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Serrin problem, constant flow property and isoparametric foliations

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Abstract. The study of overdermined elliptic problems in Euclidean space has been a very active field of research since many decades ago. In this survey paper we discuss overdetermined PDE's in the more general framework of Riemannian manifolds, and remark how the existence of isoparametric foliations of the manifold give rise to new domains supporting solutions to several overdetermined PDE's. Then we show that for a particular overdetermined parabolic problem (the constancy of the heat flow across the boundary), the existence of a solution is actually equivalent to the existence of an isoparametric foliation. The results exposed here are based on the recent articles [29], [30], [31]. The main theme of these works is in fact the link between overdetermined PDE's and the isoparametric theory.

1. Introduction

An overdetermined problem gives rise to the following question:
• can we identify the geometry of a domain Ω in a Riemannian manifold assuming the existence of a solution u of a certain PDE such that both u and its normal derivative are constant on the boundary of Ω?

Here is a class of overdetermined problems:

$$\begin{cases} \Delta u = F(u) & \text{on } \Omega \\ u = c_1 \ , \ \dfrac{\partial u}{\partial N} = c_2 & \text{on } \partial\Omega \ . \end{cases}$$

We try to derive some geometric information in the general framework of Riemannian manifolds, not just in Euclidean space.
In this paper we first focus, for simplicity of exposition, on a famous overdetermined problem, first solved by Serrin [32] in 1971. It states that the only bounded

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Euclidean domains admitting a solution to the overdetermined PDE:

$$
\begin{cases} \Delta v = 1 & \text{on } \Omega , \\ v = 0 \, , \ \dfrac{\partial v}{\partial N} = c & \text{on } \partial\Omega \end{cases} \tag{1}
$$

are balls. Such domains are often called "harmonic", because the mean value lemma holds, in the following sense: the mean value of a harmonic function on Ω equals its mean value on $\partial\Omega$.

The literature on overdetermined PDE's is vast and interesting. Other important examples include the Pompeiu problem (see the survey paper by Zalcman [35]) and also the Schiffer problem, which are still open, even in $\mathbf{R}^n$. See [12], [13], [33].

Serrin problem on manifolds, isoparametric tubes. Serrin's theorem admits a generalization to the hyperbolic space and the hemisphere (see [22]). However, on the whole sphere, there are plenty of "exotic" domains admitting a solution to (1). This was first observed by Berenstein for a related problem and then Karlovitz [18] for Clifford tori; later on Shklover [33] showed that any domain bounded by a connected isoparametric hypersurface in $\mathbf{S}^n$ is harmonic.

In this survey paper we first discuss Serrin problem on general Riemannian manifolds, and isolate a class of bounded domains, called *isoparametric tubes*, which generalize the class of spherical domains bounded by isoparametric hypersurfaces and which always host solutions to (1). These domains are foliated by parallel hypersurfaces having constant mean curvature, eventually collapsing to a minimal submanifold of possibly higher codimension. Precisely, the compact Riemannian manifold Ω with smooth boundary $\partial\Omega$ is called an *isoparametric tube* if there exists a smooth, compact submanifold P of Ω such that:

a) Ω is a tube of radius R around P, that is

$$
\Omega = \{x : d(x, P) \leq R\}
$$

b) Each equidistant from P, say

$$
\Sigma_t = \{x \in \Omega : d(x, P) = t\} \quad , \quad t \in (0, R] ,
$$

is a smooth hypersurface having constant mean curvature.
The submanifold P is called the *soul* of Ω. It can be shown that it is always minimal.

Next we ask whether the condition of Ω being harmonic is actually equivalent to Ω being an isoparametric tube. The answer is negative, which means that a classification of harmonic domains, even in a space as symmetric as the round sphere $\mathbf{S}^n$, is still an open problem.

The constant flow property. Then we consider another overdetermined (parabolic) problem, which is stronger than Serrin's, and is of its own interest. Precisely, consider the function $u(t, x)$, which is the temperature at the point $x \in \Omega$, at time $t > 0$, assuming that the initial temperature distribution is constant, equal to 1, and that the boundary is kept at zero temperature at all times (Dirichlet boundary conditions). Then $u(t, x)$ is the unique solution of the initial-boundary value problem:

$$
(2) \qquad \begin{cases} \Delta u(t,x) + \dfrac{\partial u}{\partial t}(t,x) = 0 & \text{for all } \ x \in \Omega, \ t > 0 \\ u(0,x) = 1 & \text{for all } \ x \in \Omega \\ u(t,y) = 0 & \text{for all } \ y \in \partial\Omega \ \text{ and for all } \ t > 0 \end{cases}
$$

$u(t,\cdot)$ is positive for all t. Note that:

$$
u(t,x) = \int_{\Omega} k(t,x,y)\, dy
$$

where $k(t,x,y)$ is the Dirichlet heat kernel of Ω. The normal derivative $\dfrac{\partial u}{\partial N}(t,y)$ is the heat flow at time t, at the boundary point $y \in \partial\Omega$.

• We say that Ω has the *constant flow property* if, at every fixed time $t > 0$, the heat flow is a constant function of $y \in \partial\Omega$, that is,

$$
(3) \qquad \frac{\partial u}{\partial N}(t,y) = \psi(t) \ ,
$$

for a smooth function ψ of time t only.

The condition we just introduced gives rise to an overdetermined problem, which is stronger than Serrin's, in the sense that

• any domain with the constant flow property is necessarily harmonic, that is, admits a solution to (1).

The existence of domains with the CF property is guaranteed by the following fact:

Theorem 1. *Every isoparametric tube has the constant flow property.*

One can hope to be able to give a classification under this stronger condition. In fact, this is true at least when the ambient Riemannian manifold is analytic, which provides the converse of Theorem 1 (see [30]).

Theorem 2 (Savo [30])**.** *Let Ω be a smooth, bounded domain in complete analytic manifold M. Assume that it has the constant flow property. Then Ω is an isoparametric tube around a minimal submanifold of M.*

In the body of the text we provide the ideas behind the proof, part of which is based on previous results by the author concerning the asymptotic behavior of the heat flow for small times.

Variational aspects and recent results. Overdetermined PDE's often come out of a variational problem. Let V be a smooth vector field defined in a neighborhood U of the domain Ω in M. Define a one-parameter domain deformation $f_\epsilon : \Omega \to M$ by:

$$
(4) \qquad f_\epsilon(x) = \exp_x(\epsilon V(x)) \ .
$$

For ϵ small enough, f_ϵ restricts to a diffeomorphism:

$$
f_\epsilon : \Omega \to f_\epsilon(\Omega) \doteq \Omega_\epsilon \ .
$$

We call Ω_ϵ the *one-parameter deformation of* Ω *associated to the vector field* V.

Given a geometric functional $\mathcal{F} = \mathcal{F}(\Omega)$ depending smoothly on the domain Ω we define its first variation $D\mathcal{F}(\Omega, V)$ along V as follows:

$$D\mathcal{F}(\Omega, V) \doteq \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathcal{F}(\Omega_\epsilon) \ .$$

We will say that Ω is *critical for the functional* $\mathcal{F}$ if

$$D\mathcal{F}(\Omega, V) = 0$$

for all deformations of Ω hence, for all vector fields V. However, to have a meaningful geometric problem one should (and we will) impose that the deformation f_ϵ is *volume preserving*:

$$|\Omega_\epsilon| = |\Omega|$$

for ϵ small enough. To preserve volume, the vector field V must satisfy the condition

$$\int_{\partial\Omega} \langle V, N\rangle = 0 \ . \tag{5}$$

Let v be the torsion function of the domain Ω, unique solution of

$$\begin{cases} \Delta v = 1 & \text{on } \ \Omega \ , \\ v = 0 & \text{on } \ \partial\Omega \end{cases} \tag{6}$$

The functional:

$$\Omega \mapsto \int_\Omega |\nabla v|^2$$

is called *torsional rigidity* of Ω.

