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Convex functions are p-subharmonic functions, $p > 1$, on $\mathbb{R}^n$, with applications

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Abstract. In this paper we discuss convexity, its average principle, an extrinsic average variational method in the Calculus of Variations, an average method in Partial Differential Equations, a link of convexity to p-subharmonicity, subsolutions to the p-Laplace equation, uniqueness, existence, isometric immersions in multiple settings. In particular, we show that a convex function on $\mathbb{R}^n$ is a p-subharmonic function, for every $p > 1$, and a C^2 convex function on a Riemannian manifold is a p-subharmonic function f, for every $p > 1$. We also show that a C^2 convex function which is a submersion on a Riemannian manifold is a p-subharmonic function, for every $p \geq 1$. This result is sharp. As further applications, via function growth estimates in p-harmonic geometry, we prove that every p-balanced nonnegative C^2 convex function on a complete noncompact Riemannian manifold is constant for $p > 1$. In particular, every L^q, nonnegative, convex function of class C^2 on a complete noncompact Riemannian manifold is constant for $q > p - 1 > 0$.

1. Introduction

Convex functions are fundamental objects, tools, and concepts in various branches of mathematics. In fact, the notion of convexity plays an important role in several areas of Mathematics, such as real and complex analysis, differential geometry, nonlinear potential theory, calculus of variations, partial differential equations, geometric measure theory, optimal control theory, geometric function theory, and other more. Convexity enjoys the following.

An *average* principle of convexity (resp. concavity, linearity)
(cf. [22, (8.1)]).
Let f be a convex function (resp. concave function, linear function). Then

$$\begin{aligned} f(\text{average}) &\leq \text{average } (f)\ , \\ \big(\text{resp. } f(\text{average}) &\geq \text{average } (f)\ , \\ f(\text{average}) &= \text{average } (f)\ \big)\ . \end{aligned}$$

Applying the above principle, where a convex function $f = \exp$ and "average" is taken over two positive numbers with respect to the sum, yields one of the simplest inequalities, G.M. $\leq$ A.M. This is a sharp isoperimetric inequality for

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plane rectangles that has far-reaching impacts. A dual approach from discreteness to continuity yields a sharp isoperimetric inequlality for plane curves, which is equivalent to the Sobolev inequality on $\mathbb{R}^2$ with optimal constant (cf. [22, §8]). Isoperimetric and Sobolev inequlalities are extended to Riemannian manifolds M with sharp constants and with applications to optimal sphere theorems (cf. e.g. Wei-Zhu [26]). The above *average* principle also leads to Jensen's inequalities for p-Yang-Mills energy functional and for normalized exponential Yang-Mills energy functional in Gauge Theory (cf. [22, Theorems 10.1 and 9.1]).

For a down-to-earth discussion, we recall a function $f : (a, b) \to \mathbb{R}$, on an open interval $(a, b) \subset \mathbb{R}$ is *convex* if for every interval $(c, d) \subset (a, b)$, and every linear function h with

$$f = (\text{or } \leq)\, h \quad \text{on} \quad \partial[c, d], \quad \text{then} \quad f\big((1-\lambda)c + \lambda d\big) \leq h(\lambda) \quad \text{on} \quad [c, d]\ ,$$

where $h(\lambda) = (1-\lambda)f(c) + \lambda f(d)$. It is an elegant link between geometric function theory and the theory of differential equations. Namely, if f is C^2, then f is convex on (a, b) if and only if

$$f \quad \text{is a } \textit{subsolution} \text{ of} \quad h'' = 0 \quad , \quad \text{i.e.} \quad f'' \geq 0 \quad \text{on} \quad (a, b)\ .$$

In the differentialble context the idea in a one-dimensional open interval can be extended to an infinite-dimensional Hilbert real space $\mathcal{H}$. Let $f : \mathcal{H} \to \mathbb{R}$ be a function of class C^2 (for simplicity), and $d^2 f_x(v, w)$ be the second derivative of f at $x \in \mathcal{H}$ in the directions v, w, then we have

Proposition 1.1 ([3])**.** *If $d^2 f_x(v, v) \geq 0$ for all $x, v \in \mathcal{H}$ then f is lower semiconinuous in $\mathcal{H}$. Furthermore, the subsets of $\mathcal{H}$ which are weakly compact are precisely those which are bounded and weakly closed.*

This is an abstraction of the basic work of Tonelli and Morrey on convexity properties of a variational density to insure *existence* in the calculus of variations.(cf. [15, 3]).

