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The stability of $\Phi_{(5)}$-harmonic maps

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Abstract. In this paper, we introduce the notion of $\Phi_{(5)}$-harmonic maps between Riemannian manifolds. We obtain the first and second variation formulas of the $\Phi_{(5)}$-energy functional $E_{\Phi_{(5)}}$. Then, we introduce the notion of $\Phi_{(5)}$-SSU manifolds and provide some examples of $\Phi_{(5)}$-SSU manifolds. Finally, by using the extrinsic average variational method in the calculus of variations and the second variation formula, we prove that the maps from compact $\Phi_{(5)}$-SSU manifold or into compact $\Phi_{(5)}$-SSU manifold which are stable $\Phi_{(5)}$-harmonic maps must be constant.

1. Introduction

Harmonic maps which appear in a broad spectrum of contexts in mathematics and physics, have had wide-ranging consequences and influenced developments in other fields (see, e.g. [1, 2, 3, 13]). A harmonic map $u : (M, g) \to (N, h)$ between Riemannian manifolds can be viewed as a critical point of the energy functional, given by the integral of a half of the first elementary symmetric function σ_1 of the eigenvalues of the pullback metric tensor u^*h relative to the metric g. Similarly, Φ-harmonic maps in [7] and $\Phi_{(3)}$-harmonic maps in [4] were introduced as a critical point of Φ-energy functional and $\Phi_{(3)}$-energy functional, given by a quarter of the second elementary symmetric function σ_2 and a sixth of the third elementary symmetric function σ_3 of the eigenvalues of the pullback metric tensor u^*h relative to the metric g, respectively. In [8, 9], Kawai and Nakauchi show nonconstant stable Φ-harmonic map either from $S^m(m \geq 5)$ to any manifold or from the compact minimal submanifold of S^{m+p} with $Ric_g > \frac{3}{4}mg$ to any manifold is non-existent, and they also proved nonconstant stable Φ-harmonic map either from any compact Riemannian manifold to $S^n(n \geq 5)$ or from any compact Riemannian manifold to the compact minimal submanifold of S^{n+p} with $Ric_g > \frac{3}{4}ng$ is non-existent. In [7],

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by using the extrinsic average variational method in the calculus of variations, Han and Wei generalized the results of Kawai and Nakauchi, found Φ-SSU manifolds, and proved that this manifolds can be neither the domain nor the target of any nonconstant stable Φ-harmonic maps. After defining a p-superstrongly unstable (p-SSU) manifold. By using the extrinsic average variational method in the calculus of variations, in [5], Feng-Han-Wei find $\Phi_{S,p}$-SSU manifolds and prove that any stable $\Phi_{S,p}$-harmonic map from or into a compact $\Phi_{S,p}$-SSU manifold (to or from a compact manifold) must be constant. In [4], Feng-Han-Jiang-Wei prove the maps from compact $\Phi_{(3)}$-SSU manifold or into compact $\Phi_{(3)}$-SSU manifold which are nonconstant stable $\Phi_{(3)}$-harmonic maps is non-existence.

In this paper, we extend the study of harmonic maps, Φ-harmonic maps and $\Phi_{(3)}$-harmonic maps and study the geometric properties of $\Phi_{(5)}$-harmonic maps. We give the definition of $\Phi_{(5)}$-harmonic maps. In fact, $\Phi_{(5)}$-harmonic map is a critical point of the $\Phi_{(5)}$-energy functional, given by the integral of a tenth of the fifth elementary symmetric function σ_5 of the eigenvalues of the pullback metric tensor u^*h relative to the metric g.

Let $u : (M^m, g) \to (N^n, h)$ be a smooth map between two Riemannian manifolds (M^m, g) and (N^n, h). Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame of (M^m, g). We give the definition of $\Phi_{(5)}$-harmonic map in the following.

Definition 1.1. Let $d_{(5)}u$ be a 1-from with values in the pullback bundle $u^{-1}TN$ given by

$$\begin{aligned} & d_{(5)}u(X) \\ &= \sum_{i,j,k,l=1}^{m} h\big(du(X), du(e_i)\big)h\big(du(e_i), du(e_j)h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)du(e_l) \end{aligned}$$

for any smooth vector field X on (M^m, g). The $\Phi_{(5)}$*-energy* is defined by

$$E_{\Phi_{(5)}}(u) = \int_M \frac{\|d_{(5)}u\|^2}{10}\, dv_g \ , \tag{1}$$

where $\|d_{(5)}u\|^2$ is given by

$$\begin{aligned} \|d_{(5)}u\|^2 &= \sum_{i_1=1}^{m} h\big(d_{(5)}u(e_{i_1}), du(e_{i_1})\big) \\ &= \sum_{i_1,i_2,\cdots,i_5=1}^{m} h\big(du(e_{i_1}), du(e_{i_2})\big)h\big(du(e_{i_2}), du(e_{i_3})\big)h\big(du(e_{i_3}), du(e_{i_4})\big) \\ &\qquad \times h\big(du(e_{i_4}), du(e_{i_5})\big)h\big(du(e_{i_5}), du(e_{i_1})\big) \\ &= \sum_{i_1,i_2,\cdots,i_5=1}^{m} h\big(du(e_{i_1}), du(e_{i_2})\big)\cdots h\big(du(e_{i_5}), du(e_{i_1})\big) \ . \end{aligned}$$

Definition 1.2. A smooth map u is called a $\Phi_{(5)}$*-harmonic map* if it is a critical point of the $\Phi_{(5)}$-energy functional $E_{\Phi_{(5)}}$ with respect to any smooth compactly supported variation of u.

We recall

Definition 1.3. A Riemannian manifold M^m, with the Riemannian metric $\langle\ ,\ \rangle_M$, is said to be *superstrongly unstable* (SSU) if there exists an isometric immersion of

M^m in $\mathbb{R}^q$, with second fundamental form B, such that for all unit tangent vectors $v \in T_x(M^m)$ the following symmetric linear operator Q_x^M

$$\langle Q_x^M(v), v\rangle_M = \sum_{i=1}^{m} \Big(2\big\langle B(v,e_i), B(v,e_i)\big\rangle_{\mathbb{R}^q} - \big\langle B(v,v), B(e_i,e_i)\big\rangle_{\mathbb{R}^q}\Big)$$

is negative definite. Also M is said to be *p-superstrongly unstable* (*p-SSU*) for $p \geq 2$ if the following functional

$$F_{p,x}(v) = (p-2)\big\langle B(v,v), B(v,v)\big\rangle_{\mathbb{R}^q} + \big\langle Q_x^M(v), v\big\rangle_M$$

is negative valued, where $\{e_i\}_{i=1}^m$ is a local orthonormal frame on M^m.

The notion of $\Phi_{(5)}$-SSU is defined as follows.

Definition 1.4. A Riemannian manifold M^m is said to be $\Phi_{(5)}$*-superstrongly unstable* ($\Phi_{(5)}$-SSU) if there exists an isometric immersion of M^m in $\mathbb{R}^q$, with second fundamental form B, such that for every unit tangent vector $v \in T_x(M^m)$ the following functional is always negative valued

$$F_{\Phi_{(5)x}}(v) = \sum_{i=1}^{m} \Big(10\big\langle B(v,e_i), B(v,e_i)\big\rangle_{\mathbb{R}^q} - \big\langle B(v,v), B(e_i,e_i)\big\rangle_{\mathbb{R}^q}\Big) ,$$

where $\{e_i\}_{i=1}^m$ is a local orthonormal frame on M^m.

The content of this chapter is arranged as follows: In Section 1, we present some basis materials. In Section 2, we give the first and second variation formulas of the functional $E_{\Phi_{(5)}}$ and define of the $\Phi_{(5)}$-harmonic map. In Section 3, we provide some examples of $\Phi_{(5)}$-SSU manifolds. In Section 4, by using the extrinsic average variational method in the calculus of variations and the second variation formula, we prove that the maps from compact $\Phi_{(5)}$-SSU manifold which are stable $\Phi_{(5)}$-harmonic maps must be constant. In Section 5, by using the similar method to Theorem 4.1, we prove that the maps into compact $\Phi_{(5)}$-SSU manifold which are stable $\Phi_{(5)}$-harmonic maps must be constant.

2. The First and Second Variation Formula

In this section, we will give the first and second variation formulas of the $\Phi_{(5)}$-energy functional.

Definition 2.1. Let $u : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds (M^m, g) and (N^n, h). The $\Phi_{(5)}$*-tension field of* u is defined by

$$\tau_{\Phi_{(5)}}(u) = \operatorname{div}\big(d_{(5)}u\big) = \sum_{k=1}^{m} \big(\nabla_{e_k} d_{(5)}u\big)(e_k) .$$

In the following, we will obtain the first variation formula for the $\Phi_{(5)}(u)$-energy $E_{\Phi_{(5)}}(u)$. Let ∇ and ${}^N\nabla$ denote the Levi-Civita connections of M and N respectively. Let $\widetilde{\nabla}$ be the induced connection on $u^{-1}TN$ defined by $\widetilde{\nabla}_X W = {}^N\nabla_{du(X)}W$, where X is a tangent vector of M and W is a section of $u^{-1}TN$. We will obtain the following result:

Theorem 2.2 (The first variation formula)**.** *Let $u : (M^m, g) \to (N^n, h)$ be a smooth map, and let $u_t : (M^m, g) \to (N^n, h)$ be a compactly supported variation such that $u_0 = u$ and set $V = \frac{\partial}{\partial t} u_t\Big|_{t=0}$. Then we have*

$$\frac{d}{dt} E_{\Phi_{(5)}}(u_t)\Big|_{t=0} = -\int_M h\big(V, \tau_{\Phi_{(5)}}(u)\big)\, dv_g \,. \tag{2}$$

Proof. Let $\Psi : (-\varepsilon, \varepsilon) \times M \to N$ be a smooth map defined by $\Psi(t, x) = u_t(x)$, where $(-\varepsilon, \varepsilon) \times M$ is equipped with the product metric. We extend the vector fields $\frac{\partial}{\partial t}$ on $(-\varepsilon, \varepsilon)$ and X on M naturally to $(-\varepsilon, \varepsilon) \times M$, and denote those also by $\frac{\partial}{\partial t}$, X. Then $V = d\Psi\left(\frac{\partial}{\partial t}\right)\Big|_{t=0}$. We also use the same notations ∇ and $\widetilde{\nabla}$ for the Levi-Civita connection on $(-\varepsilon, \varepsilon) \times M$ and the induced connection on $\Psi^{-1}TN$ respectively.

By directly computing, we have

$$\begin{aligned}
\frac{\partial}{\partial t} \frac{\|d_{(5)} u_t\|^2}{10} &= \frac{1}{10} \frac{\partial}{\partial t} \|d_{(5)} u_t\|^2 \\
&= \sum_{i_1, i_2, \cdots, i_5 = 1}^{m} h\big(\widetilde{\nabla}_{\frac{\partial}{\partial t}} d\Psi(e_{i_1}), d\Psi(e_{i_2})\big) h\big(d\Psi(e_{i_2}), d\Psi(e_{i_3})\big) \cdots \\
&\qquad \times h\big(d\Psi(e_{i_5}), d\Psi(e_{i_1})\big) \\
&= \sum_{i_1=1}^{m} h\left(\widetilde{\nabla}_{e_{i_1}} d\Psi\left(\frac{\partial}{\partial t}\right), d_{(5)}\Psi(e_{i_1})\right) \\
&= \sum_{i_1=1}^{m} \left[e_{i_1} h\left(d\Psi\left(\frac{\partial}{\partial t}\right), d_{(5)}\Psi(e_{i_1})\right) - h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \widetilde{\nabla}_{e_{i_1}} d_{(5)}\Psi(e_{i_1})\right)\right].
\end{aligned}$$

Here we use $\widetilde{\nabla}_{\frac{\partial}{\partial t}} d\Psi(e_{i_1}) - \widetilde{\nabla}_{e_{i_1}} d\Psi\left(\frac{\partial}{\partial t}\right) = d\Psi\left(\left[\frac{\partial}{\partial t}, e_{i_1}\right]\right) = 0$ in the third equality. Let X_t be a compactly supported vector field on M such that $g(X_t, Y) = h\left(d\Psi\left(\frac{\partial}{\partial t}\right), d_{(5)}\Psi(Y)\right)$ for any vector field Y on M. Then we have

$$\begin{aligned}
&\frac{\partial}{\partial t} \frac{\|d_{(5)} u_t\|^2}{10} \\
&= \sum_{i_1=1}^{m} \left[e_{i_1} h\left(d\Psi\left(\frac{\partial}{\partial t}\right), d_{(5)}\Psi(e_{i_1})\right) - h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \widetilde{\nabla}_{e_{i_1}} d_{(5)}\Psi(e_{i_1})\right)\right] \\
&= \sum_{i_1=1}^{m} \left[e_{i_1} g(X_t, e_{i_1}) - h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \widetilde{\nabla}_{e_{i_1}} d_{(5)}\Psi(e_{i_1})\right)\right] \\
&= \operatorname{div} X_t - h\left(d\Psi\left(\frac{\partial}{\partial t}\right), \operatorname{div}\big(d_{(5)}\Psi\big)\right) \,.
\end{aligned} \tag{3}$$

From (3) and Green's theorem, we have

$$\frac{d}{dt} E_{\Phi_{(5)}}(u_t)\Big|_{t=0} = -\int_M h\big(V, \tau_{\Phi_{(5)}}(u)\big)\, dv_g \,.$$

This completes the proof.

□

By using the definition of $\Phi_{(5)}$-harmonic maps and Theorem 2.2, we have the following.