• It is well-known that the critical points of the torsional rigidity (under volume preserving deformations) are precisely the harmonic domains, that is, domains which support a solution to the Serrin problem (9).

Given the function $u(t, x)$ defined in (2), the integral

$$H_\Omega(t) = \int_\Omega u(t, x)\, dv_g(x)$$

is called the *heat content* of Ω at time t. It measure the total heat inside Ω at time t, provided that the initial temperature is constant, equal to 1, and that the boundary is subject to absolute refrigeration at all times. The heat content has been amply studied, in particular, in connection with the following aspects: its asymptotic behaviour for small times (see [1], [2], [27], [28], [14], [15]), comparison theorems of isoperimetric nature ([3]) and its relation with the Dirichlet spectrum and the Brownian motion ([21], [20]).

In the recent paper [31], the author computed the first variation of the heat content and showed that the critical domains for this functional are precisely those which have the constant flow property. This provides a variational interpretation of this condition and then, thanks to the rigidity theorem above, a classification when the metric is analytic. The word *critical* means really *critical under volume preserving deformations.*

Theorem 3 (Savo [31])**.** *Let Ω be a smooth bounded domain in a complete Riemannian manifold (M, g). The following are equivalent:*

a) Ω is critical for the heat content $H_\Omega(t)$ at every fixed time $t > 0$.
b) Ω is critical for the k-th exit time moment $T_k(\Omega)$, for all $k \geq 1$.

c) Ω *has the constant flow property.*

If the ambient Riemannian manifold M *is* (*real*) *analytic, then a), b) and c) are in turn all equivalent to:*
d) Ω *is an isoparametric tube over a closed, minimal submanifold of* M.

A few words on the functional $T_k(\Omega)$. The torsion function v has also a probabilistic intepretation, as the *mean exit time* associated to the Brownian motion of Ω. It is then is part of a hierarchy of exit time functions E_k defined inductively as follows. We set $E_0 = 1$ and, for $k \geq 1$, we let E_k be the unique solution of

$$\begin{cases} \Delta E_k = kE_{k-1} \\ E_k = 0 \qquad\qquad \text{on } \partial\Omega \ . \end{cases} \tag{7}$$

Note that E_1 is just the torsion function. The *k-th exit time moment* of the bounded domain Ω is now defined as

$$T_k(\Omega) \doteq \int_\Omega E_k \, dv \ . \tag{8}$$

The sequence

$$m^\star(\Omega) = \{T_1(\Omega), T_2(\Omega), \cdots\}$$

is known as the *exit time moment spectrum* of Ω. These invariants have been studied in the papers [6], [7], [17],[16], [20], [21].

Conclusion. The main scope of this paper is to stress that in some cases the existence of solutions to a certain overdetermined PDE forces the existence of an isoparametric foliation of the domain.

This phenomenon is not evident when working in the Euclidean space because, (thanks to the classical Alexandrov theorem) there is only one foliation of $\mathbf{R}^n$ by compact, embedded, constant mean curvature hypersurfaces: the foliation by geodesic spheres. Therefore, the only bounded isoparametric tubes of $\mathbf{R}^n$ are geodesic balls.

Conversely, isoparametric tubes will host solutions to many overdetermined PDE's and they form a class of domains which, in many respects, generalize properties of geodesic balls in constant curvature space forms and revolution manifolds with boundary.

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2. Serrin problem

Torsion function. Consider the *torsion function* $v = v(x)$, unique solution of the boundary value problem (9). In probability, v also represents the *mean exit time function.*

For a generic domain, $\dfrac{\partial v}{\partial N}$ will not be constant on the boundary. If we add this extra assumption we then get an overdetermined problem, often called *Serrin problem*:

$$\begin{cases} \Delta v = 1 & \text{on } \ \Omega \ , \\ v = 0 \ , \ \dfrac{\partial v}{\partial N} = c & \text{on } \ \partial\Omega \ . \end{cases} \tag{9}$$

Harmonic domains. Domains for which a solution to (9) exists will be called *harmonic*, because of the following mean-value property, for which we provide the easy proof.

Proposition 4. *A domain Ω supports a solution to* (9) *if and only if the mean-value of any harmonic function h on Ω equals its mean-value on $\partial\Omega$; that is, if and only if:*

$$\frac{1}{|\Omega|}\int_\Omega h = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} h \ . \tag{10}$$

Proof. Assume that Ω is harmonic and pick the u solution to (9); integrating the identity $\dfrac{\partial u}{\partial N} = c$ over $\partial\Omega$ and applying the Green formula we see that $c = \dfrac{|\Omega|}{|\partial\Omega|}$. Let h be a harmonic function. Integrating by parts:

$$\begin{aligned} \int_\Omega h &= \int_\Omega h\Delta u \\ &= \int_\Omega u\Delta h + \int_{\partial\Omega} h\frac{\partial u}{\partial N} - \int_{\partial\Omega} u\frac{\partial h}{\partial N} \\ &= c\int_{\partial\Omega} h \\ &= \frac{|\Omega|}{|\partial\Omega|}\int_{\partial\Omega} h \ , \end{aligned}$$

which shows (10).
Conversely, assume that (10) holds for all harmonic functions h. Then, by Green formula,

$$\int_\Omega h = \int_{\partial\Omega} h\frac{\partial u}{\partial N} \ ,$$

hence

$$\frac{1}{|\Omega|}\int_\Omega h = \int_{\partial\Omega} h\frac{1}{|\Omega|}\frac{\partial u}{\partial N} \ .$$

By the hypothesis (10) we have then

$$\frac{1}{|\partial\Omega|}\int_{\partial\Omega} h = \int_{\partial\Omega} h\frac{1}{|\Omega|}\frac{\partial u}{\partial N}$$

which can be rewritten

$$\int_{\partial\Omega} h\Big(1 - \frac{|\partial\Omega|}{|\Omega|}\cdot\frac{\partial u}{\partial N}\Big) = 0 \ . \tag{11}$$

Any smooth function on $\partial\Omega$ admits a harmonic extension; (11) implies that the function in parenthesis is orhogonal to $C^\infty(\partial\Omega)$, hence it is zero. Thus $\dfrac{\partial u}{\partial N} = \dfrac{|\Omega|}{|\partial\Omega|}$ is constant.

□

2.1. **Serrin's theorem in $\mathbf{R}^n$.** First, we remark that any Euclidean ball is a harmonic domain: the torsion function is radial $u = u(r)$ (here $r = |x|$) and in fact, for the ball of radius R centered at the origin in $\mathbf{R}^n$:

$$u(r) = \frac{1}{2n}(R^2 - r^2) \ .$$

Serrin proved in 1971 the following rigidity result.

Theorem 5 (Serrin [32])**.** *Assume that $\Omega \Subset \mathbf{R}^{n+1}$ admits a solution to* (9)*. Then Ω is a ball and v is radially symmetric. That is, harmonic Euclidean domains are balls.*

He proved more generally that if there is a *positive* solution u of the problem

$$\begin{cases} \Delta u = F(u) \\ u = 0 \ , \ \dfrac{\partial u}{\partial N} = c \quad \text{on} \ \ \partial\Omega \ , \end{cases} \tag{12}$$

then Ω is a ball.

The original proof by Serrin uses the moving plane method. There is another short proof by Weinberger, which uses the maximum principle and the Pohozaev identity.

2.2. **Extension to more general Riemannian manifolds?** One could ask if Serrin's rigidity theorem extends to more general Riemannian manifolds.
Here are a number of related questions.

Existence: if M is an arbitrary Riemannian manifold, do we always have harmonic domains there? Given a positive number $\alpha < |M|$, can we always find a harmonic domain of volume α inside M?

Rigidity: do we have geometric restrictions for the existence of harmonic domains? Can we describe them?

Classification: can we actually classify harmonic domains in the general Riemannian framework?