Convexity properties also lead to the *uniqueness* of p-harmonic maps of a compact Riemannian manifold into a compact manifold with nonpositive sectional curvature, without using heat flow method (cf. [19]). When $p = 2$, this generalizes the uniqueness theorem of harmonic maps due to Hartman ([11]).

Observing Mathematics and Nature are beautifully interwoven, and are frequently two sides of the same coin, (as manifested by legendary sage Lao Tzu in his book Tao Te Ching,) we proposed an extrinsic, *average* variational method in the Calculus of Variations (cf. [17, 18]) as an approach to confront and resolve problems in global, nonlinear analysis, geometry and physics, by which we pioneered the study of p-harmonic geometry (cf. e.g., [19, 20]), and we have found new manifolds (cf. [23, 10, 5, 6, 7]). These newly found manifolds have their strong interactions with geometry, topology, analysis, partial differential equations, calculus of variations, physics, and are briefly listed in the following table (cf. [22], page 321 and references therein). This extrinsic average variational method in the Calculus of Variations is in contrast to an average method in *Partial Differential Equations* that we applied (cf. [1, Proposition 2.1]) to obtain sharp growth estimates for warping functions in multiply warped product manifolds, and to solve their *isometric immersion problems* into Riemannian manifolds, complex space forms, quaternionic space forms, etc. ([1]).

TABLE 1. An Extrinsic Average Variational Method

Mappings	Functionals	New manifolds found	Geometry	Topology
harmonic map or $\Phi_{(1)}$-harmonic map	energy functional E or $E_{\Phi_{(1)}}$	SSU manifolds or $\Phi_{(1)}$-SSU manifolds	SU or $\Phi_{(1)}$-SU	$\pi_1 = \pi_2 = 0$ $\pi_1 = \pi_2 = 0$
p-harmonic map	E_p	p-SSU manifolds	p-SU	$\pi_1 = \cdots = \pi_{[p]} = 0$
Φ-harmonic map or $\Phi_{(2)}$-harmonic map	Φ-energy functional E_Φ or $E_{\Phi_{(2)}}$	Φ-SSU manifolds or $\Phi_{(2)}$-SSU manifolds	Φ-SU or $\Phi_{(2)}$-SU	$\pi_1 = \cdots = \pi_4 = 0$ $\pi_1 = \cdots = \pi_4 = 0$
Φ_S-harmonic map	E_{Φ_S}	Φ_S-SSU manifolds	Φ_S-SU	$\pi_1 = \cdots = \pi_4 = 0$
$\Phi_{S,p}$-harmonic map	$E_{\Phi_{S,p}}$	$\Phi_{S,p}$-SSU manifolds	$\Phi_{S,p}$-SU	$\pi_1 = \cdots = \pi_{[2p]} = 0$
$\Phi_{(3)}$-harmonic map	$\Phi_{(3)}$-energy functional $E_{\Phi_{(3)}}$	$\Phi_{(3)}$-SSU manifolds	$\Phi_{(3)}$-SU	$\pi_1 = \cdots = \pi_6 = 0$

On the other hand, the idea in calculus is naturally extended to Riemannian geometry: A smooth function $f : M \to \mathbb{R}$ on a Riemannian manifold M is said to be *convex* if for each geodesic curve $c : (-\epsilon, \epsilon) \to M$, $f \circ c : (-\epsilon, \epsilon) \to \mathbb{R}$ is a convex function, or equivalently, its Hession ∇df on M is nonnegative definite, where df is the differential of f, ∇ is the Riemannian connection on M, and $\nabla df(X, Y) \equiv (\nabla_X df)(Y)$, for any smooth vector fields X and Y on M. This idea can be extended to nonlinear potential theory: An upper semicontinuous function f is said to be *p-subharmonic*, if $f : M \to \mathbb{R} \cup \{\infty\}$, $f \not\equiv \infty$ on each component of M, and for every bounded domain Ω in M, and every p-harmonic function $h \in \overline{\Omega}$, with

$$f \le h \text{ on } \partial\Omega \quad \text{then} \quad f \le h \quad \text{in} \quad \Omega\,.$$

Here a *p-harmonic function* $h \in H^{1,p}_{loc}(\Omega)$ is a continuous weak solution of p-Laplace equation

$$\operatorname{div}(|\nabla h|^{p-2}\nabla h) = 0\,.$$

Again the interplay between geometric function theory and the theory of partial differential equations indicates that if f is continuous and $1 < p < \infty$, then f is a p-subharmonic function in M if and only if f is a *subsolution of the p-Laplace equation*, i.e.

$$\operatorname{div}(|df|^{p-2}df) \equiv \sum_{i=1}^{n}(\nabla(|df|^{p-2}df))(e_i, e_i) \ge 0 \tag{1.1}$$

weakly in M, where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field on M. That is

$$\int_M \langle |df|^{p-2}df, d\phi\rangle_M \; dv \le 0 \tag{1.2}$$

whenever $\phi \in C_0^\infty(M)$ is nonnegative (cf. e.g., [8, 12, 13, 25]). Here $\langle\, ,\, \rangle_M$ and dv denote the Rimannian metric and volume element on M respectively.