Proposition 2.3. *A smooth map $u : (M^m, g) \to (N^n, h)$ is a $\Phi_{(5)}$-harmonic map if and only if u is a solution of the Euler-Lagrange equation for the $\Phi_{(5)}$-energy functional $E_{\Phi_{(5)}}(u)$, that is,*

$$\tau_{\Phi_{(5)}}(u) = \operatorname{div}\left(d_{(5)}u\right) = \sum_{k=1}^{m}\left(\nabla_{e_k} d_{(5)}u\right)(e_k) = 0 \,.$$

In the following, we calculate the second variation of the functional $E_{\Phi_{(5)}}(u)$, and obtain the following results:

Theorem 2.4 (The second variation formula)**.** *Suppose $u : (M^m, g) \to (N^n, h)$ is a $\Phi_{(5)}$-harmonic map for the functional $E_{\Phi_{(5)}}$ and $u_{s,t} : M^m \to N$ $(-\varepsilon < s,t < \varepsilon)$ is a compactly supported two-parameter variation such that $u_{0,0} = u$ and set $V = \frac{\partial}{\partial t}u_{s,t}\Big|_{s,t=0}$, $W = \frac{\partial}{\partial s}u_{s,t}\Big|_{s,t=0}$. Then we have*

$$\begin{aligned}
&\frac{\partial^2}{\partial s\partial t}E_{\Phi_{(5)}}(u_{s,t})\Big|_{s,t=0}\\
&= \int_M \sum_{i_1,i_2,i_3,i_4,i_5=1}^{m} h\left(R^N\left(V, du(e_{i_1})\right)W, d_{(5)}u(e_{i_1})\right)\, dv_g\\
&+ \int_M \sum_{i_1,i_2,i_3,i_4,i_5=1}^{m} h\left(\widetilde{\nabla}_{e_{i_1}}V, \widetilde{\nabla}_{e_{i_5}}W\right) h\big(du(e_{i_1}), du(e_{i_2})\big)\\
&\quad \times h\big(du(e_{i_2}), du(e_{i_3})\big)h\big(du(e_{i_3}), du(e_{i_4})\big)h\big(du(e_{i_4}), du(e_{i_5})\big)\, dv_g\\
&+ \int_M \sum_{i_1,i_2,i_3,i_4,i_5=1}^{m} h\left(\widetilde{\nabla}_{e_{i_1}}V, du(e_{i_5})\right)\\
&\quad \times \left[h\left(\widetilde{\nabla}_{e_{i_1}}W, du(e_{i_2})\right) + h\left(du(e_{i_1}), \widetilde{\nabla}_{e_{i_2}}W\right)\right]\\
&\quad \times h\big(du(e_{i_2}), du(e_{i_3})\big)h\big(du(e_{i_3}), du(e_{i_4})\big)h\big(du(e_{i_4}), du(e_{i_5})\big)\, dv_g\\
&+ \int_M \sum_{i_1,i_2,i_3,i_4,i_5=1}^{m} h\left(\widetilde{\nabla}_{e_{i_1}}V, du(e_{i_5})\right)\\
&\quad \times \left[h\left(\widetilde{\nabla}_{e_{i_2}}W, du(e_{i_3})\right) + h\left(du(e_{i_2}), \widetilde{\nabla}_{e_{i_3}}W\right)\right]\\
&\quad \times h\big(du(e_{i_1}), du(e_{i_2})\big)h\big(du(e_{i_3}), du(e_{i_4})\big)h\big(du(e_{i_4}), du(e_{i_5})\big)\, dv_g\\
&+ \int_M \sum_{i_1,i_2,i_3,i_4,i_5=1}^{m} h\left(\widetilde{\nabla}_{e_{i_1}}V, du(e_{i_5})\right)\\
&\quad \times \left[h\left(\widetilde{\nabla}_{e_{i_3}}W, du(e_{i_4})\right) + h\left(du(e_{i_3}), \widetilde{\nabla}_{e_{i_4}}W\right)\right]\\
&\quad \times h\big(du(e_{i_1}), du(e_{i_2})\big)h\big(du(e_{i_2}), du(e_{i_3})\big)h\big(du(e_{i_4}), du(e_{i_5})\big)\, dv_g
\end{aligned}$$

$$
\begin{aligned}
&+\int_M \sum_{i_1,i_2,i_3,i_4,i_5=1}^{m} h\left(\widetilde{\nabla}_{e_{i_1}}V, du(e_{i_5})\right) \\
&\quad\times\left[h\left(\widetilde{\nabla}_{e_{i_4}}W, du(e_{i_5})\right)+h\left(du(e_{i_4}),\widetilde{\nabla}_{e_{i_5}}W\right)\right] \\
&\quad\times h\big(du(e_{i_1}),du(e_{i_2})\big)h\big(du(e_{i_2}),du(e_{i_3})\big)h\big(du(e_{i_3}),du(e_{i_4})\big)\, dv_g\ , \qquad (4)
\end{aligned}
$$

where R^N denotes the curvature tensor of N.

We put

$$I(V,V)=\frac{\partial^2}{\partial s\partial t}E_{\Phi_{(5)}}(u_{s,t})\Big|_{s,t=0}.$$

A $\Phi_{(5)}$-harmonic map is called stable if $I(V,V)\geq 0$ for any compactly supported vector field V along u.

Proof. Let $\Psi:(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times M\to N$ be a smooth map defined by $\Psi(s,t,x)=u_{s,t}(x)$, where $(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times M$ is equipped with the product metric. We extend the vector fields $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial s}$ on $(-\varepsilon,\varepsilon)$ and X on M naturally to $(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times M$, and denote those also by $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial s}$, X. Then $V=d\Psi\left(\frac{\partial}{\partial t}\right)\Big|_{s,t=0}$, $W=d\Psi\left(\frac{\partial}{\partial s}\right)\Big|_{s,t=0}$. We also use the same notations ∇ and $\widetilde{\nabla}$ for the Levi-Civita connection on $(-\varepsilon,\varepsilon)\times(-\varepsilon,\varepsilon)\times M$ and the induced connection on $\Psi^{-1}TN$ respectively. We use $\{e_i\}_{i=1}^m$ to denote a locally orthonormal frame on M and fix any point $x_0\in M$ such that $\nabla_{e_i}e_j\big|_{x_0}=0$ for any i,j. Using (2) and Proposition 2.3, we have

$$
\begin{aligned}
&\frac{\partial^2}{\partial s\partial t}E_{\Phi_{(5)}}(u_{s,t})\Big|_{s,t=0} \\
&=-\frac{\partial}{\partial s}\int_M h\Big(d\Psi\Big(\frac{\partial}{\partial t}\Big),\tau_{\Phi_{(5)}}u_{s,t}\Big)\Big|_{s,t=0}\,dv_g \\
&=-\sum_{i_1=1}^{m}\frac{\partial}{\partial s}\int_M h\left(d\Psi\left(\frac{\partial}{\partial t}\right),\widetilde{\nabla}_{e_{i_1}}d_{(5)}u_{s,t}(e_{i_1})\right)\Big|_{s,t=0}\,dv_g \\
&=-\int_M\sum_{i_1}h\left(d\Psi\left(\frac{\partial}{\partial t}\right),\widetilde{\nabla}_{\frac{\partial}{\partial s}}\widetilde{\nabla}_{e_{i_1}}d_{(5)}u_{s,t}(e_{i_1})\right)\Big|_{s,t=0}\,dv_g \\
&=-\int_M\sum_{i_1=1}^{m}h\left(d\Psi\left(\frac{\partial}{\partial t}\right),\widetilde{\nabla}_{e_{i_1}}\widetilde{\nabla}_{\frac{\partial}{\partial s}}d_{(5)}u_{s,t}(e_{i_1})\right)\Big|_{s,t=0}\,dv_g \\
&\quad-\int_M\sum_{i_1=1}^{m}h\left(d\Psi\left(\frac{\partial}{\partial t}\right),R^N\left(d\Psi\left(\frac{\partial}{\partial s}\right),d\Psi(e_{i_1})\right)d_{(5)}u_{s,t}(e_{i_1})\right)\Big|_{s,t=0}\,dv_g \\
&=-\int_M\sum_{i_1=1}^{m}e_{i_1}\left(h\left(d\Psi\left(\frac{\partial}{\partial t}\right),\widetilde{\nabla}_{\frac{\partial}{\partial s}}d_{(5)}u_{s,t}(e_{i_1})\right)\right)\Big|_{s,t=0}\,dv_g \\
&\quad+\int_M\sum_{i_1=1}^{m}h\left(\widetilde{\nabla}_{e_{i_1}}d\Psi\left(\frac{\partial}{\partial t}\right),\widetilde{\nabla}_{\frac{\partial}{\partial s}}d_{(5)}u_{s,t}(e_{i_1})\right)\Big|_{s,t=0}\,dv_g
\end{aligned}
$$

$$(5) \quad -\int_M \sum_{i_1=1}^m h\left(d\Psi\left(\frac{\partial}{\partial t}\right), R^N\left(d\Psi\left(\frac{\partial}{\partial s}\right), d\Psi(e_{i_1})\right) d_{(5)}u_{s,t}(e_{i_1})\right)\Big|_{s,t=0} dv_g \,.$$

The second term in the right-hand side of (5),

$$\begin{aligned}
&\sum_{i_1=1}^m h\Big(\widetilde{\nabla}_{e_{i_1}} d\Psi\left(\frac{\partial}{\partial t}\right), \widetilde{\nabla}_{\frac{\partial}{\partial s}} d_{(5)}u_{s,t}(e_{i_1})\Big)\\
=& \sum_{i_1,i_2,i_3,i_4,i_5=1}^m h\Big(\widetilde{\nabla}_{e_{i_1}} d\Psi\Big(\frac{\partial}{\partial t}\Big), \widetilde{\nabla}_{\frac{\partial}{\partial s}} \Big(h\big(d\Psi(e_{i_1}), d\Psi(e_{i_2})\big)\\
&\quad \times h\big(d\Psi(e_{i_2}), d\Psi(e_{i_3})\big) h\big(d\Psi(e_{i_3}), d\Psi(e_{i_4})\big) h\big(d\Psi(e_{i_4}), d\Psi(e_{i_5})\big) d\Psi(e_{i_5})\Big)\Big)\\
=& \sum_{i_1,\cdots,i_5=1}^m h\Big(\widetilde{\nabla}_{e_{i_1}} d\Psi\left(\frac{\partial}{\partial t}\right), \widetilde{\nabla}_{e_{i_5}} d\Psi\Big(\frac{\partial}{\partial s}\Big)\Big) h\big(d\Psi(e_{i_1}), d\Psi(e_{i_2})\big)\\
&\quad \times h\big(d\Psi(e_{i_2}), d\Psi(e_{i_3})\big) h\big(d\Psi(e_{i_3}), d\Psi(e_{i_4})\big) h\big(d\Psi(e_{i_4}), d\Psi(e_{i_5})\big)\\
&+ \sum_{i_1,\cdots,i_5=1}^m h\Big(\widetilde{\nabla}_{e_{i_1}} d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_{i_5})\Big)\\
&\quad \times \Big[h\Big(\widetilde{\nabla}_{e_{i_1}} d\Psi\Big(\frac{\partial}{\partial s}\Big), d\Psi(e_{i_2})\Big) + h\Big(d\Psi(e_{i_1}), \widetilde{\nabla}_{e_{i_2}} d\Psi\Big(\frac{\partial}{\partial s}\Big)\Big)\Big]\\
&\quad \times h\big(d\Psi(e_{i_2}), d\Psi(e_{i_3})\big) h\big(d\Psi(e_{i_3}), d\Psi(e_{i_4})\big) h\big(d\Psi(e_{i_4}), d\Psi(e_{i_5})\big)\\
&+ \sum_{i_1,\cdots,i_5=1}^m h\Big(\widetilde{\nabla}_{e_{i_1}} d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_{i_5})\Big)\\
&\quad \times \Big[h\Big(\widetilde{\nabla}_{e_{i_2}} d\Psi\Big(\frac{\partial}{\partial s}\Big), d\Psi(e_{i_3})\Big) + h\Big(d\Psi(e_{i_2}), \widetilde{\nabla}_{e_{i_3}} d\Psi\Big(\frac{\partial}{\partial s}\Big)\Big)\Big]\\
&\quad \times h\big(d\Psi(e_{i_1}), d\Psi(e_{i_2})\big) h\big(d\Psi(e_{i_3}), d\Psi(e_{i_4})\big) h\big(d\Psi(e_{i_4}), d\Psi(e_{i_5})\big)\\
&+ \sum_{i_1,\cdots,i_5=1}^m h\Big(\widetilde{\nabla}_{e_{i_1}} d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_{i_5})\Big)\\
&\quad \times \Big[h\Big(\widetilde{\nabla}_{e_{i_3}} d\Psi\Big(\frac{\partial}{\partial s}\Big), d\Psi(e_{i_4})\Big) + h\Big(d\Psi(e_{i_3}), \widetilde{\nabla}_{e_{i_4}} d\Psi\Big(\frac{\partial}{\partial s}\Big)\Big)\Big]\\
&\quad \times h\big(d\Psi(e_{i_1}), d\Psi(e_{i_2})\big) h\big(d\Psi(e_{i_2}), d\Psi(e_{i_3})\big) h\big(d\Psi(e_{i_4}), d\Psi(e_{i_5})\big)\\
&+ \sum_{i_1,\cdots,i_5=1}^m h\Big(\widetilde{\nabla}_{e_{i_1}} d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_{i_5})\Big)\\
&\quad \times \Big[h\Big(\widetilde{\nabla}_{e_{i_4}} d\Psi\Big(\frac{\partial}{\partial s}\Big), d\Psi(e_{i_5})\Big) + h\Big(d\Psi(e_{i_4}), \widetilde{\nabla}_{e_{i_5}} d\Psi\Big(\frac{\partial}{\partial s}\Big)\Big)\Big]\\
(6) &\quad \times h\big(d\Psi(e_{i_1}), d\Psi(e_{i_2})\big) h\big(d\Psi(e_{i_2}), d\Psi(e_{i_3})\big) h\big(d\Psi(e_{i_3}), d\Psi(e_{i_4})\big) \,.
\end{aligned}$$

The integrand for the first term in the right-hand side of (5) is

$$\sum_{i_1=1}^m e_{i_1}\Big(h\Big(d\Psi\left(\frac{\partial}{\partial t}\right), \widetilde{\nabla}_{\frac{\partial}{\partial s}} d_{(5)}u_{s,t}(e_{i_1})\Big)\Big)$$

$$
\begin{aligned}
&= \sum_{i_1,i_2,i_3,i_4,i_5=1}^{m} e_{i_1}\Big(h\Big(d\Psi\left(\frac{\partial}{\partial t}\right), \widetilde{\nabla}_{\frac{\partial}{\partial s}}\big(h\big(d\Psi(e_{i_1}), d\Psi(e_{i_2})\big)\\
&\qquad \times h\big(d\Psi(e_{i_2}), d\Psi(e_{i_3})\big)h\big(d\Psi(e_{i_3}), d\Psi(e_{i_4})\big)h\big(d\Psi(e_{i_4}), d\Psi(e_{i_5})\big)d\Psi(e_{i_5})\big)\Big)\Big)\\
&= \sum_{i_1,\cdots,i_5=1}^{m} e_{i_1}\Big(h\Big(d\Psi\left(\frac{\partial}{\partial t}\right), \widetilde{\nabla}_{e_{i_5}} d\Psi\Big(\frac{\partial}{\partial s}\Big)\Big)h\big(d\Psi(e_{i_1}), d\Psi(e_{i_2})\big)\\
&\qquad \times h\big(d\Psi(e_{i_2}), d\Psi(e_{i_3})\big)h\big(d\Psi(e_{i_3}), d\Psi(e_{i_4})\big)h\big(d\Psi(e_{i_4}), d\Psi(e_{i_5})\big)\Big)\\
&\quad + \sum_{i_1,\cdots,i_5=1}^{m} e_{i_1}\Big(h\Big(d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_{i_5})\Big)\\
&\qquad \times\Big[h\Big(\widetilde{\nabla}_{e_{i_1}} d\Psi\Big(\frac{\partial}{\partial s}\Big), d\Psi(e_{i_2})\Big) + h\Big(d\Psi(e_{i_1}), \widetilde{\nabla}_{e_{i_2}} d\Psi\Big(\frac{\partial}{\partial s}\Big)\Big)\Big]\\
&\qquad \times h\big(d\Psi(e_{i_2}), d\Psi(e_{i_3})\big)h\big(d\Psi(e_{i_3}), d\Psi(e_{i_4})\big)h\big(d\Psi(e_{i_4}), d\Psi(e_{i_5})\big)\Big)\\
&\quad + \sum_{i_1,\cdots,i_5=1}^{m} e_{i_1}\Big(h\Big(d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_{i_5})\Big)\\
&\qquad \times\Big[h\Big(\widetilde{\nabla}_{e_{i_2}} d\Psi\Big(\frac{\partial}{\partial s}\Big), d\Psi(e_{i_3})\Big) + h\Big(d\Psi(e_{i_2}), \widetilde{\nabla}_{e_{i_3}} d\Psi\Big(\frac{\partial}{\partial s}\Big)\Big)\Big]\\
&\qquad \times h\big(d\Psi(e_{i_1}), d\Psi(e_{i_2})\big)h\big(d\Psi(e_{i_3}), d\Psi(e_{i_4})\big)h\big(d\Psi(e_{i_4}), d\Psi(e_{i_5})\big)\Big)\\
&\quad + \sum_{i_1,\cdots,i_5=1}^{m} e_{i_1}\Big(h\Big(d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_{i_5})\Big)\\
&\qquad \times\Big[h\Big(\widetilde{\nabla}_{e_{i_3}} d\Psi\Big(\frac{\partial}{\partial s}\Big), d\Psi(e_{i_4})\Big) + h\Big(d\Psi(e_{i_3}), \widetilde{\nabla}_{e_{i_4}} d\Psi\Big(\frac{\partial}{\partial s}\Big)\Big)\Big]\\
&\qquad \times h\big(d\Psi(e_{i_1}), d\Psi(e_{i_2})\big)h\big(d\Psi(e_{i_2}), d\Psi(e_{i_3})\big)h\big(d\Psi(e_{i_4}), d\Psi(e_{i_5})\big)\Big)\\
&\quad + \sum_{i_1,\cdots,i_5=1}^{m} e_{i_1}\Big(h\Big(d\Psi\left(\frac{\partial}{\partial t}\right), d\Psi(e_{i_5})\Big)\\
&\qquad \times\Big[h\Big(\widetilde{\nabla}_{e_{i_4}} d\Psi\Big(\frac{\partial}{\partial s}\Big), d\Psi(e_{i_5})\Big) + h\Big(d\Psi(e_{i_4}), \widetilde{\nabla}_{e_{i_5}} d\Psi\Big(\frac{\partial}{\partial s}\Big)\Big)\Big]\\
&\qquad \times h\big(d\Psi(e_{i_1}), d\Psi(e_{i_2})\big)h\big(d\Psi(e_{i_2}), d\Psi(e_{i_3})\big)h\big(d\Psi(e_{i_3}), d\Psi(e_{i_4})\big)\Big)\,. \qquad (7)
\end{aligned}
$$