We have seen that if M is Euclidean space then the answer to the first question is positive (any ball is a harmonic domain) and we have a strong rigidity result: the only harmonic domains are balls.

Naturally, the next step is to examine these questions in the other (simply connected) manifolds of constant curvature: hyperbolic space and the sphere.
About existence, Fall and Minlend prove in [8] that in any compact Riemannian manifold there exists at least one harmonic domain of *small volume*, constructed by suitably perturbing small geodesic balls.

About rigidity, one could expect that the Serrin problem, at least in constant curvature, will display a kind of geometric rigidity similar to the one in Euclidean space.

We will see that this is true in hyperbolic space and in the hemisphere, but, perhaps surprisingly, not in the whole (round) sphere.

2.3. **Serrin problem in hyperbolic space and the hemisphere.** It is not restrictive to study the case of the unique simply connected manifold M_K of constant curvature $K \in \{1, -1\}$. Then $M_1 = \mathbf{S}^n$ and $M_{-1} = \mathbf{H}^n$.

Existence. Geodesic balls in M_K are harmonic domains: in fact, the mean exit time function is radial. Here is a formula for the explicit expression valid in all space forms (r is the distance function to a fixed reference point):

$$u(r) = \int_r^R \frac{1}{\theta(s)} \int_0^s \theta(t)\, dt\, ds$$

where

$$\theta(r) = \begin{cases} \sin^{n-1} r & \text{if } M = \mathbf{S}^n \\ r^{n-1} & \text{if } = \mathbf{R}^n \\ \sinh^{n-1} r & \text{if } M = \mathbf{H}^n \,. \end{cases}$$

As for rigidity, the result in Euclidean space extends to the hyperbolic space and the hemisphere without change (see [22] and [19]).

Theorem 6. *The only harmonic domains in $\mathbf{H}^n$ and $\mathbf{S}^n_+$ are geodesic balls.*

Method of proof: moving planes (again). Now, the usual method of moving planes works in the hemisphere, but not in the whole sphere; then, one could ask if the restriction to the hemisphere is an essential hypothesis, or is just assumed for technical reasons. In other words:

Is it true that the only harmonic domains in $\mathbf{S}^n$ are geodesic balls?

Perhaps a bit surprisingly, the answer is: no.

2.4. **Exotic harmonic domains in spheres.** It seems that the first example was given by C. Berenstein for a related problem. Consider the 2-surface (*Clifford torus*) isometrically embedded in $\mathbf{S}^3$:

$$\Sigma = \mathbf{S}^1(\frac{1}{\sqrt{2}}) \times \mathbf{S}^1(\frac{1}{\sqrt{2}}) \,.$$

It is easy to show that Σ is the common boundary of two spherical domains Ω_1 and Ω_2.

Berenstein shows that both these domains are harmonic.

As the boundary of each is a torus, which is topologically different from a sphere, it is clear that Ω_1 and Ω_2 are not isometric to geodesic balls. This gives the desired counterexample.

His proof is analytical; we provide here a simpler proof and enlarge the set of examples.
More generally, for positive numbers a, b such that $a^2 + b^2 = 1$, consider the *Clifford torus*

$$\Sigma_{a,b} = \mathbf{S}^1(a) \times \mathbf{S}^1(b) \,,$$

where

$$\mathbf{S}^1(a) = \{(x_1, x_2) : x_1^2 + x_2^2 = a^2\} \quad , \quad \mathbf{S}^1(b) = \{(x_3, x_4) : x_3^2 + x_4^2 = b^2\} \,.$$

Then the natural map $\phi : \Sigma_{a,b} \to \mathbf{S}^3$ defined by

$$\phi((x_1, x_2), (x_3, x_4)) = (x_1, x_2, x_3, x_4)$$

is an isometric embedding.
Consider the domain $\Omega_{a,b} \subseteq \mathbf{S}^3$ defined by the inequalities:

$$\Omega_{a,b} : \begin{cases} x_1^2 + x_2^2 \leq a^2 \\ x_3^2 + x_4^2 \geq b^2 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \end{cases}$$

so that $\partial\Omega_{a,b} = \Sigma_{a,b}$.

Proposition 7. *The spherical domain $\Omega_{a,b}$ is harmonic for all a, b.*

Proof. The group $\mathbf{SO}(2)$ acts by rotations on each circle in the (x_1, x_2)-plane, and also on each circle in the (x_3, x_4)-plane; hence the group $G = \mathbf{SO}(2) \times \mathbf{SO}(2)$ acts by isometries on Ω. This action restricts to a transitive action on $\partial\Omega = \Sigma_{a,b}$. We need to show that the solution of

$$\begin{cases} \Delta u = 1 & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{13}$$

has constant normal derivative. Now G acts on $C^\infty(\Omega)$ as follows: for $g \in G$ define $g \cdot u \in C^\infty(\Omega)$ by

$$(g \cdot u)(x) = u(g^{-1} \cdot x) .$$

As the Laplacian commutes with isometries, one has

$$\Delta(g \cdot u) = g \cdot \Delta u = g \cdot 1 = 1$$

and clearly $g \cdot u = 0$ on $\partial\Omega$. Hence $g \cdot u$ is also a solution of (13). By uniqueness,

$$g \cdot u = u$$

for all $g \in G$. Therefore u is G-invariant, that is, it is constant on the orbits of G. An isometry is in particular a conformal map, hence the action of the group preserves the unit normal vector field N; as the action of G is transitive on $\partial\Omega$, given $p, q \in \partial\Omega$ we can always find $g \in G$ such that $g \cdot p = q$. The invariance of u shows that then

$$\frac{\partial u}{\partial N}(p) = \frac{\partial u}{\partial N}(q) .$$

As p and q are arbitrary, this implies that $\dfrac{\partial u}{\partial N}$ is constant on $\partial\Omega$, as asserted. □

The previous construction gives a whole one-parameter family of harmonic domains not isometric to balls.

• Actually, the above example generalizes to get the following class of examples:

Let Ω be any compact Riemannian manifold with boundary on which G acts by isometries. Assume that the action of G restricts to a transitive action on $\partial\Omega$. Then Ω is a harmonic domain.

$\Omega_{a,b}$ is an example of "isoparametric tube". Observe that the domain $\Omega_{a,b}$ above is foliated by a one parameter family of parallel, equidistant Clifford tori:

$$\Sigma_t = \mathbf{S}^1(t) \times \mathbf{S}^1(\sqrt{1 - t^2}) .$$

where $t \in (0, a]$ which, as $t \to 0$, collapses to the great circle P (which is a minimal submanifold of $\mathbf{S}^3$):

$$\begin{cases} x_1 = x_2 = 0 \\ x_3^2 + x_4^2 = 1 \ . \end{cases}$$

Each leaf Σ_t has constant mean curvature. Note also that $\Omega_{a,b}$ is a tube of constant radius around P, and all the equidistants to P have constant mean curvature. In our language, $\Omega_{a,b}$ is an *isoparametric tube around P*.

3. Isoparametric tubes in Riemannian manifolds

We have a general notion of isoparametric tube, which gives rise to a family of domains which always support a solution to the Serrin problem.

By definition, the compact Riemannian manifold Ω with smooth boundary $\partial\Omega$ is called an *isoparametric tube* if there exists a smooth, compact submanifold P of Ω such that:

a) Ω is a tube of radius R around P, that is

$$\Omega = \{x : d(x, P) \leq R\} \ .$$

b) Each equidistant from P, say

$$\Sigma_t = \{x \in \Omega : d(x, P) = t\} \quad , \quad t \in (0, R] \ ,$$

is a smooth hypersurface having constant mean curvature.

• The submanifold P is called the *soul* of Ω. It can be shown that it is always minimal.