In fact, there are many

Example 1.1 (of p-subharmonic functions)**.** Most commonly seen functions such as the exponential function e^x on $\mathbb{R}$, $e^{|x|}$ on $\mathbb{R}^n$, and the distance function squared in Cartan-Hadmard manifolds are p-subharmonic functions for every $p > 1$.

These examples are natural generalizations of convex functions. In particular, we have

Theorem 1.1. *A C^2 convex function on a Riemannian manifold M is p-subharmonic, for every $p > 1$.*

Theorem 1.1 is a real analog of a well-known result in complex geometry due to Greene and Wu [9]

Theorem 1.2. *On a Kähler manifold, every C^2 convex function is plurisubharmonic.*

Recall a C^2 real-valued function f on a complex manifold is said to be *plurisubharmonic* if the Levi form Lf of f

$$Lf \equiv 4 \sum_{\alpha,\beta} \frac{\partial^2 f}{\partial z^\alpha \partial \overline{z}^\beta} dz^\alpha d\overline{z}^\beta \geq 0 \ ,$$

where $\{z^\alpha = x^\alpha + \sqrt{-1}\, y^\alpha\}$ is a local (complex) coordinate system in M, $dz^\alpha = dx^\alpha + \sqrt{-1}\, dy^\alpha$, $\dfrac{\partial}{\partial z^\alpha} = \dfrac{1}{2}(\dfrac{\partial}{\partial x^\alpha} - \sqrt{-1}\,\dfrac{\partial}{\partial y^\alpha})$, and $d\overline{z}^\alpha$ and $\dfrac{\partial}{\partial \overline{z}^\alpha}$ are complex conjugates of dz^α and $\dfrac{\partial}{\partial z^\alpha}$ respectively.

If f is a submersion, i.e. $|df| \neq 0$ everywhere, then one can extend the range of p:

Theorem 1.3. *A C^2 convex function on a Riemannian manifold M that is a submersion, is p-subharmonic for every $p \geq 1$.*

This result is sharp (cf. Counter-Example 4.1). As immediate consequences of Theorems 1.1 and 1.3, we have

Corollary 1.1. *Every C^2 concave function on a Riemannian manifold M is p-superharmonic, for any $p > 1$, and every C^2 concave submersive function on M is p-superharmonic, for any $p \geq 1$.*

Corollary 1.2. *Let f_i, $i = 1, 2$, and p be as in the assumption and conclusion of Theorem 1.1 or 1.3 respectively. Let $\lambda > 0$, then λf_1, $f_1 + f_2$, and $\max\{f_1, f_2\}$ are p-subharmonic functions.*

Corollary 1.3. *Let an increasing sequence of functions $\{f_i\}_{i=1}^\infty$ and p be as in the Corollary 1.2. Then $\lim_{i\to\infty} f_i$ is p-subharmonic.*

If M is Euclidean space $\mathbb{R}^n$, then one can drop the C^2 assumption on f:

Theorem 1.4. *A convex function on $\mathbb{R}^n$ is p-subharmonic, for every $p > 1$, and a convex function on $\mathbb{R}^n$ with the n-dimensional Lebesgue measure $\mathcal{L}^n(\{x \in \mathbb{R}^n : |df| = 0\}) = 0$, is p-subharmonic, for every $p \geq 1$,*

This result is sharp (cf. Counter-Example 4.1).

In this paper we combine the link between convex functions and p-subharmonic functions, and the estimates on the growth of p-subharmonic functions (cf. [25], or §2) to prove Liouville type theorems for convex functions. We recall for a given $q \in \mathbb{R}$, a function or a differential form or a bundle-valued differential form f has p-balanced growth (or, simply, is p-balanced) if f has one of the following: p-finite, p-mild, p-obtuse, p-moderate, or p-small growth, and has p-imbalanced growth (or, simply, is p-imbalanced) otherwise (cf. [25], or §2). As further applications, we have the following.