Let X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8 and X_9 be compactly supported vector fields on M and Y be any vector field on M. We have

$$
\begin{aligned}
g(X_1, Y) &= \sum_{i_2,i_3,i_4,i_5=1}^{m} h(V, \widetilde{\nabla}_{e_{i_5}} W)h\big(du(Y), du(e_{i_2})\big)h\big(du(e_{i_2}), du(e_{i_3})\big)\\
&\qquad \times h\big(du(e_{i_3}), du(e_{i_4})\big)h\big(du(e_{i_4}), du(e_{i_5})\big)\ ,\\
g(X_2, Y) &= \sum_{i_2,i_3,i_4,i_5=1}^{m} h\big(V, du(e_{i_5})\big)h\big(\widetilde{\nabla}_Y W, du(e_{i_2})\big)h\big(du(e_{i_2}), du(e_{i_3})\big)\\
&\qquad \times h\big(du(e_{i_3}), du(e_{i_4})\big)h\big(du(e_{i_4}), du(e_{i_5})\big)\ ,
\end{aligned}
$$

$$g(X_3,Y)=\sum_{i_2,i_3,i_4,i_5=1}^{m} h\big(V,du(e_{i_5})\big)h\big(du(Y),\widetilde{\nabla}_{e_{i_2}}W\big)h\big(du(e_{i_2}),du(e_{i_3})\big)$$
$$\times h\big(du(e_{i_3}),du(e_{i_4})\big)h\big(du(e_{i_4}),du(e_{i_5})\big)\ ,$$

$$g(X_4,Y)=\sum_{i_2,i_3,i_4,i_5=1}^{m} h\big(V,du(e_{i_5})\big)h\big(\widetilde{\nabla}_{e_{i_2}}W,du(e_{i_3})\big)h\big(du(Y),du(e_{i_2})\big)$$
$$\times h\big(du(e_{i_3}),du(e_{i_4})\big)h\big(du(e_{i_4}),du(e_{i_5})\big)\ ,$$

$$g(X_5,Y)=\sum_{i_2,i_3,i_4,i_5=1}^{m} h\big(V,du(e_{i_5})\big)h\big(du(e_{i_2}),\widetilde{\nabla}_{e_{i_3}}W\big)h\big(du(Y),du(e_{i_2})\big)$$
$$\times h\big(du(e_{i_3}),du(e_{i_4})\big)h\big(du(e_{i_4}),du(e_{i_5})\big)\ ,$$

$$g(X_6,Y)=\sum_{i_2,i_3,i_4,i_5=1}^{m} h\big(V,du(e_{i_5})\big)h\big(\widetilde{\nabla}_{e_{i_3}}W,du(e_{i_4})\big)h\big(du(Y),du(e_{i_2})\big)$$
$$\times h\big(du(e_{i_2}),du(e_{i_3})\big)h\big(du(e_{i_4}),du(e_{i_5})\big)\ ,$$

$$g(X_7,Y)=\sum_{i_2,i_3,i_4,i_5=1}^{m} h\big(V,du(e_{i_5})\big)h\big(du(e_{i_3}),\widetilde{\nabla}_{e_{i_4}}W\big)h\big(du(Y),du(e_{i_2})\big)$$
$$\times h\big(du(e_{i_2}),du(e_{i_3})\big)h\big(du(e_{i_4}),du(e_{i_5})\big)\ ,$$

$$g(X_8,Y)=\sum_{i_2,i_3,i_4,i_5=1}^{m} h\big(V,du(e_{i_5})\big)h\big(\widetilde{\nabla}_{e_{i_4}}W,du(e_{i_5})\big)h\big(du(Y),du(e_{i_2})\big)$$
$$\times h\big(du(e_{i_2}),du(e_{i_3})\big)h\big(du(e_{i_3}),du(e_{i_4})\big)\ ,$$

$$g(X_9,Y)=\sum_{i_2,i_3,i_4,i_5=1}^{m} h\big(V,du(e_{i_5})\big)h\big(du(e_{i_4}),\widetilde{\nabla}_{e_{i_5}}W\big)h\big(du(Y),du(e_{i_2})\big)$$
$$\times h\big(du(e_{i_2}),du(e_{i_3})\big)h\big(du(e_{i_3}),du(e_{i_4})\big)\ .$$

Hence, when $s,t=0$, (7) becomes

$$\sum_{e_{i_1}=1}^{m}\big(e_{i_1}g(X_1,e_{i_1})+e_{i_1}g(X_2,e_{i_1})+e_{i_1}g(X_3,e_{i_1})+\cdots+e_{i_1}g(X_9,e_{i_1})\big)$$
$$=\operatorname{div}(X_1)+\operatorname{div}(X_2)+\operatorname{div}(X_3)+\cdots+\operatorname{div}(X_9)\ . \tag{8}$$

By Green's theorem and the integral of (8) vanishes. The theorem follows from (5) -(8).

□

3. Examples of $\Phi_{(5)}$-SSU manifolds

In this section, we obtain some examples of $\Phi_{(5)}$-SSU manifolds.

Theorem 3.1. *Let $M^m\subset\mathbb{R}^{m+1}$, $(m\geq 10)$ be the compact hypersurface. Assuming that the principal curvatures λ_i of M^m satisfy $0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_m$ and $9\lambda_m<\sum_{i=1}^{m-1}\lambda_i$. Then M is $\Phi_{(5)}$-SSU.*

Proof. Similar to the proof of Theorem 5.1 in [6] from the definition of the $\Phi_{(5)}$-SSU. We have

$$\sum_{i=1}^{m}\Big(10\big\langle B(v,e_i),B(v,e_i)\big\rangle_{\mathbb{R}^q}-\big\langle B(v,v),B(e_i,e_i)\big\rangle_{\mathbb{R}^q}\Big)$$
$$\leq \lambda_i\left(10\lambda_m-\sum_{i=1}^{m}\lambda_i\right)=\lambda_i\left(9\lambda_m-\sum_{i=1}^{m-1}\lambda_i\right)<0\ .$$

This completes the proof. □

Using theorem 3.1 we have

Corollary 3.2. *The standard sphere S^m is $\Phi_{(5)}$-SSU if and only if $m > 10$.*

Proof. Since S^m is a compact convex hypersurface in $\mathbb{R}^{m+1}$. By Theorem 3.1 and its principle curvatures satisfy

$$\lambda_1=\lambda_2=\cdots=\lambda_m=1\ .$$

That is, $m > 10$. This completes the proof. □

Corollary 3.3. *The graph of $f(x) = x_1^2 + \cdots + x_m^2$, $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$ is $\Phi_{(5)}$-SSU if and only if $m > 10$.*

Lemma 3.4 ([12])**.** *A Euclidean hypersurface M is p-SSU if and only if its principle curvatures satisfy*

$$0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_m \quad \textit{and} \quad (p-1)\lambda_m<\sum_{i=1}^{m-1}\lambda_i\ .$$

Theorem 3.5. *Every $\Phi_{(5)}$-SSU manifold M is p-SSU for any $2 \leq p \leq 10$.*

Proof. By definition, $\Phi_{(5)}$-SSU manifold satisfy

$$F_{\Phi_{(5)x}}(v)=\sum_{i=1}^{m}\Big(10\big\langle B(v,e_i),B(v,e_i)\big\rangle_{\mathbb{R}^q}-\big\langle B(v,v),B(e_i,e_i)\big\rangle_{\mathbb{R}^q}\Big)<0$$

for all unit tangent vector $v \in T_x(M)$. It follows that

$$\begin{aligned}
F_{p,x}(v) &= (p-2)\big\langle B(v,v),B(v,v)\big\rangle_{\mathbb{R}^q}+\big\langle Q_x^M(v),v\big\rangle_M\\
&= (p-2)\big\langle B(v,v),B(v,v)\big\rangle_{\mathbb{R}^q}\\
&\quad +\sum_{i=1}^{m}\Big(2\big\langle B(v,e_i),B(v,e_i)\big\rangle_{\mathbb{R}^q}-\big\langle B(v,v),B(e_i,e_i)\big\rangle_{\mathbb{R}^q}\Big)\\
&\leq (p-2)\sum_{i=1}^{m}\big\langle B(v,e_i),B(v,e_i)\big\rangle_{\mathbb{R}^q}\\
&\quad +\sum_{i=1}^{m}\Big(2\big\langle B(v,e_i),B(v,e_i)\big\rangle_{\mathbb{R}^q}-\big\langle B(v,v),B(e_i,e_i)\big\rangle_{\mathbb{R}^q}\Big)\\
&= \sum_{i=1}^{m}\Big(p\big\langle B(v,e_i),B(v,e_i)\big\rangle-\big\langle B(v,v),B(e_i,e_i)\big\rangle\Big)
\end{aligned}$$

$$(9) \qquad \leq \sum_{i=1}^{m} \Big(10\big\langle B(v,e_i), B(v,e_i)\big\rangle - \big\langle B(v,v), B(e_i,e_i)\big\rangle \Big) < 0 \, .$$

Thus, M is p-SSU for any $2 \leq p \leq 10$.

□

Theorem 3.6. *Every compact $\Phi_{(5)}$-SSU manifold M is 10-connected, i.e.,*

$$\pi_1(M) = \pi_2(M) = \cdots = \pi_{10}(M) = 0 \, .$$

Proof. Because all compact p-SSU manifolds are $[p]$-connected (cf. Theorem 3.10 in [11]) and $p = 10$. By the previous theorem, the result follows.

□

Theorem 3.7. *The dimension of any compact $\Phi_{(5)}$-SSU manifold M is greater than* 10*.*

Proof. Assuming that $m \leq 10$, then M is not a 10-SSU manifold. By the previous theorem, M is not a $\Phi_{(5)}$-SSU manifold. Thus, the dimension of any compact $\Phi_{(5)}$-SSU manifold M is greater than 10. This completes the proof.

□

Theorem 3.8. *Let $\widetilde{M} \in \mathbb{R}^q$ is a compact convex hypersurface and its principal curvatures satisfy*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{q-1} \, .$$

If

$$\big\langle Ric^M(v), v\big\rangle > \frac{9}{10} k \lambda_{q-1}^2$$

then M is $\Phi_{(5)}$-SSU, where $M \in \widetilde{M}$ is a compact connected minimal k-submanifold and v is any unit tangent vector to M.

Proof. Let B, B_1 and $\widetilde{B}$ denote the second fundamental form of $M \in \mathbb{R}^q$, $M \in \widetilde{M}$ and $\widetilde{M} \in \mathbb{R}^q$, respectively. According to Gauss equation, we obtain

$$B(X,Y) = B_1(X,Y) + \widetilde{B}(X,Y)\mu \, ,$$

where μ is the unit normal field of $\widetilde{M} \in \mathbb{R}^q$. By the definition of minimal submanifold, we have

$$\sum_{i=1}^{k} B(e_i,e_i) = \sum_{i=1}^{k} B_1(e_i,e_i) + \sum_{i=1}^{k} \widetilde{B}(e_i,e_i)\mu = \sum_{i=1}^{k} \widetilde{B}(e_i,e_i)\mu \, ,$$

where $\{e_i\}_{i=1}^k$ is a local orthonormal frame on M. Let $\widetilde{B}(e_i,e_j) = \lambda_i \delta_{ij}$.

Hence,

$$\begin{aligned} &\sum_{i=1}^{k} \Big(10\big\langle B(v,e_i), B(v,e_i)\big\rangle - \big\langle B(v,v), B(e_i,e_i)\big\rangle \Big) \\ &= -10\big\langle Ric^M(v), v\big\rangle + \sum_{i=1}^{k} \Big(10\big\langle B(v,v), B(e_i,e_i)\big\rangle - \big\langle B(v,v), B(e_i,e_i)\big\rangle \Big) \\ &= -10\big\langle Ric^M(v), v\big\rangle + \sum_{i=1}^{k} 9\big\langle B(v,v), B(e_i,e_i)\big\rangle \end{aligned}$$

$$= -10\big\langle Ric^M(v), v\big\rangle + 9\sum_{i=1}^{k}\big\langle \widetilde{B}(v,v), \widetilde{B}(e_i,e_i)\big\rangle$$

$$\leq -10\big\langle Ric^M(v), v\big\rangle + 9\sum_{i=1}^{k}\lambda_i\lambda_{q-1}$$

$$\leq -10\big\langle Ric^M(v), v\big\rangle + 9k\lambda_{q-1}^2 < 0 \ , \tag{10}$$

where in the first equality we use Gauss equation. This completes the proof. □

4. Stable $\Phi_{(5)}$-harmonic maps from $\Phi_{(5)}$-SSU manifolds

In this section, we first recall some definitions and facts of submanifolds, which will be used in the following results(cf. [7]).

We assume M^m is isometrically immersed in the Euclidean space $\mathbb{R}^q$. Let $\overline{\nabla}$ and ∇ denote the standard flat connection on $\mathbb{R}^q$ and the Riemannian connection on M respectively, and B is the second fundamental form of M^m in $\mathbb{R}^q$. These are related by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y) \ ,$$

where X, Y are smooth vector fields on M. If $T^{\perp}M$ is the normal bundle of the M in $\mathbb{R}^q$, η is a smooth section of $T^{\perp}M$. Then the tensors A and B are related by

$$\langle A^{\eta}X, Y\rangle = \langle B(X,Y), \eta\rangle \ ,$$

where $A^{\eta}X$ is the Weingarten map.