3.1. **Examples.**

• We will see below that, in $\mathbf{S}^n$, any domain bounded by a connected isoparametric hypersurface (that is, a hypersurface with constant principal curvatures) is an isoparametric tube. The soul is the focal submanifold of $\partial\Omega$.

• Geodesic balls in constant curvature spaces (or, more generally, in a harmonic manifold) are isoparametric tubes. The soul is a point (the center of the ball).

• A spherical shell in $\mathbf{R}^n$ is *not* an isoparametric tube.

• An isoparametric tube might have more than one boundary component: for example, a tubular neighborhood of an equator of the sphere. However, it cannot have more than two boundary components (otherwise it is not even a smooth tube).

• A solid revolution torus in $\mathbf{R}^3$ is a smooth tube, but it is not isoparametric: the soul is circle, and equidistant do not have constant mean curvature.

Here is the relevant result, proved in [30].

Theorem 8. *Any isoparametric tube is a harmonic domain.*

The aim is to show that the mean exit time function u:

$$\begin{cases} \Delta u = 1 & \text{on } \ \Omega \\ u = 0 & \text{on } \ \partial\Omega \end{cases}$$

has constant normal derivative. In order to do that, we introduce the family of radial functions, and then an averaging operator.

3.2. **Radial functions.** On any isoparametric tube we have the family of *radial functions*: these are the functions which are constant on every equidistant.
If we write ρ for the distance function to P:

$$\rho(x) = d(x, P)$$

then we say that the function f on Ω is radial if and only if it can be expressed

$$f = \psi \circ \rho \ ,$$

for a smooth function $\psi : [0, R] \to \mathbf{R}$. It is clear that a radial function has constant normal derivative; in fact on the boundary one has:

$$\frac{\partial f}{\partial N} = \langle \nabla f, N \rangle = -\langle \nabla f, \nabla \rho \rangle = -\psi'(R) \ ,$$

which is constant. Hence, to prove the theorem, it is enough to show that the torsion function is radial.

Averaging operator (radialization). For details we refer to [30]. We define an operator

$$\mathcal{A} : C^\infty(\Omega) \to C^\infty(\Omega)$$

which will take a function to its *radialization*. Recall that Ω is foliated by the equidistants (level sets of the distance function ρ), hence any point x belongs to a unique equidistant; if $x \in \Omega \setminus P$ this will be the regular hypersurface

$$\Sigma_x = \rho^{-1}(\rho(x)) \ ;$$

and if $x \in P$ then it will be simply P (the unique singular leaf).

Given $f \in C^\infty(\Omega)$ and $x \in \Omega \setminus P$ we define $\mathcal{A}f(x)$ as the average of f on the equidistant through x:

$$\mathcal{A}f(x) = \frac{1}{|\Sigma_x|} \int_{\Sigma_x} f \ ;$$

if $x \in P$ we simply define

$$\mathcal{A}f(x) = \frac{1}{|P|} \int_P f \ ,$$

where of course we use in both cases the measure induced by the Riemannian metric on Σ_x and P, respectively. If $f \in C^\infty(\Omega)$ its radialization $\mathcal{A}f$ has the following properties:

- *$\mathcal{A}f$ is smooth as well*;
- *$\mathcal{A}f$ is radial*;
- *f is radial if and only if $\mathcal{A}f = f$.*

The crucial property of an isoparametric foliation is the constancy of the mean curvature of the leaves. This has the following important consequence.

- *The radialization commutes with the Laplacian: $\mathcal{A}\Delta f = \Delta\mathcal{A}f$.*

3.3. **Any isoparametric tube is a harmonic domain.** We now prove Theorem 8. It is enough to prove that the torsion function u is radial; for that it is enough to show that

$$\mathcal{A}u = u \ .$$

Let $\hat{u} = \mathcal{A}u$. Then as the radialization commutes with the Laplacian:

$$\Delta\hat{u} = \Delta\mathcal{A}u = \mathcal{A}\Delta u = \mathcal{A}1 = 1 \ ,$$

and of course $\hat{u} = 0$ on $\partial\Omega$. Then u and $\hat{u}$ are two solutions of the boundary value problem

$$\begin{cases} \Delta u = 1 & \text{on} \ \ \Omega \\ u = 0 & \text{on} \ \ \partial\Omega \end{cases}$$

By uniqueness, they must coincide, hence $\mathcal{A}u = u$, as asserted.

3.4. **Is the converse true? Classification of harmonic domains?** We have seen that every isoparametric tube is a harmonic domain. Is the converse also true? If true, this will give a quite rigid and significant geometric structure to such manifolds. The answer, however, is *negative*, and we point out an interesting class of counterexamples: minimal free boundary immersions with many boundary components.

3.5. **Minimal free boundary immersions.** Recall that a hypersurface of a Riemannian manifold is minimal if it has everywhere vanishing mean curvature. If Σ is a minimal hypersurface of $\mathbf{R}^n$, and $B(x_0, R)$ is a Euclidean ball, the manifold with boundary (or, a connected component of it)

$$\Omega = \Sigma \cap B(x_0, R)$$

is called an *extrinsic ball*. Note that Ω has dimension $n-1$ and that the boundary $\partial\Omega$ is contained in the sphere $\partial B(x_0, R)$. An interesting case occurs when, at each point of the boundary, the tangent space to Ω is orthogonal to the sphere: we then say that Ω meets ∂B orthogonally.

In what follows, we let B^n be the unit ball of $\mathbf{R}^n$ centered at the origin.

• A *minimal free boundary hypersurface* is a minimal hypersurface Ω of B^n such that $\partial\Omega \subseteq \partial B^n$ and Ω meets ∂B^n orthogonally.

Examples in B^3.

• A *flat disk* (cross-section of B^3 with a plane through the origin).

• The *critical catenoid* (the unique catenoid which meets ∂B^3 orthogonally).

Up until few years ago, these were the only known minimal free boundary hypersurfaces. Then new examples by Fraser and Schoen [11] were discovered:

• *Given any positive integer k, there exists a* (*minimal*) *free boundary embedding of a* (*genus zero*) *surface Ω_k with k boundary components into B^3.*

By then other existence results were proved. We have the following interesting fact.

Theorem 9. *Let Σ be a minimal free boundary immersion in B^n. Then Σ is a harmonic domain.*

Proof. We sketch the main arguments. First, recall that the restriction of any coordinate function x_j to a minimal hypersurface is harmonic. Let x be the position vector, so that $|x|^2 = \sum_j x_j^2$. We restrict the function $|x|^2$ to Ω, and denote Δ the Laplacian of Ω. Using the above fact one shows easily that:

$$\Delta|x|^2 = -(2n-2) .$$

Clearly, $|x|^2 = 1$ on $\partial\Omega$ hence $1 - |x|^2$ vanishes on $\partial\Omega$. The conclusion is that the mean exit time function of Ω is given by:

$$u(x) = \frac{1}{2n-2}(1 - |x|^2) .$$

It remains to show that the normal derivative of u is constant on $\partial\Omega$. This follows because, on $\partial\Omega$, the vector ∇u is collinear with x, and that $N = -x$ because Ω meets ∂B orthogonally. Then $\dfrac{\partial u}{\partial N} = \dfrac{1}{n-1}$, a constant.

□

3.6. **Some counterexamples.**

Theorem 10. *Let Ω be any minimal free boundary immersion in B^3 with at least 3 boundary components. Then Ω is harmonic but cannot be an isoparametric tube.*

In fact, Ω is a harmonic domain by the previous theorem but cannot be an isoparametric tube, for topological reasons: in fact any smooth tube over a connected submanifold can have at most two boundary components (the boundary of a solid tube around P is homeomorphic to the unit normal bundle of P), see [30].

The classification problem of harmonic domains in Riemannian manifolds is an open and interesting problem, because it will imply, in particular, a classification of minimal free boundary immersions.

In fact Fall, Minlend and Weth [9] constructed new examples of harmonic domains in $\mathbf{S}^n$, by perturbing tubular neighborhoods of totally geodesic hypersurfaces.

