^2 - 4} = \mp s + C'\ , \tag{3.39}$$

where C' is another constant. We thus get that the solution of Eq. (3.35) is of the form

$$\lambda(s) = f^{-2} = \frac{4C}{C^2(s + C_1)^2 + 16}\ , \tag{3.40}$$

where C and $C_1 := \mp C'$ are arbitrary constants.

Acknowledgements.

The first author wishes to thank Professor Yu-Xin Dong for his encouragement and invitation on the special volume. The authors also wish to thank Professor Cezar Oniciuc for his very useful suggestions to improve Theorem 1.3. This work is supported by Liaoning Provincial Education Department Project (No. LJKMZ20221561), the NSFC (No. 12101083) and Natural Science Foundation of Jiangsu Province (No. BK20210936).

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