For each $x \in M$, let $\{e_\alpha\}_{\alpha=m+1}^{q}$ denote an orthonormal basis for the normal space $T^{\perp}M_x$ to M at x. Define the Ricci tensor $Ric^M : T_x(M) \to T_x(M)$ by

$$Ric^M(v) = \sum_{i=1}^{m} R(v, e_i)e_i \ .$$

Then the Guass curvature equation implies

$$Ric^M = \sum_{\alpha=m+1}^{q} \operatorname{trace}\big(A^{\alpha}\big)A^{\alpha} - \sum_{\alpha=m+1}^{q} A^{\alpha}A^{\alpha} \ ,$$

where $v \in T_x(M)$.

Theorem 4.1. *Let $u : (M^m, g) \to (N^n, h)$ be a $\Phi_{(5)}$-harmonic map, and let (M^m, g) be a compact $\Phi_{(5)}$-SSU manifold and (N^n, h) be a Riemannian manifold. Then all stable $\Phi_{(5)}$-harmonic map u is constant.*

Proof. Now we choose an orthonormal basis $\{e_i\}_{i=1}^{q}$ of $\mathbb{R}^q$, such that $\{e_i\}_{i=1}^{m}$ are tangent to M^m, $\{e_\alpha\}_{\alpha=m+1}^{q}$ are normal to M^m and $\nabla_{e_i}e_j\big|_{x_0} = 0$, where x_0 is a fixed point of M^m. Meanwhile, we take a fixed orthonormal basis of $\mathbb{R}^q$ denoted by $\{E_A\}_{A=1}^{q}$, and set

$$V_A = \sum_{i=1}^{m} v_A^i e_i \quad , \quad v_A^i = \langle E_A, e_i\rangle \quad , \quad v_A^{\alpha} = \langle E_A, e_\alpha\rangle \ , \tag{11}$$

where $\langle \cdot\, , \cdot \rangle$ denotes the canonical Euclidean inner product. Then $du(V_A) \in \Gamma(u^{-1}TN)$ and

$$\sum_{A=1}^{q} v_A^i v_A^j = \delta_{ij}, i, j = 1, \cdots , m \tag{12}$$

$$\nabla_{e_i} V_A = \sum_{\alpha=m+1}^{q} \sum_{j=1}^{m} B_{ij}^{\alpha} v_A^{\alpha} e_j \ , \tag{13}$$

$$\widetilde{\nabla}_{e_i} du(V_A) = \sum_{\alpha=m+1}^{q} \sum_{k=1}^{m} v_A^{\alpha} B_{ik}^{\alpha} du(e_k) + \sum_{k=1}^{m} v_A^k \widetilde{\nabla}_{e_i} du(e_k) \ , \tag{14}$$

where B_{ij}^{α} is the components of the second fundamental form B of M^m in $\mathbb{R}^q$.

Now let $u : M \to N$ be $\Phi_{(5)}$-harmonic map. By using the condition $\operatorname{div}\big(d_{(5)}(u)\big) = -\delta\big(d_{(5)}(u)\big) = 0$, and (12), we have

$$\begin{aligned}
&\sum_{A=1}^{q} \int_M \langle (\Delta du)(V_A), d_{(5)}(V_A)\rangle \, dv_g \\
&= \sum_{i,j=1}^{m} \sum_{A=1}^{q} \int_M \langle (\Delta du)(v_A^i e_i), d_{(5)}(v_A^j e_j)\rangle \, dv_g \\
&= \sum_{i,j=1}^{m} \sum_{A=1}^{q} \int_M v_A^i v_A^j \langle (\Delta du)(e_i), d_{(5)} u(e_j)\rangle \, dv_g \\
&= \sum_{i=1}^{m} \int_M \langle (\Delta du)(e_i), d_{(5)} u(e_i)\rangle \, dv_g \\
&= \sum_{i=1}^{m} \int_M \Big\langle \delta du(e_i), \delta d_{(5)} u(e_i) \Big\rangle \, dv_g = 0 \ .
\end{aligned} \tag{15}$$

By using the Weitzenböck formula, we have

$$-\sum_{k=1}^{m} R^N\big(du(X), du(e_k)\big)du(e_k) + du\big(Ric^M(X)\big) = \big(\Delta du\big)(X) + \big(\nabla^2 du\big)(X) \ , \tag{16}$$

where X is any smooth vector field on (M^m, g), and $\big(\nabla^2 du\big) = \sum_{i=1}^{m} \big(\nabla_{e_i}\nabla_{e_i} du - \nabla_{\nabla_{e_i} e_i} du\big)$ with respect to the variational vector field $du(V_A)$ along u. We choose $i, j, k, l, p \in \{1, \cdots , m\}$, $\alpha, \beta \in \{m + 1, \cdots , q\}$. Thus

$$\begin{aligned}
&\sum_A I\big(du(V_A), du(V_A)\big) \\
&= -\int_M \sum_{i=1}^{m} h\big(du\big(Ric^M(e_i)\big), d_{(5)} u(e_i)\big) dv_g \\
&\quad + \int_M \sum_{i=1}^{m} h\big(\big(\nabla^2 du\big)(e_i), d_{(5)} u(e_i)\big) \, dv_g
\end{aligned}$$

$$+\int_M \sum_{i,j,k,l,p,A} h\big(\widetilde{\nabla}_{e_i} du(V_A), \widetilde{\nabla}_{e_p} du(V_A)\big) h\big(du(e_i), du(e_j)\big)$$
$$\times h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big)\ dv_g$$
$$+\int_M \sum_{i,j,k,l,p,A} h\big(\widetilde{\nabla}_{e_i} du(V_A), du(e_p)\big)$$
$$\times\Big[h\big(\widetilde{\nabla}_{e_i} du(V_A), du(e_j)\big) + h\big(du(e_i), \widetilde{\nabla}_{e_j}$$
$$\times h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big)\ dv_g$$
$$+\int_M \sum_{i,j,k,l,p,A} h\big(\widetilde{\nabla}_{e_i} du(V_A), du(e_p)\big)$$
$$\times\Big[h\big(\widetilde{\nabla}_{e_j} du(V_A), du(e_k)\big) + h\big(du(e_j), \widetilde{\nabla}_{e_k} du(V_A)\big)\Big]$$
$$\times h\big(du(e_i), du(e_j) big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big)\ dv_g$$
$$+\int_M \sum_{i,j,k,l,p,A} h\big(\widetilde{\nabla}_{e_i} du(V_A), du(e_p)\big)$$
$$\times\Big[h\big(\widetilde{\nabla}_{e_k} du(V_A), du(e_l)\big) + h\big(du(e_k), \widetilde{\nabla}_{e_l} du(V_A)\big)\Big]$$
$$\times h\big(du(e_i), du(e_j)\big) h\big(du(e_j), du(e_k)\big) h\big(du(e_l), du(e_p)\big)\ dv_g$$
$$+\int_M \sum_{i,j,k,l,p,A} h\big(\widetilde{\nabla}_{e_i} du(V_A), du(e_p)\big)$$
$$\times\Big[h\big(\widetilde{\nabla}_{e_l} du(V_A), du(e_p)\big) + h\big(du(e_l), \widetilde{\nabla}_{e_p} du(V_A)\big)\Big]$$
$$\times h\big(du(e_i), du(e_j)\big) h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_l)\big)\ dv_g\ . \tag{17}$$

At x_0, we compute

$$\sum_i h\big(du\big(Ric^M(e_i)\big), d_{(5)}u(e_i)\big)$$
$$= \sum_i \sum_{\alpha=m+1}^{q} h\Big(du\Big(\big(\mathrm{trace}\big(A^\alpha\big) A^\alpha - A^\alpha A^\alpha\big)(e_i)\Big), d_{(5)}u(e_i)\Big)$$
$$= \sum_{i,\alpha} h\big(du\big(\mathrm{trace}\big(A^\alpha\big) A^\alpha(e_i)\big), d_{(5)}u(e_i)\big)$$
$$- \sum_{i,\alpha} h\big(du\big(A^\alpha A^\alpha(e_i)\big), d_{(5)}u(e_i)\big) \tag{18}$$

and

$$\sum_{i=1}^{m} h\big(\big(\nabla^2 du\big)(e_i), d_{(5)}u(e_i)\big)$$
$$= \sum_{i,j,k,l,p} h\big(\big(\nabla^2 du\big)(e_i), du(e_p)\big) h\big(du(e_i), du(e_j)\big)$$
$$\times h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big)$$

$$
\begin{aligned}
&= \sum_{i,j,k,l,p,a} h\big(\big(\nabla_{e_a}\nabla_{e_a} du\big)(e_i), du(e_p)\big) h\big(du(e_i), du(e_j)\big) \\
&\qquad \times h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&= \sum_{i,j,k,l,p,a} e_a \Big[h\big(\nabla_{e_a} du(e_i), du(e_p)\big) h\big(du(e_i), du(e_j)\big) \\
&\qquad \times h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \Big] \\
&\quad - \sum_{i,j,k,l,p,a} h\big(\nabla_{e_a} du(e_i), \nabla_{e_a} du(e_p)\big) h\big(du(e_i), du(e_j)\big) \\
&\qquad \times h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&\quad - \sum_{i,j,k,l,p,a} h\big(\nabla_{e_a} du(e_i), du(e_p)\big) \\
&\qquad \times \Big[h\big(\nabla_{e_a} du(e_i), du(e_j)\big) + h\big(du(e_i), \nabla_{e_a} du(e_j)\big) \Big] \\
&\qquad \times h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&\quad - \sum_{i,j,k,l,p,a} h\big(\nabla_{e_a} du(e_i), du(e_p)\big) \\
&\qquad \times \Big[h\big(\nabla_{e_a} du(e_j), du(e_k)\big) + h\big(du(e_j), \nabla_{e_a} du(e_k)\big) \Big] \\
&\qquad \times h\big(du(e_i), du(e_j)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&\quad - \sum_{i,j,k,l,p,a} h\big(\nabla_{e_a} du(e_i), du(e_p)\big) \\
&\qquad \times \Big[h\big(\nabla_{e_a} du(e_k), du(e_l)\big) + h\big(du(e_k), \nabla_{e_a} du(e_l)\big) \Big] \\
&\qquad \times h\big(du(e_i), du(e_j)\big) h\big(du(e_j), du(e_k)\big) h\big(du(e_l), du(e_p)\big) \\
&\quad - \sum_{i,j,k,l,p,a} h\big(\nabla_{e_a} du(e_i), du(e_p)\big) \\
&\qquad \times \Big[h\big(\nabla_{e_a} du(e_l), du(e_p)\big) + h\big(du(e_l), \nabla_{e_a} du(e_p)\big) \Big] \\
&\qquad \times h\big(du(e_i), du(e_j)\big) h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_l)\big) \qquad (19)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i,j,k,l,p,A} h\big(\widetilde{\nabla}_{e_i} du(V_A), \widetilde{\nabla}_{e_p} du(V_A)\big) h\big(du(e_i), du(e_j)\big) \\
&\qquad \times h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&= \sum_{A,\alpha,\beta,i,j,k,l,p,a,b} h\left(v_A^\alpha B_{ia}^\alpha du(e_a) + v_A^a \widetilde{\nabla}_{e_i} du(e_a), v_A^\beta B_{pb}^\beta du(e_b) + v_A^b \widetilde{\nabla}_{e_p} du(e_b) \right) \\
&\qquad \times h\big(du(e_i), du(e_j)\big) h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&= \sum_{A,\alpha,\beta,i,j,k,l,p,a,b} \Big[v_A^\alpha v_A^\beta B_{ia}^\alpha B_{pb}^\beta h\big(du(e_a), du(e_b)\big) + v_A^a v_A^b B_{ia}^\alpha h\big(du(e_a), \widetilde{\nabla}_{e_p} du(e_b)\big) \\
&\qquad + v_A^a v_A^\beta B_{pb}^\beta h\big(\widetilde{\nabla}_{e_i} du(e_a), du(e_b)\big) + v_A^a v_A^b h\big(\widetilde{\nabla}_{e_i} du(e_a), \widetilde{\nabla}_{e_p} du(e_b)\big) \Big]
\end{aligned}
$$

$$\times h\big(du(e_i), du(e_j)\big)h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)$$

$$= \sum_{\alpha,i,j,k,l,p,a,b} \Big[B^{\alpha}_{ia}B^{\alpha}_{pb}h\big(du(e_a), du(e_b)\big) + h\big((\nabla_{e_i}du)(e_a), (\nabla_{e_p}du)(e_a)\big)\Big]$$

$$\times h\big(du(e_i), du(e_j)\big)h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)$$

$$= \sum_{\alpha,i,j,k,l,p,a} \Big[h\big(du\big(A^{\alpha}(e_i)\big), du\big(A^{\alpha}(e_p)\big)\big) + h\big((\nabla_{e_a}du)(e_i), (\nabla_{e_a}du)(e_p)\big)\Big]$$

$$\times h\big(du(e_i), du(e_j)\big)h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big) \tag{20}$$

and

$$\sum_{i,j,k,l,p,A} h\big(\widetilde{\nabla}_{e_i}du(V_A), du(e_p)\big)$$

$$\times\Big[h\big(\widetilde{\nabla}_{e_i}du(V_A), du(e_j)\big) + h\big(du(e_i), \widetilde{\nabla}_{e_j}du(V_A)\big)\Big]$$

$$\times h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)$$

$$= \sum_{A,\alpha,\beta,i,j,k,l,p,a,b} h\big(v^{\alpha}_A B^{\alpha}_{ia}du(e_a) + v^a_A\widetilde{\nabla}_{e_i}du(e_a), du(e_p)\big)$$

$$\times\Big[h\big(v^{\alpha}_A B^{\alpha}_{ia}du(e_a) + v^a_A\widetilde{\nabla}_{e_i}du(e_a), du(e_j)\big)$$

$$+h\big(du(e_i), v^{\beta}_A B^{\beta}_{jb}du(e_b) + v^b_A\widetilde{\nabla}_{e_j}du(e_b)\big)\Big]$$

$$\times h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)$$

$$= \sum_{A,\alpha,\beta,i,j,k,l,p,a,b} \Big[v^{\alpha}_A B^{\alpha}_{ia}h\big(du(e_a), du(e_p)\big) + v^a_A h\big(\widetilde{\nabla}_{e_i}du(e_a), du(e_p)\big)\Big]$$

$$\times\Big[v^{\alpha}_A B^{\alpha}_{ia}h\big(du(e_a), du(e_j)\big) + v^a_A h\big(\widetilde{\nabla}_{e_i}du(e_a), du(e_j)\big)$$

$$+v^{\beta}_A B^{\beta}_{jb}h\big(du(e_i), du(e_b)\big) + v^b_A h\big(du(e_i), \widetilde{\nabla}_{e_j}du(e_b)\big)\Big]$$

$$\times h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)$$

$$= \sum_{\alpha,i,j,k,l,p,a} B^{\alpha}_{ia}B^{\alpha}_{ia}h\big(du(e_a), du(e_p)\big)h\big(du(e_a), du(e_j)\big)h\big(du(e_j), du(e_k)\big)$$

$$\times h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)$$

$$+ \sum_{i,j,k,l,p,a} h\big(\widetilde{\nabla}_{e_i}du(e_a), du(e_p)\big)h\big(\widetilde{\nabla}_{e_i}du(e_a), du(e_j)\big)h\big(du(e_j), du(e_k)\big)$$