These examples are not isoparametric tubes.

So, a classification seems to be at the moment out of reach, even in spaces as simple as the round sphere.

In the next section we will consider another overdetermined problem, which is stronger than the Serrin problem, and for which a classification in the Riemannian case is actually possible.

4. The heat equation

Let Ω be a domain in a Riemannian manifold. Assume that at time $t = 0$ the temperature is prescribed by the function $\phi_0(x)$ on Ω, and that the boundary is kept at zero temperature at all times.

If for example ϕ_0 is positive, heat will flow away from the domain because of refrigeration and eventually, at infinite time, the temperature will be constant, equal to zero, at all points of Ω.

But what is precisely the evolution of temperature?

Denote by $\phi(t,x)$ the temperature at the point $x \in \Omega$, at time $t > 0$. Then $\phi(t,x)$ is a solution of the heat equation:

$$\Delta\phi(t,x) + \frac{\partial\phi}{\partial t}(t,x) = 0 \ .$$

where the Laplacian is acting on the space variable x.
In conclusion, the temperature function is a solution of the following initial-boundary value problem:

$$\begin{cases} \Delta\phi(t,x) + \dfrac{\partial\phi}{\partial t}(t,x) = 0 & \text{for all } \ x \in \Omega \ , \quad t > 0 \\ \phi(0,x) = \phi_0(x) & \\ \phi(t,y) = 0 & \text{for all } \ y \in \partial\Omega \ \text{ and for all } \ t > 0 \end{cases}$$

In the last line we see the Dirichlet boundary condition, which correspond to boundary refrigeration at all times.

One shows that indeed, once the initial data and the boundary conditions are chosen, the solution $\phi(t,x)$ *exists and is unique*.

Let us write $\phi_t(x) \doteq \phi(t,x)$. The maximum principle implies that, if ϕ_0 is a positive function, then ϕ_t will be positive in the interior of Ω for all $t > 0$.

4.1. **The heat kernel.** The initial data could be a distribution: when the initial data is the Dirac mass concentrated at a point $y \in \Omega$ then we have the *fundamental solution*, or *heat kernel* of the heat equation.

This corresponds to the case when a unit of heat is initially placed at the given point y, which is the limit case of a real physical situation. If one assumes, as usual, boundary refrigeration, the temperature at time t, at the generic point x will be denoted by the symbol

$$k(t,x,y) \ .$$

It is perhaps a curious fact the the heat kernel is actually *symmetric* in the spatial entries:

$$k(t,x,y) = k(t,y,x) \ .$$

The fact that the initial data is the Dirac distribution δ_y translates into the following statement. For all continuous functions $\phi_0(x)$ one has:

$$\lim_{t\to 0} \int_\Omega k(t,x,y)\phi_0(x)\, dx = \phi_0(y) \ .$$

In fact, the word *fundamental* means that the solution of the heat equation with initial data $\phi_0(x)$ is given by convolution with the heat kernel:

$$\phi(t,x) = \int_\Omega k(t,x,y)\phi_0(y)\, dy \ .$$

The heat kernel is, at all times, a positive function of x, y in the interior of Ω, it vanishes whenever x or y are on the boundary, and has positive normal derivative on the boundary:

$$\frac{\partial k}{\partial N}(t,x,y) > 0$$

for all $y \in \partial\Omega$.

4.2. **When the initial temperature is constant.** If the initial data is the constant unit function 1, we will denote by $u_t(x) = u(t,x)$ the corresponding temperature function. In other words, $u(t,x)$ is the unique solution of the problem:

$$(14)\qquad \begin{cases} \Delta u(t,x) + \dfrac{\partial u}{\partial t}(t,x) = 0 & \text{for all } \ x \in \Omega\ , \ \ t > 0 \\ u(0,x) = 1 & \text{for all } \ x \in \Omega \\ u(t,y) = 0 & \text{for all } \ y \in \partial\Omega \ \text{ and for all } \ t > 0 \end{cases}$$

Then u_t will be positive for all t. Note that:

$$u(t,x) = \int_\Omega k(t,x,y)\,dy\ ,$$

in particular, u_t is positive for all t. The physical interpretation is the following: $u(t,x)$ is the temperature at time $t > 0$, at the point $x \in \Omega$, assuming the initial temperature is uniformly constant, equal to 1, and that the boundary is subject to absolute refrigeration.

4.3. **The constant flow property.** We now introduce an overdetermined problem for the heat equation (14). We first turn our attention to the *heat content* of Ω, which is the function of time:

$$H(t) = \int_\Omega u_t\ .$$

It is smooth on $(0,\infty)$ and it is a decreasing function of t. In fact,

$$H'(t) = \frac{d}{dt}\int_\Omega u_t = \int_\Omega \frac{\partial u_t}{\partial t} = -\int_\Omega \Delta u_t = -\int_{\partial\Omega} \frac{\partial u_t}{\partial N}\ .$$

The function

$$\frac{\partial u_t}{\partial N}(y)$$

gives, at any point $y \in \partial\Omega$, the *pointwise heat flow at* y. It is a smooth, positive function defined on $\partial\Omega$.

We expect that, for a general domain, the heat flow $\dfrac{\partial u_t}{\partial N}$ is not constant on $\partial\Omega$.

• We say that Ω has the *constant flow property* if, for all fixed $t > 0$, the normal derivative $\dfrac{\partial u_t}{\partial N}$ is a constant function on $\partial\Omega$.

This additional request gives rise to an overdetermined problem, which can then be written:

$$(15)\qquad \begin{cases} \Delta u_t + \dfrac{\partial u_t}{\partial t} = 0 & \\ u_0 = 1 & \text{on } \ \Omega \\ u_t = 0\ ,\ \dfrac{\partial u_t}{\partial N} = \psi(t) & \text{on } \ \partial\Omega \ \text{ for all } \ t > 0 \end{cases}$$

for a suitable smooth function ψ of the only variable $t \in (0,\infty)$.

4.4. **Perfect heat diffusers.** Domains with the constant flow property are *perfect heat diffusers* in the following sense (see [29]). Given a smooth function $\phi \in C^\infty(\partial\Omega)$, define

$$\hat{\phi}_t(x) \doteq \hat{\phi}(t,x)$$

as the solution of the heat equation on Ω with boundary conditions prescribed by the function $\phi(x)$ (at all times) and zero initial conditions.

That is, $\hat{\phi}_t$ is the unique solution of the problem:

$$\begin{cases} \Delta\hat{\phi}_t + \dfrac{\partial\hat{\phi}_t}{\partial t} = 0 \\ \hat{\phi}_0 = 0 \\ \hat{\phi}_t = \phi \quad \text{on } \partial\Omega\,, \quad \text{for all } \; t > 0 \end{cases}$$

Let

$$H_\phi(t) = \int_\Omega \hat{\phi}(t,x)\,dx$$

be the heat content at time t with boundary data ϕ. Clearly $H_\phi(0) = 0$. The following has been proved in [29]:

Theorem 11. *Ω has the constant flow property if and only if $H_\phi(t) = 0$ for all $t \geq 0$ and for all $\phi \in C_0^\infty(\partial\Omega)$ (smooth functions on $\partial\Omega$ with zero mean).*

The theorem says that if Ω has the constant flow property, and if the total boundary heat is zero at all times, then also the total inner heat content is identically zero at all times.

At $x \in \partial\Omega$ the boundary acts as a refrigerator if $\phi(x) < 0$, and acts as a heater if $\phi(x) > 0$. Then the constant flow property holds if and only if the *incoming heat flow* is perfectly balanced, at all times, by the *outgoing heat flow*.

4.5. **Constant flow implies harmonic.** It turns out that the constant flow property is stronger than harmonicity (see [29]):

Theorem 12. *Any domain with the constant flow property is also harmonic, that is, it supports a solution to the Serrin problem.*

This can be justified as follows. Introduce the function

$$v(x) = \int_0^\infty u(t,x)\,dt\ .$$

Note that the integral converges because $u(t,x)$ decreases to zero exponentially fast as $t \to \infty$.
In fact it can be shown that, as $t \to \infty$

$$u(t,x) \sim ce^{-\lambda_1 t}\phi_1(x)$$

where λ_1 is the first Dirichlet eigenvalue of Ω and $\phi_1(x)$ is the positive first eigenfunction with unit L^2-norm. One sees that $\Delta v = 1$ and obviously $v = 0$ on $\partial\Omega$. Therefore, v is the mean-exit time function of Ω. If Ω has the constant flow property one sees that, differentiating in the normal direction, one has $\dfrac{\partial v}{\partial N} = c$ as well. By Serrin's result and the above, we have:

The only domains in $\mathbf{R}^n, \mathbf{H}^n$ *and* $\mathbf{S}^n_+$ *having the constant flow property are geodesic balls.*

But what about existence of domains with constant flow property? Here considerations similar to the Serrin problem apply.

Theorem 13. *Any isoparametric tube has the constant flow property.*

For the proof, recall that the radialization $\mathcal{A}$ commutes with the Laplacian. To show that the temperature function u_t has constant normal derivative (for all t), it is enough to show that it is radial, or that:

$$\mathcal{A}u_t = u_t \ .$$

The argument is the same as before: the function $\mathcal{A}u_t$ is still a solution of the heat equation, with initial condition equal to $\mathcal{A}u_0 = 1$ and, obviously, Dirichlet boundary conditions.

As the initial and boundary values data of $\mathcal{A}u_t$ are the same as those of u_t, we have by uniqueness $\mathcal{A}u_t = u_t$.

4.6. **Geometric rigidity.** So far we have isolated a whole class of manifolds with the constant flow property, the isoparametric tubes.

We can ask if there are other "exotic" examples.

We prove that, for analytic metrics, isoparametricity is also a necessary condition for having the constant flow property. Here is central result, proved in [30], and considerably more difficult than Theorem 13.

Theorem 14 (Savo [30])**.** *Let* Ω *be a compact analytic manifold with smooth boundary. Assume that it has the constant flow property. Then* Ω *is an isoparametric tube around a minimal submanifold of* M*.*

Then we have a complete characterization, in the analytic case, of the class of domains with the constant flow property: this class coincides with the class of isoparametric tubes.

This also gives an analytic characterization of the isoparametric condition.

4.7. **Main steps of the proof of Theorem 14.** Details can be found in [30]. We assume that Ω has the constant flow property.

Step 1. *The mean curvature is radial.*

Define a function η in a neighborhood of $\partial\Omega$ as follows:

$$\eta(x) = \text{mean curvature of } \Sigma_x \text{ at } x$$

where Σ_x is the equidistant to the boundary containing x; if we let ρ denote the distance function to the boundary, then

$$\Sigma_x = \{y \in \Omega : \rho(y) = \rho(x)\} \ .$$

Observe that ρ is continuous, and smooth a.e. on Ω, in particular, it is smooth near the boundary of Ω. Precisely, ρ is singular on the so-called *cut-locus*, which is a closed set of zero measure, supported away from $\partial\Omega$. We will discuss the cut-locus below.

The aim in the first step is to show that η is radial, that is, it is constant on equidistants to the boundary, and the following theorem is the main technical step, proved in [29].

Theorem 15. *Let Ω be a smooth (not necessarily analytic) Riemannian manifold. Assume that Ω has the CFP. Then:*

$$\frac{\partial^k \eta}{\partial N^k} = c_k = \textit{constant on } \partial\Omega$$

for all $k = 0, 1, 2, \cdots$.

The proof of the theorem relies on the asymptotic expansion of the heat flow, which has been obtained by the author in [28] and [27]. We will sketch the proof in the next section.

Given Theorem 15, we proceed as follows. Fix a point x on the boundary and consider the unit speed geodesic arc $\gamma_x(t)$ exiting x in the inner normal direction. As η is an analytic function, its restriction to γ_x is also analytic, and by Theorem 15 the Taylor expansion of this restriction does not depend on the base point x.

In conclusion, the value of η is the same at all points at fixed distance to the boundary (that is, at the regular points of the normal exponential map). In particular:

• *near the boundary*, η depends only on the distance to $\partial\Omega$, hence the equidistants close to the boundary have constant mean curvature.

One could also say that, *near the boundary*, Ω is isoparametric.

Now we have to show the global result, namely that Ω is isoparametric not only near the boundary but also at points far from it, and that is a tube over a regular submanifold.

Typically, this involves the study of the singularity of the normal exponential map, that is, the study of the *cut-locus* Cut_Ω.

4.8. **Step 2: study of the cut-locus.** Given $x \in \partial\Omega$, consider as before the geodesic $\gamma_x : [0, L] \to \Omega$ which starts at x and goes in the unit normal direction:

$$\gamma_x(0) = x \quad , \quad \gamma_x'(0) = N(x) .$$

Clearly γ_x can be written $\gamma_x(t) = \exp_x(tN(x))$. For small t, γ_x will minimize distance to $\partial\Omega$, in the sense that $d(\gamma_x(t), \partial\Omega) = t$.

As Ω is bounded, γ_x cannot measure distance for every t, and there will be a maximum value $c(x)$ for which this happens. Hence, we define $c(x)$ as follows:

$$d(\gamma_x(t), \partial\Omega) = t \quad \text{if and only if} \quad t \in [0, c(x)] .$$

$c(x)$ is called the *cut-radius at $x \in \partial\Omega$* and $\gamma_x(c(x))$ is the *cut point* at x.

The cut-locus is then the union of all cut-points:

$$\mathrm{Cut}_\Omega = \{\gamma_x(c(x)) : x \in \partial\Omega\} .$$

It is known that the cut-locus is a *closed* subset of Ω, and it has *measure zero*. Moreover, Cut_Ω is a deformation retract of Ω and the distance function to the boundary is smooth on $\Omega \setminus \mathrm{Cut}_\Omega$.

In general, the cut-locus of $\partial\Omega$ is far from being a regular subset; however, if Ω has the constant flow property then the cut-locus is a nice smooth, embedded and even minimal submanifold of Ω.

4.9. **End of proof.** Using the constant flow property one shows that Cut_Ω coincides with the set of points of Ω which are at (constant) maximum distance to the boundary; in fact Cut_Ω coincides with the critical set of the torsion function.

One then verifies that the normal exponential map has locally constant rank, which implies, after some work, that Cut_Ω is a compact, connected, regular submanifold of Ω.

We now set

$$P = \mathrm{Cut}_\Omega\,.$$

The equidistants from P coincide with the equidistants to $\partial\Omega$; as these have constant mean curvature, we see that Ω is an isoparametric tube over P, as asserted.

4.10. **Sketch of proof of Theorem 15.** Recall that we have to show that, if Ω has the constant flow property, then

$$\frac{\partial^k \eta}{\partial N^k} = c_k = \text{constant on } \ \partial\Omega$$

for all $k = 0, 1, 2, \cdots$. For details we refer to [29].

The main tool is an asymptotic study of the heat flow for small times. The heat flow at $y \in \partial\Omega$ admits an asymptotic expansion for $t \to 0$, of type:

$$\begin{aligned} \frac{\partial u_t}{\partial N}(y) &\sim \sum_{k=1}^{\infty} B_k(y) \cdot t^{k/2-1} \\ &\sim B_1(y) \cdot \frac{1}{\sqrt{t}} + B_2(y) + B_3(y)\sqrt{t} + \cdots \end{aligned}$$

for certain heat flow invariants $B_k(y) \in C^\infty(\partial\Omega)$.