$$\times h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)$$

$$+ \sum_{\alpha,i,j,k,l,p,a,b} B^{\alpha}_{ia}B^{\alpha}_{jb}h\big(du(e_a), du(e_p)\big)h\big(du(e_i), du(e_b)\big)$$

$$\times h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)$$

$$+ \sum_{i,j,k,l,p,a} h\big(\widetilde{\nabla}_{e_i}du(e_a), du(e_p)\big)h\big(du(e_i), \widetilde{\nabla}_{e_j}du(e_a)\big)h\big(du(e_j), du(e_k)\big)$$

$$\times h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)$$

$$
\begin{aligned}
&= \sum_{\alpha,i,j,k,l,p} h\big(du\big(A^\alpha(e_i)\big), du(e_p)\big) h\big(du\big(A^\alpha(e_i)\big), du(e_j)\big) h\big(du(e_j), du(e_k)\big) \\
&\quad \times h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&\quad + \sum_{i,j,k,l,p,a} h\big(\big(\nabla_{e_a} du\big)(e_i), du(e_p)\big) h\big(\big(\nabla_{e_a} du\big)(e_i), du(e_j)\big) h\big(du(e_j), du(e_k)\big) \\
&\quad \times h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&\quad + \sum_{\alpha,i,j,k,l,p} h\big(du\big(A^\alpha(e_i)\big), du(e_p)\big) h\big(du\big(e_i\big), du\big(A^\alpha(e_j)\big)\big) h\big(du(e_j), du(e_k)\big) \\
&\quad \times h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&\quad + \sum_{i,j,k,l,p,a} h\big(\big(\nabla_{e_a} du\big)(e_i), du(e_p)\big) h\big(du(e_i), \big(\nabla_{e_a} du\big)(e_j)\big) h\big(du(e_j), du(e_k)\big) \\
&\quad \times h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \qquad (21)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i,j,k,l,p,A} h\big(\widetilde{\nabla}_{e_i} du(V_A), du(e_p)\big) \\
&\quad \times \Big[h\big(\widetilde{\nabla}_{e_j} du(V_A), du(e_k)\big) + h\big(du(e_j), \widetilde{\nabla}_{e_k} du(V_A)\big)\Big] \\
&\quad \times h\big(du(e_i), du(e_j)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&= \sum_{A,\alpha,\beta,i,j,k,l,p,a,b} \Big[h\big(v_A^\alpha B_{ia}^\alpha du(e_a), du(e_p)\big) + v_A^a h\big(\widetilde{\nabla}_{e_i} du(e_a), du(e_p)\big)\Big] \\
&\quad \times \Big[h\big(v_A^\beta B_{jb}^\beta du(e_b), du(e_k)\big) + h\big(v_A^b \widetilde{\nabla}_{e_j} du(e_b), du(e_k)\big) \\
&\quad + h\big(du(e_j), v_A^\beta B_{kb}^\beta du(e_b)\big) + h\big(du(e_j), v_A^b \widetilde{\nabla}_{e_k} du(e_b)\big)\Big] \\
&\quad \times h\big(du(e_i), du(e_j)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&= \sum_{\alpha,i,j,k,l,p,a,b} B_{ia}^\alpha B_{jb}^\alpha h\big(du(e_a), du(e_p)\big) h\big(du(e_b), du(e_k)\big) h\big(du(e_i), du(e_j)\big) \\
&\quad \times h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&\quad + \sum_{\alpha,i,j,k,l,p,a} B_{ia}^\alpha B_{kb}^\alpha h\big(du(e_a), du(e_p)\big) h\big(du(e_j), du(e_b)\big) h\big(du(e_i), du(e_j)\big) \\
&\quad \times h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&\quad + \sum_{i,j,k,l,p,a,b} h\big(\big(\nabla_{e_a} du\big)(e_i), du(e_p)\big) h\big(\big(\nabla_{e_a} du\big)(e_j), du(e_k)\big) h\big(du(e_i), du(e_j)\big) \\
&\quad \times h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&\quad + \sum_{i,j,k,l,p,a} h\big(\big(\nabla_{e_a} du\big)(e_i), du(e_p)\big) h\big(du(e_j), \big(\nabla_{e_a} du\big)(e_k)\big) h\big(du(e_i), du(e_j)\big) \\
&\quad \times h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&= \sum_{A,i,j,k,l,p} h\big(du\big(A^\alpha(e_i)\big), du(e_p)\big) h\big(du\big(A^\alpha(e_j)\big), du(e_k)\big) h\big(du(e_i), du(e_j)\big)
\end{aligned}
$$

$$
\begin{aligned}
&\times h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big)\\
&+ \sum_{A,i,j,k,l,p} h\big(du\big(A^\alpha(e_i)\big), du(e_p)\big) h\big(du(e_j), du\big(A^\alpha(e_k)\big)\big) h\big(du(e_i), du(e_j)\big)\\
&\times h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big)\\
&+ \sum_{i,j,k,l,p,a} h\big(\big(\nabla_{e_a} du\big)(e_i), du(e_p)\big) h\big(\big(\nabla_{e_a} du\big)(e_j), du(e_k)\big) h\big(du(e_i), du(e_j)\big)\\
&\times h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big)\\
&+ \sum_{i,j,k,l,p,a} h\big(\big(\nabla_{e_a} du\big)(e_i), du(e_p)\big) h\big(du(e_j), \big(\nabla_{e_a} du\big)(e_k)\big) h\big(du(e_i), du(e_j)\big)\\
&\times h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \qquad (22)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i,j,k,l,p,A} h\big(\widetilde{\nabla}_{e_i} du(V_A), du(e_p)\big)\\
&\quad\times \Big[h\big(\widetilde{\nabla}_{e_k} du(V_A), du(e_l)\big) + h\big(du(e_k), \widetilde{\nabla}_{e_l} du(V_A)\big)\Big]\\
&\quad\times h\big(du(e_i), du(e_j)\big) h\big(du(e_j), du(e_k)\big) h\big(du(e_l), du(e_p)\big)\\
=& \sum_{A,\alpha,\beta,i,j,k,l,p,a,b} \Big[h\big(v_A^\alpha B_{ia}^\alpha du(e_a), du(e_p)\big) + v_A^a h\big(\widetilde{\nabla}_{e_i} du(e_a), du(e_p)\big)\Big]\\
&\quad\times \Big[h\big(v_A^\beta B_{kb}^\beta du(e_b), du(e_l)\big) + h\big(v_A^b \widetilde{\nabla}_{e_k} du(e_b), du(e_l)\big)\\
&\quad + h\big(v_A^\beta B_{lb}^\beta du(e_b), du(e_k)\big) + h\big(v_A^b \widetilde{\nabla}_{e_l} du(e_b), du(e_k)\big)\Big]\\
&\quad\times h\big(du(e_i), du(e_j)\big) h\big(du(e_j), du(e_k)\big) h\big(du(e_l), du(e_p)\big)\\
=& \sum_{\alpha,i,j,k,l,p,a,b} B_{ia}^\alpha B_{kb}^\alpha h\big(du(e_a), du(e_p)\big) h\big(du(e_b), du(e_l)\big) h\big(du(e_i), du(e_j)\big)\\
&\quad\times h\big(du(e_j), du(e_k)\big) h\big(du(e_l), du(e_p)\big)\\
&+ \sum_{\alpha,i,j,k,l,p,a,b} B_{ia}^\alpha B_{lb}^\alpha h\big(du(e_a), du(e_p)\big) h\big(du(e_b), du(e_k)\big) h\big(du(e_i), du(e_j)\big)\\
&\quad\times h\big(du(e_j), du(e_k)\big) h\big(du(e_l), du(e_p)\big)\\
&+ \sum_{i,j,k,l,p,a} h\big(\widetilde{\nabla}_{e_i} du(e_a), du(e_p)\big) h\big(\widetilde{\nabla}_{e_k} du(e_a), du(e_l)\big) h\big(du(e_i), du(e_j)\big)\\
&\quad\times h\big(du(e_j), du(e_k)\big) h\big(du(e_l), du(e_p)\big)\\
&+ \sum_{i,j,k,l,p,a} h\big(\widetilde{\nabla}_{e_i} du(e_a), du(e_p)\big) h\big(\widetilde{\nabla}_{e_l} du(e_a), du(e_k)\big) h\big(du(e_i), du(e_j)\big)\\
&\quad\times h\big(du(e_j), du(e_k)\big) h\big(du(e_l), du(e_p)\big)\\
=& \sum_{A,\alpha,i,j,k,l,p} h\big(du\big(A^\alpha(e_i)\big), du(e_p)\big) h\big(du\big(A^\alpha(e_k)\big), du(e_l)\big) h\big(du(e_i), du(e_j)\big)\\
&\quad\times h\big(du(e_j), du(e_k)\big) h\big(du(e_l), du(e_p)\big)
\end{aligned}
$$

$$+\sum_{A,\alpha,i,j,k,l,p} h\big(du\big(A^\alpha(e_i)\big),du(e_p)\big)h\big(du\big(A^\alpha(e_l)\big),du(e_k)\big)h\big(du(e_i),du(e_j)\big)$$
$$\times h\big(du(e_j),du(e_k)\big)h\big(du(e_l),du(e_p)\big)$$
$$+\sum_{i,j,k,l,p,a} h\big(\big(\nabla_{e_a}du\big)(e_i),du(e_p)\big)h\big(\big(\nabla_{e_a}du\big)(e_k),du(e_l)\big)h\big(du(e_i),du(e_j)\big)$$
$$\times h\big(du(e_j),du(e_k)\big)h\big(du(e_l),du(e_p)\big)$$
$$+\sum_{i,j,k,l,p,a} h\big(\big(\nabla_{e_a}du\big)(e_i),du(e_p)\big)h\big(\big(\nabla_{e_a}du\big)(e_l),du(e_k)\big)h\big(du(e_i),du(e_j)\big)$$
$$\times h\big(du(e_j),du(e_k)\big)h\big(du(e_l),du(e_p)\big) \tag{23}$$

and

$$\sum_{i,j,k,l,p,A} h\big(\widetilde{\nabla}_{e_i}du(V_A),du(e_p)\big)$$
$$\times\Big[h\big(\widetilde{\nabla}_{e_l}du(V_A),du(e_p)\big)+h\big(du(e_l),\widetilde{\nabla}_{e_p}du(V_A)\big)\Big]$$
$$\times h\big(du(e_i),du(e_j)\big)h\big(du(e_j),du(e_k)\big)h\big(du(e_k),du(e_l)\big)$$
$$=\sum_{A,\alpha,\beta,i,j,k,l,p,a,b} h\big(v_A^\alpha B_{ia}^\alpha du(e_a)+v_A^a\widetilde{\nabla}_{e_i}du(e_a),du(e_p)\big)$$
$$\times\Big[h\big(v_A^\beta B_{lb}^\beta du(e_b),du(e_p)\big)+h\big(v_A^b\widetilde{\nabla}_{e_l}du(e_b),du(e_p)\big)$$
$$+h\big(du(e_l),v_A^\beta B_{pb}^\beta du(e_b)\big)+h\big(du(e_l),v_A^b\widetilde{\nabla}_{e_p}du(e_b)\big)\Big]$$
$$\times h\big(du(e_i),du(e_j)\big)h\big(du(e_j),du(e_k)\big)h\big(du(e_k),du(e_l)\big)$$
$$=\sum_{\alpha,i,j,k,l,p,a,b} B_{ia}^\alpha B_{lb}^\alpha h\big(du(e_a),du(e_p)\big)h\big(du(e_b),du(e_p)\big)h\big(du(e_i),du(e_j)\big)$$
$$\times h\big(du(e_j),du(e_k)\big)h\big(du(e_k),du(e_l)\big)$$
$$+\sum_{\alpha,i,j,k,l,p,a,b} B_{ia}^\alpha B_{pb}^\alpha h\big(du(e_a),du(e_p)\big)h\big(du(e_l),du(e_b)\big)h\big(du(e_i),du(e_j)\big)$$
$$\times h\big(du(e_j),du(e_k)\big)h\big(du(e_k),du(e_l)\big)$$
$$+\sum_{i,j,k,l,p,a} h\big(\widetilde{\nabla}_{e_i}du(e_a),du(e_p)\big)h\big(\widetilde{\nabla}_{e_l}du(e_a),du(e_p)\big)h\big(du(e_i),du(e_j)\big)$$
$$\times h\big(du(e_j),du(e_k)\big)h\big(du(e_k),du(e_l)\big)$$
$$+\sum_{i,j,k,l,p,a} h\big(\widetilde{\nabla}_{e_i}du(e_a),du(e_p)\big)h\big(du(e_l),\widetilde{\nabla}_{e_p}du(e_a)\big)h\big(du(e_i),du(e_j)\big)$$
$$\times h\big(du(e_j),du(e_k)\big)h\big(du(e_k),du(e_l)\big)$$
$$=\sum_{A,\alpha,i,j,k,l,p} h\big(du\big(A^\alpha(e_i)\big),du(e_p)\big)h\big(du\big(A^\alpha(e_l)\big),du(e_p)\big)h\big(du(e_i),du(e_j)\big)$$
$$\times h\big(du(e_j),du(e_k)\big)h\big(du(e_k),du(e_l)\big)$$
$$+\sum_{A,\alpha,i,j,k,l,p} h\big(du\big(A^\alpha(e_i)\big),du(e_p)\big)h\big(du(e_l),du\big(A^\alpha(e_p)\big)\big)h\big(du(e_i),du(e_j)\big)$$

$$\times h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)$$
$$+ \sum_{i,j,k,l,p,a} h\big((\nabla_{e_a} du)(e_i), du(e_p)\big)h\big((\nabla_{e_a} du)(e_l), du(e_p)\big)h\big(du(e_i), du(e_j)\big)$$
$$\times h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)$$
$$+ \sum_{i,j,k,l,p,a} h\big((\nabla_{e_a} du)(e_i), du(e_p)\big)h\big(du(e_l), (\nabla_{e_a} du)(e_p)\big)h\big(du(e_i), du(e_j)\big)$$
$$\times h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big) \ . \tag{24}$$

From (17) to (24) and Green's theorem, we have

$$\sum_A I\big(du(V_A), du(V_A)\big)$$
$$= -\int_M \sum_{i,\alpha} h\big(du(\mathrm{trace}(A^\alpha)A^\alpha(e_i)), d_{(5)}u(e_i)\big)\ dv_g$$
$$+ \int_M \sum_{i,\alpha} h\big(du(A^\alpha A^\alpha(e_i)), d_{(5)}u(e_i)\big)\ dv_g$$
$$+ \int_M \sum_{i,j,k,l,p,\alpha} h\big(du(A^\alpha(e_i)), du(A^\alpha(e_p))\big)h\big(du(e_i), du(e_j)\big)$$
$$\times h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)\ dv_g$$
$$+ \int_M \sum_{i,j,k,l,p,\alpha} h\big(du(A^\alpha(e_i)), du(e_p)\big)$$
$$\times \big[h\big(du(A^\alpha(e_i)), du(e_j)\big) + h\big(du(A^\alpha(e_j)), du(e_i)\big)\big]$$
$$\times h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)\ dv_g$$
$$+ \int_M \sum_{i,j,k,l,p,\alpha} h\big(du(A^\alpha(e_i)), du(e_p)\big)$$
$$\times \big[h\big(du(A^\alpha(e_j)), du(e_k)\big) + h\big(du(A^\alpha(e_k)), du(e_j)\big)\big]$$
$$\times h\big(du(e_i), du(e_j)\big)h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)\ dv_g$$
$$+ \int_M \sum_{i,j,k,l,p,\alpha} h\big(du(A^\alpha(e_i)), du(e_p)\big)$$
$$\times \big[h\big(du(A^\alpha(e_k)), du(e_l)\big) + h\big(du(A^\alpha(e_l)), du(e_k)\big)\big]$$
$$\times h\big(du(e_i), du(e_j)\big)h\big(du(e_j), du(e_k)\big)h\big(du(e_l), du(e_p)\big)\ dv_g$$
$$+ \int_M \sum_{i,j,k,l,p,\alpha} h\big(du(A^\alpha(e_i)), du(e_p)\big)$$
$$\times \big[h\big(du(A^\alpha(e_l)), du(e_p)\big) + h\big(du(A^\alpha(e_p)), du(e_l)\big)\big]$$
$$\times h\big(du(e_i), du(e_j)\big)h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)\ dv_g \ . \tag{25}$$