We see that, if Ω has the CFP, then every B_k is constant on $\partial\Omega$. The existence of the asymptotic series follows from the work done by van den Berg and Gilkey in [2]; the first few terms are:

$$\begin{aligned} B_1 &= \frac{2}{\sqrt{\pi}} \\ B_2 &= -\frac{1}{2}\eta \\ B_3 &= -\frac{1}{6\sqrt{\pi}}\Big(2\mathrm{tr}(R_N + S^2) - \eta^2\Big) \\ B_4 &= \frac{1}{16}\Big(\eta\mathrm{tr}(R_N + S^2) - \mathrm{tr}(\nabla_N R_N + 2S \circ R_N + 2S^3) + \Delta^{\partial\Omega}\eta\Big) \end{aligned}$$

where R_N is the Jacobi operator $R_N(X) = R(N,X)N$ and S is the shape operator of $\partial\Omega$ relative to the inner unit normal N.

Given that, we see that the CFP immediately implies some valuable informations.

• The mean curvature must be constant: hence the only Euclidean or hyperbolic domains with the CFP property are balls (by the Alexandrov theorem).

• If Ω has constant curvature (or, more generally, if it is an Einstein manifold) then the scalar curvature of $\partial\Omega$ must be constant.

However, to prove that all the jets of the mean curvature are constant on $\partial\Omega$ one needs a complete information about the invariants B_k, for all k; moreover it would be desirable to give a different presentation of B_k. In [28] the author obtained a recursive formula which computes all invariants B_k. We now describe the outcome and the way the invariants B_k are presented.

4.11. **Complete asymptotic expansion of the heat flow.** This is the work [28] (which improves the calculation in [27]). On a small tubular neighborhood U of $\partial\Omega$, the distance function ρ is smooth, and then we can define the first order operator:

$$N\phi = 2\langle\nabla\phi, \nabla\rho\rangle - \phi\Delta\rho \ ,$$

for all $\phi \in C^\infty(U)$. Note that $\nabla\rho$ is normal to equidistants and gives rise to a vector field N which, restricted to the boundary, is precisely the unit normal vector. Remark also that $\Delta\rho = \eta$.

Hence the operator $N\phi$ can also be written:

$$N\phi = 2\frac{\partial\phi}{\partial N} - \eta\phi \ .$$

In particular:

$$N1 = -\eta \ .$$

Let Δ be the Laplacian of the ambient manifold Ω. We let $\mathcal{A}$ be the algebra of operators acting on $C^\infty(U)$ and generated by the operators N and Δ. Here is the main calculation.

Theorem 16 (Savo [28]). *For all $k = 1, 2, \cdots$ there exists an operator $D_k \in \mathcal{A}$ (that is, a polynomial in N and Δ of homogeneous degree $k-1$) such that:*

$$B_k = D_k 1\big|_{\partial\Omega} \ .$$

The sequence of operators $\{D_k\}$ is explicitly computable by a recursive formula.

Here are the first few operators:

$$\begin{aligned} D_1 &= \frac{2}{\sqrt{\pi}} \cdot I \\ D_2 &= \frac{1}{2}N \\ D_3 &= \frac{1}{6\sqrt{\pi}}(N^2 - 4\Delta) \\ D_4 &= -\frac{1}{16}(\Delta N + 3N\Delta) \\ D_5 &= -\frac{1}{240\sqrt{\pi}}(N^4 + 16N^2\Delta + 8N\Delta N - 48\Delta^2) \ . \end{aligned}$$

In particular, we obtain the following presentation of the invariants $B_1, \cdots, B_5$: note in fact that $N1 = -\eta$ etc. In what follows, $\Delta^T, \nabla^T, \delta^T$ etc. will denote the tangential operators (that is, the operators defined on each equidistant with the induced metric).

$$
\begin{aligned}
B_1 &= \frac{2}{\sqrt{\pi}} \\
B_2 &= -\frac{1}{2}\eta \\
B_3 &= \frac{1}{6\sqrt{\pi}}\Big(-2\frac{\partial\eta}{\partial N}+\eta^2\Big) \\
B_4 &= -\frac{1}{16}\Big(\frac{\partial^2\eta}{\partial N^2}-\eta\frac{\partial\eta}{\partial N}-\Delta^T\eta\Big) \\
B_5 &= -\frac{1}{240\sqrt{\pi}}\Big(-20\frac{\partial^3\eta}{\partial N^3}+40\eta\frac{\partial^2\eta}{\partial N^2}+28\Big(\frac{\partial\eta}{\partial N}\Big)^2-20\eta^2\partial\eta\partial N+\eta^4 \\
&\quad +12\Delta^T(\frac{\partial\eta}{\partial N})+16\left|\nabla^T\eta\right|^2+4\delta^T(S\nabla^T\eta)\Big)\ .
\end{aligned}
$$

It is now easy to show that, if $B_1,\cdots,B_5$ are constant, then η and $\dfrac{\partial^k\eta}{\partial N^k}$ are constant for $k\leq 3$.

4.12. **Final iteration.** Iterating the above argument and using the recursive formulae one proves (see [29]):

$$
B_k = c_k\frac{\partial^{k-2}\eta}{\partial N^{k-2}} + \text{ terms involving normal derivatives of lower order}
$$

where c_k is a constant.

Important: *one needs to show that $c_k\neq 0$ for all k!*

This is a rather involved combinatorial fact, obtained using the algorithm for the computation of the operators D_k. Having done that, it follows by induction that:

if all B_k are constant, then all normal derivatives of η are also constant.

This ends the proof.

5. Appendix: Isoparametric tubes in $\mathbf{S}^n$

In this appendix we discuss the classification of isoparametric tubes in the round sphere $\mathbf{S}^n$. By definition, the boundary of an isoparametric tube is an isoparametric hypersurface, and thus we can apply the classification of isoparametric hypersurfaces of the sphere, a fascinating field which started from Segre and Cartan and gave rise to beautiful constructions. After many efforts, and many intermediate results, the classification was completed only recently [5].
Standard references for general facts on the isoparametric theory are [4] and [34].

After recalling the main definitions, we outline the main results concerning the Cartan polynomials.

5.1. **Shape operator and principal curvatures.** We briefly recall the basic definitions of shape operator and principal curvatures. Let M^n be a Riemannian manifold and Σ an hypersurface of M. We can define, at least locally, a unit normal vector field N to Σ. A way to see how curved is Σ in M is to examine the rate of change of the normal field N infinitesimally, along a curve γ in Σ.

At any point $p \in \Sigma$ and for any tangent vector $X \in T_p\Sigma$ we can define the covariant derivative of N along X, denoted $S_p(X)$:

$$S_p(X) = -\nabla_X N .$$

It is easy to see that this defines a linear operator

$$S_p : T_p\Sigma \to T_p\Sigma$$

which is self-adjoint with respect to the inner product given by the metric.

S_p is called the *shape operator* of Σ at p. Its eigenvalues $k_1, \cdots, k_{n-1}$ are called *principal curvatures* of Σ at p, and its trace, divided by $n-1$, is the *mean curvature*:

$$H(p) = \frac{1}{n-1}(k_1 + \cdots + k_{n-1}) .$$

That is,

$$H = \frac{1}{n-1}\mathrm{tr}S .$$

5.2. **When all principal curvatures are constant.** It is well-known that hypersurfaces with constant mean curvature are critical points for the area functional. But, what is the meaning of the constancy of the *principal curvatures*? Let M be a constant curvature space form (up to homotheties, $\mathbf{R}^n, \mathbf{H}^n$ and $\mathbf{S}^n$) and let Σ be a hypersurface of M.

Definition 17. We say that Σ is *isoparametric* if it has constant principal curvatures, that is, if the characteristic polynomial of Σ is the same at all points of Σ.

Obvious examples: geodesic spheres are isoparametric (all principal curvatures are the same). Here is one of the first results in the theory (Cartan):

In $\mathbf{R}^n, \mathbf{H}^n$ *and the hemisphere* $\mathbf{S}^n_+$ *the only* compact *isoparametric hypersurfaces are geodesic spheres.*

So, the theory is rather basic in those spaces: for some interesting facts we need to look at $\mathbf{S}^n$. Survey papers in the argument are [4], [33] and [34].

5.3. **Some examples in $\mathbf{S}^n$.** We start from *Clifford tori.*

Fix positive numbers a, b such that $a^2 + b^2 = 1$, and consider the Riemannian product:

$$\Sigma = \mathbf{S}^p(a) \times \mathbf{S}^q(b) .$$

This manifold has a natural isometric embedding into $\mathbf{S}^{p+q+1}$. It is easy to see that Σ admits only two distinct principal curvatures, namely

$$\begin{cases} \lambda = \dfrac{b}{a} & p \ \text{ times} \\ \mu = -\dfrac{a}{b} & q \ \text{ times} \end{cases}$$

which are constant on Σ. Thus, Clifford tori are isoparametric.

- *In* $\mathbf{S}^3$ *the only isoparametric surfaces are geodesic spheres and Clifford tori.*

What about higher dimensions? Here is a fancier example.

Consider the polynomial function $F : \mathbf{R}^5 \to \mathbf{R}$:

$$\begin{aligned} F(x_1, x_2, x_3, x_4, x_5) = \; & x_5^3 + \frac{3}{2} x_5 (x_1^2 - 2x_2^2 + x_3^2 - 2x_4^2) \\ & + \frac{3\sqrt{3}}{2} x_4 (x_1^2 - x_3^2) - 3\sqrt{3} x_1 x_2 x_3 \ . \end{aligned}$$

and restrict it to $\mathbf{S}^4$ to get a function $\widetilde{F} : \mathbf{S}^4 \to \mathbf{R}$.

Fact: *Any regular level set of* $\widetilde{F}$ *is an isoparametric hypersurface of* $\mathbf{S}^4$.

We see that there are more isoparametric hypersurface besides spheres and Clifford tori.

5.4. **Classification of isoparametric hypersurfaces, some algebraic facts.** It turns out that every isoparametric hypersurface is a regular level set of the restriction to $\mathbf{S}^n$ of a polynomial in $\mathbf{R}^{n+1}$, which we will define below. Hence *isoparametric hypersurfaces are smooth algebraic varieties.*

• A classification is done according to the number g of distinct principal curvatures, which, by a celebrated result of Münzner [23, 24], proved by topological methods, can be only $1, 2, 3, 4, 6$.

• The case $g = 1$ corresponds to the family of geodesic spheres, and $g = 2$ corresponds to Clifford tori.

• It is a remarkable and perhaps surprising fact that when $g = 4$ there exists non-homogenous isoparametric hypersurfaces (see [34]).

• For each Σ there exist two regular, connected submanifolds Σ_+, Σ_- of $\mathbf{S}^n$ such that Σ is the surface of the tube with radius r_+ (resp. r_-) around Σ_+ (resp. Σ_-). These submanifolds are called the *focal submanifolds* of Σ, and are minimal in $\mathbf{S}^n$.

• Every isoparametric hypersurface Σ belongs to a one-parameter family, which gives rise to what is known to be an *isoparametric foliation* of $\mathbf{S}^n$. This foliation has precisely two singular leaves (the focal submanifolds Σ_+ and Σ_-) and contains exactly one minimal isoparametric hypersurface: when $g = 1$ it is the unique equator of the family (which is totally geodesic), and when $g = 2$, for fixed p and q, it is the minimal Clifford torus defined by the identity $pb^2 = qa^2$.

Other geometric properties. In what follows, Σ is an isoparametric hypersurface of $\mathbf{S}^n$.
List the distinct principal curvatures of Σ in decreasing order:

$$k_1 > k_2 > \cdots > k_g \ ,$$

where g denotes the number of distinct principal curvatures. It turns out that $k_i = \cot \theta_i$ for a sequence $0 < \theta_1 < \cdots < \theta_g < \pi$ such that

$$\theta_j = \theta_1 + \frac{j-1}{g} \pi \ .$$

If m_j is the multiplicity of k_j then we have a cyclic behavior:

$$m_{j+2} = m_j \quad (\text{modulo } g)$$

which implies that the sequence of multiplicities $m_1, m_2, \cdots$ is determined by the pair

$$(m_1, m_2)\ .$$

In particular, $m_1 = m_2 = \cdots = m_g$ whenever g is odd. Set

$$c = \frac{1}{2}(m_2 - m_1)g^2\ . \tag{16}$$

Note that $c = 0$ if and only if all multiplicities are equal; this holds whenever g is odd and also when $g = 6$, by a result of Münzner. When $g = 2$ one has $c = 0$ if and only if n is odd and Σ is a Clifford torus $\mathbf{S}^p(a) \times \mathbf{S}^p(b)$, that is, $p = q$.

It turns out that Σ is always a (smooth) real algebraic variety: in fact, Σ is a regular level set of the restriction to $\mathbf{S}^n$ of a homogeneous polynomial $F : \mathbf{R}^{n+1} \to \mathbf{R}$ of degree g which satisfies the conditions

$$\begin{cases} |\overline{\nabla} F|^2 &= g^2 |x|^{2g-2} \\ \overline{\Delta} F &= c|x|^{g-2} \end{cases} \tag{17}$$

where c is as in (16) and $\bar{\Delta}, \bar{\nabla}$ are the Laplacian and the gradient in $\mathbf{R}^{n+1}$.

• A polynomial F with the properties (17) is an example of *Cartan-Münzner polynomial.*

Conversely, any Cartan-Münzner polynomial of degree g with a constant $c \neq \pm(n-1)g$ gives rise to an isoparametric foliation of $\mathbf{S}^n$ with g distinct principal curvatures.

With that in mind, the geometric classification of isoparametric foliations reduces to the algebraic problem of classifying all Cartan-Münzner polynomials.

5.5. **Inhomogeneous examples.** It is remarkable that there exists isoparametric hypersurfaces which are not homogeneous. These occur for the value $g = 4$, and were first found by Ozeki and Takeuchi [25, 26]. Later on Ferus, Karcher and Münzner [10] were able to construct a much larger family, called hypersurfaces of FKM-type, using representation of Clifford algebras. This construction gives rise to an infinite family of non-congruent inhomogeneous isoparametric foliations of $\mathbf{S}^n$. It turns out that an isoparametric hypersurface is either honogeneous or of FKM-type, and the final steps in the classification were recently completed by Chi [5].

5.6. **Classification of isoparametric tubes in $\mathbf{S}^n$.** As an easy consequence of the classification of isoparametric hypersurfaces, we remark the following fact whose proof can be found in [31]:

Theorem 18. *Let Ω be a domain in $\mathbf{S}^n$. Then Ω is an isoparametric tube if and only if:*

a) either Ω is the domain bounded by a connected isoparametric hypersurface,

b) or Ω is a tube around a minimal isoparametric hypersurface Σ such that all its distinct principal curvatures have the same multiplicity (that is, Σ is minimal with $c = 0$).

In the first case the boundary is connected and the soul is a focal submanifold of $\partial\Omega$; in the second case the soul is Σ and the boundary consists of two parallel isoparametric hypersurfaces, which are at the same distance to Σ and have the same mean curvature (with respect to the inner normal vector).

For what we have just said we see that, in low dimensions:

Corollary 19. a) *An isoparametric tube in* $\mathbf{S}^2$ *is either a geodesic disk or a tube around an equator.*
b) *An isoparametric tube in* $\mathbf{S}^3$ *is congruent to one of the following: a geodesic ball, a tube around the equator, a domain bounded by a Clifford torus or a tube around the minimal Clifford torus* $\mathbf{S}^1(\frac{1}{\sqrt{2}}) \times \mathbf{S}^1(\frac{1}{\sqrt{2}})$.

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