Because the matrix $h\big(du(e_i), du(e_j)\big)$ is symmetric, we can take a local orthonormal basis $\{e_i\}_{i=1}^m$ such that $h\big(du(e_i), du(e_j)\big) = \lambda_i^2\delta_{ij}$, $i, j = 1, \cdots, m$. Then we have

$$\begin{aligned}
&\sum_{i,\alpha} h\big(du(\text{trace}(A^\alpha)A^\alpha(e_i)), d_{(5)}u(e_i)\big)\\
&= \sum_{i,j,k,l,p,\alpha} h\big(du(\text{trace}(A^\alpha)A^\alpha(e_i)), du(e_p)\big)h\big(du(e_i), du(e_j)\big)\\
&\quad \times h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)\\
&= \sum_{i,j,k,l,p,\alpha,a,b} \big\langle A^\alpha(e_a), e_a\big\rangle\big\langle A^\alpha(e_i), e_b\big\rangle h\big(du(e_b), du(e_p)\big)\\
&\quad \times h\big(du(e_i), du(e_j)\big)h\big(du(e_j), du(e_k)\big)\\
&\quad \times h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)\\
&= \sum_{i,j,k,l,p,a,b} \big\langle B(e_a, e_a), B(e_i, e_b)\big\rangle \lambda_b^2\delta_{bp}\lambda_i^2\delta_{ij}\lambda_j^2\delta_{jk}\lambda_k^2\delta_{kl}\lambda_l^2\delta_{lp}\\
&= \sum_{i,j} \lambda_i^{10}\big\langle B(e_i, e_i), B(e_j, e_j)\big\rangle \qquad (26)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i,\alpha} h\big(du(A^\alpha A^\alpha(e_i)), d_{(5)}u(e_i)\big)\\
&= \sum_{i,j,k,l,p,\alpha} h\big(du(A^\alpha A^\alpha(e_i)), du(e_p)\big)h\big(du(e_i), du(e_j)\big)h\big(du(e_j), du(e_k)\big)\\
&\quad \times h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)\\
&= \sum_{i,j,k,l,p,\alpha,a,b} \big\langle A^\alpha(e_i), e_b\big\rangle\big\langle A^\alpha(e_b), e_a\big\rangle h\big(du(e_a), du(e_p)\big)h\big(du(e_i), du(e_j)\big)\\
&\quad \times h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)\\
&= \sum_{i,j,k,l,p,a,b} \big\langle B(e_i, e_b), B(e_b, e_a)\big\rangle \lambda_a^2\delta_{ap}\lambda_i^2\delta_{ij}\lambda_j^2\delta_{jk}\lambda_k^2\delta_{kl}\lambda_l^2\delta_{lp}\\
&= \sum_{i,j} \lambda_i^{10}\big\langle B(e_i, e_j), B(e_i, e_j)\big\rangle \qquad (27)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i,j,k,l,p,\alpha} h\big(du(A^\alpha(e_i)), du(A^\alpha(e_p))\big)h\big(du(e_i), du(e_j)\big)\\
&\quad \times h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)\\
&= \sum_{i,j,k,l,p,\alpha,a,b} \big\langle A^\alpha(e_i), e_a\big\rangle\big\langle A^\alpha(e_p), e_b\big\rangle h\big(du(e_a), du(e_b)\big)\\
&\quad \times h\big(du(e_i), du(e_j)\big)h\big(du(e_j), du(e_k)\big)\\
&\quad \times h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)\\
&= \sum_{i,j,k,l,p,a,b} \big\langle B(e_i, e_a), B(e_p, e_b)\big\rangle \lambda_a^2\delta_{ab}\lambda_i^2\delta_{ij}\lambda_j^2\delta_{jk}\lambda_k^2\delta_{kl}\lambda_l^2\delta_{lp}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j} \lambda_i^8 \lambda_j^2 \big\langle B(e_i,e_j), B(e_i,e_j)\big\rangle \\
&\leq \sum_{i,j} \Big(\frac{4}{5}\lambda_i^{10} + \frac{1}{5}\lambda_j^{10}\Big)\big\langle B(e_i,e_j), B(e_i,e_j)\big\rangle \\
&= \sum_{i,j} \lambda_i^{10}\big\langle B(e_i,e_j), B(e_i,e_j)\big\rangle \qquad (28)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i,j,k,l,p,\alpha} h\big(du(A^\alpha(e_i)), du(e_p)\big) \\
&\quad \times\big[h\big(du(A^\alpha(e_i)), du(e_j)\big) + h\big(du(A^\alpha(e_j)), du(e_i)\big)\big] \\
&\quad \times h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&= \sum_{i,j,k,l,p,\alpha,a,b} \big[\big\langle A^\alpha(e_i), e_a\big\rangle \big\langle A^\alpha(e_i), e_b\big\rangle \\
&\quad \times h\big(du(e_a), du(e_p)\big) h\big(du(e_b), du(e_j)\big) \\
&\quad + \big\langle A^\alpha(e_i), e_a\big\rangle \big\langle A^\alpha(e_j), e_b\big\rangle \\
&\quad \times h\big(du(e_a), du(e_p)\big) h\big(du(e_b), du(e_i)\big)\big] \\
&\quad \times h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&= \sum_{i,j,k,l,p,a,b} \Big[\big\langle B(e_i,e_a), B(e_i,e_b)\big\rangle \lambda_a^2 \delta_{ap} \lambda_b^2 \delta_{bj} \\
&\quad + \big\langle B(e_i,e_a), B(e_j,e_b)\big\rangle \lambda_a^2 \delta_{ap} \lambda_b^2 \delta_{ib}\Big] \lambda_j^2 \delta_{jk} \lambda_k^2 \delta_{kl} \lambda_l^2 \delta_{lp} \\
&= \sum_{i,j} \lambda_i^{10}\big\langle B(e_i,e_j), B(e_i,e_j)\big\rangle + \lambda_i^2 \lambda_j^8 \big\langle B(e_i,e_j), B(e_i,e_j)\big\rangle \\
&\leq \sum_{i,j} \Big(\lambda_i^{10}\big\langle B(e_i,e_j), B(e_i,e_j)\big\rangle + \Big(\frac{1}{5}\lambda_i^{10} + \frac{4}{5}\lambda_j^{10}\Big) \\
&\quad \times \big\langle B(e_i,e_j), B(e_i,e_j)\big\rangle\Big) \\
&= \sum_{i,j} 2\lambda_i^{10}\big\langle B(e_i,e_j), B(e_i,e_j)\big\rangle \qquad (29)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i,j,k,l,p,\alpha} h\big(du(A^\alpha(e_i)), du(e_p)\big) \\
&\quad \times\big[h\big(du(A^\alpha(e_j)), du(e_k)\big) + h\big(du(A^\alpha(e_k)), du(e_j)\big)\big] \\
&\quad \times h\big(du(e_i), du(e_j)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big) \\
&= \sum_{i,j,k,l,p,\alpha,a,b} \big\langle A^\alpha(e_i), e_a\big\rangle h\big(du(e_a), du(e_p)\big) \\
&\quad \times\Big[\big\langle A^\alpha(e_j), e_b\big\rangle h\big(du(e_b), du(e_k)\big) \\
&\quad + \big\langle A^\alpha(e_k), e_b\big\rangle h\big(du(e_b), du(e_j)\big)\Big] \\
&\quad \times h\big(du(e_i), du(e_j)\big) h\big(du(e_k), du(e_l)\big) h\big(du(e_l), du(e_p)\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i,j,k,l,p,a,b} \Big[\big\langle B(e_i,e_a),B(e_j,e_b)\big\rangle \lambda_a^2\delta_{ap}\lambda_b^2\delta_{bk} \\
&\quad +\big\langle B(e_i,e_a),B(e_k,e_b)\big\rangle \lambda_a^2\delta_{ap}\lambda_b^2\delta_{bj}\Big]\lambda_i^2\delta_{ij}\lambda_k^2\delta_{kl}\lambda_l^2\delta_{lp} \\
&= \sum_{i,j}\Big(\lambda_i^2\lambda_j^8\big\langle B(e_i,e_j),B(e_i,e_j)\big\rangle + \lambda_i^4\lambda_j^6\big\langle B(e_i,e_j),B(e_i,e_j)\big\rangle\Big) \\
&\le \sum_{i,j}\Big[\Big(\frac{1}{5}\lambda_i^{10}+\frac{4}{5}\lambda_j^{10}\Big)+\Big(\frac{2}{5}\lambda_i^{10}+\frac{3}{5}\lambda_j^{10}\Big)\Big]\big\langle B(e_i,e_j),B(e_i,e_j)\big\rangle \\
&= \sum_{i,j} 2\lambda_i^{10}\big\langle B(e_i,e_j),B(e_i,e_j)\big\rangle
\end{aligned} \tag{30}
$$

and

$$
\begin{aligned}
&\sum_{i,j,k,l,p,\alpha} h\big(du(A^\alpha(e_i)),du(e_p)\big) \\
&\quad \times\big[h\big(du(A^\alpha(e_k)),du(e_l)\big)+h\big(du(A^\alpha(e_l)),du(e_k)\big)\big] \\
&\quad \times h\big(du(e_i),du(e_j)\big)h\big(du(e_j),du(e_k)\big)h\big(du(e_l),du(e_p)\big) \\
&= \sum_{i,j,k,l,p,\alpha,a,b} \big\langle A^\alpha(e_i),e_a\big\rangle h\big(du(e_a),du(e_p)\big) \\
&\quad \times\Big[\big\langle A^\alpha(e_k),e_b\big\rangle h\big(du(e_b),du(e_l)\big) \\
&\quad +\big\langle A^\alpha(e_l),e_b\big\rangle h\big(du(e_b),du(e_k)\big)\Big] \\
&\quad \times h\big(du(e_i),du(e_j)\big)h\big(du(e_j),du(e_k)\big)h\big(du(e_l),du(e_p)\big) \\
&= \sum_{i,j,k,l,p,a,b} \Big[\big\langle B(e_i,e_a),B(e_k,e_b)\big\rangle \lambda_a^2\delta_{ap}\lambda_b^2\delta_{bl} \\
&\quad +\big\langle B(e_i,e_a),B(e_l,e_b)\big\rangle \lambda_a^2\delta_{ap}\lambda_b^2\delta_{bk}\Big]\lambda_i^2\delta_{ij}\lambda_j^2\delta_{jk}\lambda_l^2\delta_{lp} \\
&= \sum_{i,j}\Big(\lambda_i^4\lambda_j^6\big\langle B(e_i,e_j),B(e_i,e_j)\big\rangle + \lambda_i^6\lambda_j^4\big\langle B(e_i,e_j),B(e_i,e_j)\big\rangle\Big) \\
&\le \sum_{i,j}\Big[\Big(\frac{2}{5}\lambda_i^{10}+\frac{3}{5}\lambda_j^{10}\Big)+\Big(\frac{3}{5}\lambda_i^{10}+\frac{2}{5}\lambda_j^{10}\Big)\Big]\big\langle B(e_i,e_j),B(e_i,e_j)\big\rangle \\
&= \sum_{i,j} 2\lambda_i^{10}\big\langle B(e_i,e_j),B(e_i,e_j)\big\rangle
\end{aligned} \tag{31}
$$

and

$$
\begin{aligned}
&\sum_{i,j,k,l,p,\alpha} h\big(du(A^\alpha(e_i)),du(e_p)\big) \\
&\quad \times\big[h\big(du(A^\alpha(e_l)),du(e_p)\big)+h\big(du(A^\alpha(e_p)),du(e_l)\big)\big] \\
&\quad \times h\big(du(e_i),du(e_j)\big)h\big(du(e_j),du(e_k)\big)h\big(du(e_k),du(e_l)\big) \\
&= \sum_{i,j,k,l,p,\alpha,a,b} \big\langle A^\alpha(e_i),e_a\big\rangle h\big(du(e_a),du(e_p)\big) \\
&\quad \times\Big[\big\langle A^\alpha(e_l),e_b\big\rangle h\big(du(e_b),du(e_p)\big) \\
&\quad +\big\langle A^\alpha(e_p),e_b\big\rangle h\big(du(e_l),du(e_b)\big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\times h\big(du(e_i), du(e_j)\big)h\big(du(e_j), du(e_k)\big)h\big(du(e_k), du(e_l)\big)\\
=& \sum_{i,j,k,l,p,a,b} \Big[\big\langle B(e_i, e_a), B(e_l, e_b)\big\rangle \lambda_a^2\delta_{ap}\lambda_b^2\delta_{bp}\\
&+\big\langle B(e_i, e_a), B(e_p, e_b)\big\rangle \lambda_a^2\delta_{ap}\lambda_l^2\delta_{lb}\Big]\lambda_i^2\delta_{ij}\lambda_j^2\delta_{jk}\lambda_k^2\delta_{kl}\\
=& \sum_{i,j}\Big(\lambda_i^6\lambda_j^4\big\langle B(e_i, e_j), B(e_i, e_j)\big\rangle + \lambda_i^8\lambda_j^2\big\langle B(e_i, e_j), B(e_i, e_j)\big\rangle\Big)\\
\le& \sum_{i,j}\Big[\Big(\frac{3}{5}\lambda_i^{10} + \frac{2}{5}\lambda_j^{10}\Big) + \Big(\frac{4}{5}\lambda_i^{10} + \frac{1}{5}\lambda_j^{10}\Big)\Big]\big\langle B(e_i, e_j), B(e_i, e_j)\big\rangle\\
=& \sum_{i,j} 2\lambda_i^{10}\big\langle B(e_i, e_j), B(e_i, e_j)\big\rangle\ . \qquad (32)
\end{aligned}
$$

From (25) to (32), we have

$$
\begin{aligned}
&\sum_A I\big(du(V_A), du(V_A)\big)\\
\le& \int_M \sum_{i,j} \lambda_i^{10}\big(10\big\langle B(e_i, e_j), B(e_i, e_j)\big\rangle - \big\langle B(e_i, e_i), B(e_j, e_j)\big\rangle\big)\, dv_g\ . \qquad (33)
\end{aligned}
$$

If u is not a constant, then we have

$$\sum_A I\big(du(V_A), du(V_A)\big) < 0$$

in M. Hence, u is a constant. This completes the proof.

□

5. Stable $\Phi_{(5)}$-harmonic maps into $\Phi_{(5)}$-SSU manifolds

In this section, we obtain the following result.

Theorem 5.1. *Let $u : (M^m, g) \to (N^n, h)$ be a $\Phi_{(5)}$-harmonic map, and suppose (N^n, h) is a compact $\Phi_{(5)}$-SSU manifold and (M^m, g) is any compact manifold. Then all stable $\Phi_{(5)}$-harmonic map u is constant.*

Proof. Now we take an orthonormal frame field $\{\varepsilon_i\}_{i=1}^q$ of $\mathbb{R}^q$ such that $\{\varepsilon_a\}_{a=1}^n$ are tangent to N^n, $\{\varepsilon_\alpha\}_{\alpha=n+1}^q$ are normal to N^n and $\nabla_{\varepsilon_a}\varepsilon_b\big|_{u(x_0)} = 0$, where x_0 is a fixed point of N^n and we take a local orthonormal frame field $\{e_i\}_{i=1}^m$ of M. Meanwhile, we take a fixed orthonormal basis of $\mathbb{R}^q$ denoted by $\{E_A\}_{A=1}^q$ and $B_{ab}^\alpha = \big\langle B(\varepsilon_a, \varepsilon_b), \varepsilon_\alpha\big\rangle$ denote the components of the second fundamental form B of N^n and set

$$V_A = \sum_{a=1}^n v_A^a\varepsilon_a \quad , \quad v_A^a = \big\langle E_A, \varepsilon_a\big\rangle\ , \qquad (34)$$

$$v_A^\alpha = \big\langle E_A, \varepsilon_\alpha\big\rangle\ , \qquad (35)$$

$$\nabla_{\varepsilon_a} V_A = \sum_{\alpha=n+1}^q \sum_{b=1}^n v_A^\alpha B_{ab}^\alpha \varepsilon_b\ , \qquad (36)$$

where $\langle\cdot,\cdot\rangle$ denotes the canonical Euclidean inner product.

Set $du(e_i)=\sum_a u_i^a\varepsilon_a$, then the product matrix $(u_i^a)^\top\cdot(u_i^a)$ is an $n\times n$ symmetric matrix given by

$$\left(\sum_{i=1}^m u_i^a u_i^b\right)_{a,b=1,\cdots,n}, \tag{37}$$

where $(u_i^a)^\top$ is the transpose of (u_i^a). Next we choose a local orthonormal basis $\{\varepsilon_a\}_{a=1}^n$ such that

$$\sum_{i=1}^m u_i^a u_i^b=\lambda_a^2\delta_{ab}\ . \tag{38}$$

Suppose that $i,j,k,l,p\in\{1,\cdots,m\}, a,b,c,d,e,f,r,s,t,x,y,z\in\{1,\cdots,n\}$ and $\alpha,\beta\in\{n+1,\cdots,q\}$. With respect to the variational vector field V_A, by the second variation formula, we have

$$\begin{aligned}
&\sum_A I(V_A,V_A)\\
&=\int_M h\left(R^N\big(V_A,du(e_i)\big)V_A,d_{(5)}u(e_i)\right)\,dv_g\\
&\quad+\int_M h\left(\widetilde\nabla_{e_i}V_A,\widetilde\nabla_{e_p}V_A\right)h\big(du(e_i),du(e_j)\big)h\big(du(e_j),du(e_k)\big)\\
&\quad\times h\big(du(e_k),du(e_l)\big)h\big(du(e_l),du(e_p)\big)\,dv_g\\
&\quad+\int_M h\left(\widetilde\nabla_{e_i}V_A,du(e_p)\right)\Big[h\big(\widetilde\nabla_{e_i}V_A,du(e_j)\big)+h\big(du(e_i),\widetilde\nabla_{e_j}V_A\big)\Big]\\
&\quad\times h\big(du(e_j),du(e_k)\big)h\big(du(e_k),du(e_l)\big)h\big(du(e_l),du(e_p)\big)\,dv_g\\
&\quad+\int_M h\left(\widetilde\nabla_{e_i}V_A,du(e_p)\right)\Big[h\big(\widetilde\nabla_{e_j}V_A,du(e_k)\big)+h\big(du(e_j),\widetilde\nabla_{e_k}V_A\big)\Big]\\
&\quad\times h\big(du(e_i),du(e_j)\big)h\big(du(e_k),du(e_l)\big)h\big(du(e_l),du(e_p)\big)\,dv_g\\
&\quad+\int_M h\left(\widetilde\nabla_{e_i}V_A,du(e_p)\right)\Big[h\big(\widetilde\nabla_{e_k}V_A,du(e_l)\big)+h\big(du(e_k),\widetilde\nabla_{e_l}V_A\big)\Big]\\
&\quad\times h\big(du(e_i),du(e_j)\big)h\big(du(e_j),du(e_k)\big)h\big(du(e_l),du(e_p)\big)\,dv_g\\
&\quad+\int_M h\left(\widetilde\nabla_{e_i}V_A,du(e_p)\right)\Big[h\big(\widetilde\nabla_{e_l}V_A,du(e_p)\big)+h\big(du(e_l),\widetilde\nabla_{e_p}V_A\big)\Big]\\
&\quad\times h\big(du(e_i),du(e_j)\big)h\big(du(e_j),du(e_k)\big)h\big(du(e_k),du(e_l)\big)\,dv_g\ .
\end{aligned}\tag{39}$$

Hence, at x_0, we have

$$\begin{aligned}
&\sum h\left(R^N\big(V_A,du(e_i)\big)V_A,d_{(5)}u(e_i)\right)\\
&=\sum h\left(R^N\big(V_A,du(e_i)\big)V_A,d_u(e_p)\right)h\big(du(e_i),du(e_j)\big)\\
&\qquad\times h\big(du(e_j),du(e_k)\big)h\big(du(e_k),du(e_l)\big)h\big(du(e_l),du(e_p)\big)\\
&=\sum h\big(R^N\big(v_A^a\varepsilon_a,u_i^b\varepsilon_b\big)v_A^c\varepsilon_c,u_p^d\varepsilon_d\big)h\big(u_i^e\varepsilon_e,u_j^f\varepsilon_f\big)
\end{aligned}$$

$$
\begin{aligned}
&\times h\big(u_j^r\varepsilon_r, u_k^s\varepsilon_s\big)h\big(u_k^t\varepsilon_t, u_l^x\varepsilon_x\big)h\big(u_l^y\varepsilon_y, u_p^z\varepsilon_z\big)\\
&= \sum v_A^a v_A^c u_i^b u_i^e u_j^f u_j^r u_k^s u_k^t u_l^x u_l^y u_p^d u_p^z\\
&\quad \times h\big(R^N(\varepsilon_a,\varepsilon_b)\varepsilon_c,\varepsilon_d\big)h(\varepsilon_e,\varepsilon_f)h(\varepsilon_r,\varepsilon_s)h(\varepsilon_t,\varepsilon_x)h(\varepsilon_y,\varepsilon_z)\\
&= \sum \lambda_b^2\lambda_f^2\lambda_s^2\lambda_x^2\lambda_d^2\, h\big(R^N(\varepsilon_a,\varepsilon_b)\varepsilon_a,\varepsilon_d\big)\\
&\quad \times\delta_{be}\delta_{fr}\delta_{st}\delta_{xy}\delta_{dz}\delta_{ef}\delta_{rs}\delta_{tx}\delta_{yz}\\
&= \sum \lambda_a^{10} h\big(R^N(\varepsilon_a,\varepsilon_b)\varepsilon_a,\varepsilon_b\big)\\
&= \sum \lambda_a^{10}\left(\big\langle B(\varepsilon_a,\varepsilon_b), B(\varepsilon_a,\varepsilon_b)\big\rangle - \big\langle B(\varepsilon_a,\varepsilon_a), B(\varepsilon_b,\varepsilon_b)\big\rangle\right)
\end{aligned}
\tag{40}
$$

and

$$
\begin{aligned}
&h\left(\widetilde{\nabla}_{e_i}V_A, \widetilde{\nabla}_{e_p}V_A\right) h\big(du(e_i), du(e_j)\big)h\big(du(e_j), du(e_k)\big)\\
&\quad\times h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)\\
&= \sum u_i^a u_p^b\, h\big(\nabla_{\varepsilon_a}V_A, \nabla_{\varepsilon_b}V_A\big)h\big(u_i^e\varepsilon_e, u_j^f\varepsilon_f\big)h\big(u_j^r\varepsilon_r, u_k^s\varepsilon_s\big)\\
&\quad\times h\big(u_k^t\varepsilon_t, u_l^x\varepsilon_x\big)h\big(u_l^y\varepsilon_y, u_p^z\varepsilon_z\big)\\
&= \sum u_i^a u_p^b\, h\big(v_A^\alpha B_{ac}^\alpha\varepsilon_c, v_A^\beta B_{bd}^\beta\varepsilon_d\big)u_i^e u_j^f u_j^r u_k^s u_k^t u_l^x u_l^y u_p^z\delta_{ef}\delta_{rs}\delta_{tx}\delta_{yz}\\
&= \sum \lambda_a^2\lambda_f^2\lambda_s^2\lambda_x^2\lambda_b^2\delta_{ae}\delta_{fr}\delta_{st}\delta_{xy}\delta_{bz}\delta_{cd}\delta_{ef}\delta_{rs}\delta_{tx}\delta_{yz}B_{ac}^\alpha B_{bd}^\alpha\\
&= \sum \lambda_a^{10}B_{ab}^\alpha B_{ab}^\alpha\\
&= \sum \lambda_a^{10}\big\langle B(\varepsilon_a,\varepsilon_b), B(\varepsilon_a,\varepsilon_b)\big\rangle
\end{aligned}
\tag{41}
$$

and

$$
\begin{aligned}
&\sum h\left(\widetilde{\nabla}_{e_i}V_A, du(e_p)\right) h\left(\widetilde{\nabla}_{e_i}V_A, du(e_j)\right) h\big(du(e_j), du(e_k)\big)\\
&\quad\times h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)\\
&= \sum u_i^a h\big(\nabla_{\varepsilon_a}V_A, u_p^b\varepsilon_b\big)h\big(u_i^e\nabla_{\varepsilon_e}V_A, u_j^f\varepsilon_f\big)h\big(u_j^r\varepsilon_r, u_k^s\varepsilon_s\big)\\
&\quad\times h\big(u_k^t\varepsilon_t, u_l^x\varepsilon_x\big)h\big(u_l^y\varepsilon_y, u_p^z\varepsilon_z\big)\\
&= \sum u_i^a h\big(v_A^\alpha B_{ac}^\alpha\varepsilon_c, u_p^b\varepsilon_b\big)h\big(u_i^e v_A^\beta B_{ed}^\beta\varepsilon_d, u_j^f\varepsilon_f\big)\\
&\quad\times u_j^r u_k^s u_k^t u_l^x u_l^y u_p^z\delta_{rs}\delta_{tx}\delta_{yz}\\
&= \sum B_{ac}^\alpha B_{ed}^\alpha u_i^a u_i^e u_j^f u_j^r u_k^s u_k^t u_l^x u_l^y u_p^b u_p^z\delta_{cb}\delta_{df}\delta_{rs}\delta_{tx}\delta_{yz}\\
&= \sum \lambda_a^2\lambda_f^2\lambda_s^2\lambda_x^2\lambda_b^2\delta_{ae}\delta_{fr}\delta_{st}\delta_{xy}\delta_{bz}\delta_{cb}\delta_{df}\delta_{rs}\delta_{tx}\delta_{yz}B_{ac}^\alpha B_{ed}^\alpha\\
&= \sum \lambda_a^2\lambda_b^8\big\langle B(\varepsilon_a,\varepsilon_b), B(\varepsilon_a,\varepsilon_b)\big\rangle\\
&\le \sum \left(\frac{1}{5}\lambda_a^{10} + \frac{4}{5}\lambda_b^{10}\right)\big\langle B(\varepsilon_a,\varepsilon_b), B(\varepsilon_a,\varepsilon_b)\big\rangle\\
&= \sum \lambda_a^{10}\big\langle B(\varepsilon_a,\varepsilon_b), B(\varepsilon_a,\varepsilon_b)\big\rangle
\end{aligned}
\tag{42}
$$

and

$$
\sum h\left(\widetilde{\nabla}_{e_i}V_A, du(e_p)\right) h\left(du(e_i), \widetilde{\nabla}_{e_j}V_A\right) h\big(du(e_j), du(e_k)\big)
$$

$$
\begin{aligned}
&\quad \times h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)\\
&= \sum u_i^a h\big(\nabla_{\varepsilon_a} V_A, u_p^b \varepsilon_b\big) h\big(u_i^e \varepsilon_e, u_j^f \nabla_{\varepsilon_f} V_A\big) h\big(u_j^r \varepsilon_r, u_k^s \varepsilon_s\big)\\
&\quad \times h\big(u_k^t \varepsilon_t, u_l^x \varepsilon_x\big) h\big(u_l^y \varepsilon_y, u_p^z \varepsilon_z\big)\\
&= \sum u_i^a h(v_A^\alpha B_{ac}^\alpha \varepsilon_c, u_p^b \varepsilon_b) h(u_i^e \varepsilon_e, u_j^f v_A^\beta B_{fd}^\beta \varepsilon_d)\\
&\quad \times u_j^r u_k^s u_k^t u_l^x u_l^y u_p^z \delta_{rs}\delta_{tx}\delta_{yz}\\
&= \sum B_{ac}^\alpha B_{fd}^\alpha u_i^a u_i^e u_j^f u_j^r u_k^s u_k^t u_l^x u_l^y u_p^b u_p^z \delta_{cb}\delta_{ed}\delta_{rs}\delta_{tx}\delta_{yz}\\
&= \sum \lambda_a^2 \lambda_f^2 \lambda_s^2 \lambda_x^2 \lambda_b^2 \delta_{ae}\delta_{fr}\delta_{st}\delta_{xy}\delta_{bz}\delta_{cb}\delta_{ed}\delta_{rs}\delta_{tx}\delta_{yz} B_{ae}^\alpha B_{fd}^\alpha\\
&= \sum \lambda_a^2 \lambda_b^8 \big\langle B(\varepsilon_a, \varepsilon_b), B(\varepsilon_a, \varepsilon_b)\big\rangle\\
&\leq \sum \left(\frac{1}{5}\lambda_a^{10} + \frac{4}{5}\lambda_b^{10}\right) \big\langle B(\varepsilon_a, \varepsilon_b), B(\varepsilon_a, \varepsilon_b)\big\rangle\\
&= \sum \lambda_a^{10} \big\langle B(\varepsilon_a, \varepsilon_b), B(\varepsilon_a, \varepsilon_b)\big\rangle \qquad (43)
\end{aligned}
$$

and

$$
\begin{aligned}
&h\left(\widetilde{\nabla}_{e_i} V_A, du(e_p)\right) h\left(du(e_j), \widetilde{\nabla}_{e_k} V_A\right) h\big(du(e_i), du(e_j)\big)\\
&\quad \times h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)\\
&= \sum u_i^a h\big(\nabla_{\varepsilon_a} V_A, u_p^b \varepsilon_b\big) h\big(u_j^c \nabla_{\varepsilon_c} V_A, u_k^d \varepsilon_d\big) h\big(u_i^r \varepsilon_r, u_j^s \varepsilon_s\big)\\
&\quad \times h\big(u_k^t \varepsilon_t, u_l^x \varepsilon_x\big) h\big(u_l^y \varepsilon_y, u_p^z \varepsilon_z\big)\\
&= \sum u_i^a h\big(v_A^\alpha B_{ae}^\alpha \varepsilon_e, u_p^b \varepsilon_b\big) h\big(u_j^c v_A^\beta B_{cf}^\beta \varepsilon_f, u_k^d \varepsilon_d\big)\\
&\quad \times u_i^r u_j^s u_k^t u_l^x u_l^y u_p^z \delta_{rs}\delta_{tx}\delta_{yz}\\
&= \sum B_{ae}^\alpha B_{cf}^\alpha u_i^a u_i^r u_j^c u_j^s u_k^d u_k^t u_l^x u_l^y u_p^b u_p^z \delta_{eb}\delta_{fd}\delta_{rs}\delta_{tx}\delta_{yz}\\
&= \sum \lambda_a^2 \lambda_c^2 \lambda_d^2 \lambda_x^2 \lambda_b^2 \delta_{ar}\delta_{cs}\delta_{dt}\delta_{xy}\delta_{bz}\delta_{eb}\delta_{fd}\delta_{rs}\delta_{tx}\delta_{yz} B_{ac}^\alpha B_{fd}^\alpha\\
&= \sum \lambda_a^4 \lambda_b^6 \big\langle B(\varepsilon_a, \varepsilon_b), B(\varepsilon_a, \varepsilon_b)\big\rangle\\
&\leq \sum \left(\frac{2}{5}\lambda_a^{10} + \frac{3}{5}\lambda_b^{10}\right) \big\langle B(\varepsilon_a, \varepsilon_b), B(\varepsilon_a, \varepsilon_b)\big\rangle\\
&= \sum \lambda_a^{10} \big\langle B(\varepsilon_a, \varepsilon_b), B(\varepsilon_a, \varepsilon_b)\big\rangle \qquad (44)
\end{aligned}
$$

and

$$
\begin{aligned}
&h\left(\widetilde{\nabla}_{e_i} V_A, du(e_p)\right) h\left(\widetilde{\nabla}_{e_j} V_A, du(e_k)\right) h\big(du(e_i), du(e_j)\big)\\
&\quad \times h\big(du(e_k), du(e_l)\big)h\big(du(e_l), du(e_p)\big)\\
&= \sum u_i^a h\big(\nabla_{\varepsilon_a} V_A, u_p^b \varepsilon_b\big) h\big(u_j^c \varepsilon_c, u_k^d \nabla_{\varepsilon_d} V_A\big) h\big(u_i^r \varepsilon_r, u_j^s \varepsilon_s\big)\\
&\quad \times h\big(u_k^t \varepsilon_t, u_l^x \varepsilon_x\big) h\big(u_l^y \varepsilon_y, u_p^z \varepsilon_z\big)\\
&= \sum u_i^a h\big(v_A^\alpha B_{ae}^\alpha \varepsilon_e, u_p^b \varepsilon_b\big) h\big(u_j^c \varepsilon_c, u_k^d v_A^\beta B_{df}^\beta \varepsilon_f\big)\\
&\quad \times u_i^r u_j^s u_k^t u_l^x u_l^y u_p^z \delta_{rs}\delta_{tx}\delta_{yz}\\
&= \sum B_{ae}^\alpha B_{df}^\alpha u_i^a u_i^r u_j^c u_j^s u_k^d u_k^t u_l^x u_l^y u_p^b u_p^z \delta_{eb}\delta_{cf}\delta_{rs}\delta_{tx}\delta_{yz}
\end{aligned}
$$

$$= \sum \lambda_a^2\lambda_c^2\lambda_d^2\lambda_x^2\lambda_b^2\delta_{ar}\delta_{cs}\delta_{dt}\delta_{xy}\delta_{bz}\delta_{eb}\delta_{cf}\delta_{rs}\delta_{tx}\delta_{yz}B^\alpha_{ae}B^\alpha_{df}$$
$$= \sum \lambda_a^4\lambda_b^6\big\langle B(\varepsilon_a,\varepsilon_b), B(\varepsilon_a,\varepsilon_b)\big\rangle$$
$$\leq \sum \left(\frac{2}{5}\lambda_a^{10} + \frac{3}{5}\lambda_b^{10}\right)\big\langle B(\varepsilon_a,\varepsilon_b), B(\varepsilon_a,\varepsilon_b)\big\rangle$$
$$= \sum \lambda_a^{10}\big\langle B(\varepsilon_a,\varepsilon_b), B(\varepsilon_a,\varepsilon_b)\big\rangle \tag{45}$$

and

$$h\left(\widetilde{\nabla}_{e_i}V_A, du(e_p)\right) h\left(\widetilde{\nabla}_{e_k}V_A, du(e_l)\right) h\big(du(e_i), du(e_j)\big) \times h\big(du(e_j), du(e_k)\big)h\big(du(e_l), du(e_p)\big)$$
$$= \sum u_i^a h\big(\nabla_{\varepsilon_a}V_A, u_p^b\varepsilon_b\big)h\big(u_k^c\nabla_{\varepsilon_c}V_A, u_l^d\varepsilon_d\big)h\big(u_i^e\varepsilon_e, u_j^f\varepsilon_f\big) \times h\big(u_j^r\varepsilon_r, u_k^s\varepsilon_s\big)h\big(u_l^t\varepsilon_t, u_p^x\varepsilon_x\big)$$
$$= \sum u_i^a h\big(v_A^\alpha B^\alpha_{ay}\varepsilon_y, u_p^b\varepsilon_b\big)h\big(u_k^c v_A^\beta B^\beta_{cz}\varepsilon_z, u_l^d\varepsilon_d\big) \times u_i^e u_j^f u_j^r u_k^s u_l^t u_p^x\delta_{ef}\delta_{rs}\delta_{tx}$$
$$= \sum B^\alpha_{ay}B^\alpha_{cz}u_i^a u_i^e u_j^f u_j^r u_k^c u_k^s u_l^d u_l^t u_p^b u_p^x\delta_{yb}\delta_{zd}\delta_{ef}\delta_{rs}\delta_{tx}$$
$$= \sum \lambda_a^2\lambda_f^2\lambda_c^2\lambda_d^2\lambda_b^2\delta_{ae}\delta_{fr}\delta_{cs}\delta_{dt}\delta_{bx}\delta_{yb}\delta_{zd}\delta_{ef}\delta_{rs}\delta_{tx}B^\alpha_{ay}B^\alpha_{cz}$$
$$= \sum \lambda_a^6\lambda_b^4\big\langle B(\varepsilon_a,\varepsilon_b), B(\varepsilon_a,\varepsilon_b)\big\rangle$$
$$\leq \sum \left(\frac{3}{5}\lambda_a^{10} + \frac{2}{5}\lambda_b^{10}\right)\big\langle B(\varepsilon_a,\varepsilon_b), B(\varepsilon_a,\varepsilon_b)\big\rangle$$
$$= \sum \lambda_a^{10}\big\langle B(\varepsilon_a,\varepsilon_b), B(\varepsilon_a,\varepsilon_b)\big\rangle \tag{46}$$

and

$$h\left(\widetilde{\nabla}_{e_i}V_A, du(e_p)\right) h\left(du(e_k), \widetilde{\nabla}_{e_l}V_A\right) h\big(du(e_i), du(e_j)\big) \times h\big(du(e_j), du(e_k)\big)h\big(du(e_l), du(e_p)\big)$$
$$= \sum u_i^a h\big(\nabla_{\varepsilon_a}V_A, u_p^b\varepsilon_b\big)h\big(u_k^c\varepsilon_c, u_l^d\nabla_{\varepsilon_d}V_A\big)h\big(u_i^e\varepsilon_e, u_j^f\varepsilon_f\big) \times h\big(u_j^r\varepsilon_r, u_k^s\varepsilon_s\big)h\big(u_l^t\varepsilon_t, u_p^x\varepsilon_x\big)$$
$$= \sum u_i^a h\big(v_A^\alpha B^\alpha_{ay}\varepsilon_y, u_p^b\varepsilon_b\big)h\big(u_k^c\varepsilon_c, u_l^d v_A^\beta B^\beta_{dz}\varepsilon_z\big) \times u_i^e u_j^f u_j^r u_k^s u_l^t u_p^x\delta_{ef}\delta_{rs}\delta_{tx}$$
$$= \sum B^\alpha_{ay}B^\alpha_{dz}u_i^a u_i^e u_j^f u_j^r u_k^c u_k^s u_l^d u_l^t u_p^b u_p^x\delta_{yb}\delta_{zd}\delta_{ef}\delta_{rs}\delta_{tx}$$
$$= \sum \lambda_a^2\lambda_f^2\lambda_c^2\lambda_d^2\lambda_b^2\delta_{ae}\delta_{fr}\delta_{cs}\delta_{dt}\delta_{bx}\delta_{yb}\delta_{cz}\delta_{ef}\delta_{rs}\delta_{tx}B^\alpha_{ay}B^\alpha_{dz}$$
$$= \sum \lambda_a^6\lambda_b^4\big\langle B(\varepsilon_a,\varepsilon_b), B(\varepsilon_a,\varepsilon_b)\big\rangle$$
$$\leq \sum \left(\frac{3}{5}\lambda_a^{10} + \frac{2}{5}\lambda_b^{10}\right)\big\langle B(\varepsilon_a,\varepsilon_b), B(\varepsilon_a,\varepsilon_b)\big\rangle$$
$$= \sum \lambda_a^{10}\big\langle B(\varepsilon_a,\varepsilon_b), B(\varepsilon_a,\varepsilon_b)\big\rangle \tag{47}$$

and

$$
\begin{aligned}
& h\left(\widetilde{\nabla}_{e_i} V_A, du(e_p)\right) h\left(du(e_l), \widetilde{\nabla}_{e_p} V_A\right) h\big(du(e_i), du(e_j)\big) \\
& \quad \times h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_l)\big) \\
= & \sum u_i^a h\big(\nabla_{\varepsilon_a} V_A, u_p^b \varepsilon_b\big) h\big(u_l^c \nabla_{\varepsilon_c} V_A, u_p^d \varepsilon_d\big) h\big(u_i^e \varepsilon_e, u_j^f \varepsilon_f\big) \\
& \quad \times h\big(u_j^r \varepsilon_r, u_k^s \varepsilon_s\big) h\big(u_k^t \varepsilon_t, u_l^x \varepsilon_x\big) \\
= & \sum u_i^a h\big(v_A^\alpha B_{ay}^\alpha \varepsilon_y, u_p^b \varepsilon_b\big) h\big(u_l^c v_A^\beta B_{cz}^\beta \varepsilon_z, u_p^d \varepsilon_d\big) \\
& \quad \times u_i^e u_j^f u_j^r u_k^s u_k^t u_l^x \delta_{ef} \delta_{rs} \delta_{tx} \\
= & \sum B_{ay}^\alpha B_{cz}^\alpha u_i^a u_i^e u_j^f u_j^r u_k^s u_k^t u_l^x u_l^c u_p^b u_p^d \delta_{yb} \delta_{zd} \delta_{ef} \delta_{rs} \delta_{tx} \\
= & \sum \lambda_a^2 \lambda_f^2 \lambda_s^2 \lambda_x^2 \lambda_b^2 \delta_{ae} \delta_{fr} \delta_{st} \delta_{xc} \delta_{bd} \delta_{yb} \delta_{zd} \delta_{ef} \delta_{rs} \delta_{tx} B_{ay}^\alpha B_{cz}^\alpha \\
= & \sum \lambda_a^8 \lambda_b^2 \big\langle B(\varepsilon_a, \varepsilon_b), B(\varepsilon_a, \varepsilon_b)\big\rangle \\
\leq & \sum \left(\frac{4}{5} \lambda_a^{10} + \frac{1}{5} \lambda_b^{10}\right) \big\langle B(\varepsilon_a, \varepsilon_b), B(\varepsilon_a, \varepsilon_b)\big\rangle
\end{aligned}
$$

$$
= \sum \lambda_a^{10} \big\langle B(\varepsilon_a, \varepsilon_b), B(\varepsilon_a, \varepsilon_b)\big\rangle \tag{48}
$$

and

$$
\begin{aligned}
& h\left(\widetilde{\nabla}_{e_i} V_A, du(e_p)\right) h\left(\widetilde{\nabla}_{e_l} V_A, du(e_p)\right) h\big(du(e_i), du(e_j)\big) \\
& \quad \times h\big(du(e_j), du(e_k)\big) h\big(du(e_k), du(e_l)\big) \\
= & \sum u_i^a h\big(\nabla_{\varepsilon_a} V_A, u_p^b \varepsilon_b\big) h\big(u_l^c \varepsilon_c, u_p^d \nabla_{\varepsilon_d} V_A\big) h\big(u_i^e \varepsilon_e, u_j^f \varepsilon_f\big) \\
& \quad \times h\big(u_j^r \varepsilon_r, u_k^s \varepsilon_s\big) h\big(u_k^t \varepsilon_t, u_l^x \varepsilon_x\big) \\
= & \sum u_i^a h\big(v_A^\alpha B_{ay}^\alpha \varepsilon_y, u_p^b \varepsilon_b\big) h\big(u_l^c \varepsilon_c, u_p^d v_A^\beta B_{dz}^\beta \varepsilon_z\big) \\
& \quad \times u_i^e u_j^f u_j^r u_k^s u_k^t u_l^x \delta_{ef} \delta_{rs} \delta_{tx} \\
= & \sum B_{ay}^\alpha B_{dz}^\alpha u_i^a u_i^e u_j^f u_j^r u_k^s u_k^t u_l^x u_l^c u_p^b u_p^d \delta_{yb} \delta_{cz} \delta_{ef} \delta_{rs} \delta_{tx} \\
= & \sum \lambda_a^2 \lambda_f^2 \lambda_s^2 \lambda_x^2 \lambda_b^2 \delta_{ae} \delta_{fr} \delta_{st} \delta_{xc} \delta_{bd} \delta_{yb} \delta_{cz} \delta_{ef} \delta_{rs} \delta_{tx} B_{ay}^\alpha B_{dz}^\alpha \\
= & \sum \lambda_a^8 \lambda_b^2 \big\langle B(\varepsilon_a, \varepsilon_b), B(\varepsilon_a, \varepsilon_b)\big\rangle \\
\leq & \sum \left(\frac{4}{5} \lambda_a^{10} + \frac{1}{5} \lambda_b^{10}\right) \big\langle B(\varepsilon_a, \varepsilon_b), B(\varepsilon_a, \varepsilon_b)\big\rangle
\end{aligned}
$$

$$
= \sum \lambda_a^{10} \big\langle B(\varepsilon_a, \varepsilon_b), B(\varepsilon_a, \varepsilon_b)\big\rangle . \tag{49}
$$

From (37) to (49), we have

$$
\begin{aligned}
& \sum_A I(V_A, V_A) \\
\leq & \int_M \sum_{a,b} \lambda_a^{10} \big(10 \langle B(\varepsilon_a, \varepsilon_b), B(\varepsilon_a, \varepsilon_b)\rangle - \langle B(\varepsilon_a, \varepsilon_a), B(\varepsilon_b, \varepsilon_b)\rangle\big) \, dv_g .
\end{aligned} \tag{50}
$$

Since N is a $\Phi_{(5)}$-SSU manifold, we know that if u is not a constant, then

$$\sum_A I(V_A, V_A) < 0\ .$$

Hence, u is a constant. This completes the proof.

□

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