Theorem 1.5 (Louiville type theorem for convex functions)**.** *Every p-balanced nonnegative C^2 convex function on a complete noncompact Riemannian manifold is constant for $p>1$.*

Corollary 1.4. *Every L^q- nonnegative C^2 convex function on a complete noncompact Riemannian manifold is constant for $q>p-1>0$.*

2. Preliminaries

Let (M,g) be a smooth Riemannian manifold. Let $\xi : E \to M$ be a smooth Riemannian vector bundle over (M,g), i.e. a vector bundle such that at each fiber is equipped with a positive inner product $\langle\ ,\ \rangle_E$. Set $A^k(\xi)=\Gamma(\Lambda^k T^*M \otimes E)$ the space of smooth $k-$forms on M with values in the vector bundle $\xi : E \to M$. For two forms $\Omega, \Omega' \in A^k(\xi)$, the induced inner product $\langle \Omega, \Omega' \rangle$ is defined as in follows:

$$\begin{aligned}\langle \Omega, \Omega' \rangle &= \sum_{i_1<\cdots<i_k} \langle \Omega(e_{i_1},\dots,e_{i_k}), \Omega'(e_{i_1},\dots,e_{i_k})\rangle_E \\ &= \frac{1}{k!}\sum_{i_1,\dots,i_k} \langle \Omega(e_{i_1},\dots,e_{i_k}), \Omega'(e_{i_1},\dots,e_{i_k})\rangle_E \ ,\end{aligned}$$

where $\{e_1,\cdots e_n\}$ is a local orthonormal frame field on (M,g). For $\Omega \in A^k(\xi)$, set $|\Omega|^2=\langle \Omega,\Omega\rangle$. Then $|\Omega|^q=\langle \Omega,\Omega\rangle^{q/2}$. Following [21], we introduce the following notions.

Definition 2.1. For a given $q \in \mathbb{R}$ a function or a differential form or a bundle-valued differential form f has *p-finite growth* (or, simply, is *p-finite*) if there exists $x_0 \in M$ such that

$$\liminf_{r\to\infty} \frac{1}{r^p}\int_{B(x_0;r)} |f|^q\, dv < \infty \ , \tag{2.1}$$

and has *p-infinite growth* (or, simply, is *p-infinite*) otherwise.

For a given $q \in \mathbb{R}$, a function or a differential form or a bundle-valued differential form f has *p-mild growth* (or, simply, is *p-mild*) if there exist $x_0 \in M$ and a strictly increasing sequence of $\{r_j\}_0^\infty$ going to infinity, such that for every $l_0>0$, we have

$$\sum_{j=\ell_0}^{\infty}\left(\frac{(r_{j+1}-r_j)^p}{\int_{B(x_0;r_{j+1})\backslash B(x_0;r_j)} |f|^q\, dv}\right)^{1/(p-1)} = \infty \ , \tag{2.2}$$

and has *p-severe growth* (or, simply, is *p-severe*) otherwise.

For a given $q \in \mathbb{R}$, a function or a differential form or a bundle-valued differential form f has *p-obtuse growth* (or, simply, is *p-obtuse*) if there exists $x_0 \in M$ such that for every $a > 0$, we have

$$\int_a^\infty \left(\frac{1}{\displaystyle\int_{\partial B(x_0;r)} |f|^q \, ds} \right)^{1/(p-1)} dr = \infty \,, \tag{2.3}$$

and has *p-acute growth* (or, simply, is *p-acute*) otherwise.

For a given $q \in \mathbb{R}$, a function or a differential form or a bundle-valued differential form f has *p-moderate growth* (or, simply, is *p-moderate*) if there exist $x_0 \in M$, and $\psi(r) \in \mathcal{F}$, such that

$$\limsup_{r \to \infty} \frac{1}{r^p \psi^{p-1}(r)} \int_{B(x_0;r)} |f|^q \, dv < \infty \,, \tag{2.4}$$

and has *p-immoderate growth* (or, simply, is *p-immoderate*) otherwise, where

$$\mathcal{F} = \{\psi : [a,\infty) \to (0,\infty) \mid \int_a^\infty \frac{dr}{r\psi(r)} = \infty \quad \text{for some} \ \ a \geq 0\} \,. \tag{2.5}$$

(Notice that the functions in $\mathcal{F}$ are not necessarily monotone.)

For a given $q \in \mathbb{R}$, a function or a differential form or a bundle-valued differential form f has *p-small growth* (or, simply, is *p-small*) if there exists $x_0 \in M$, such that for every $a > 0$, we have

$$\int_a^\infty \left(\frac{r}{\displaystyle\int_{B(x_0;r)} |f|^q \, dv} \right)^{1/(p-1)} dr = \infty \,, \tag{2.6}$$

and has *p-large growth* (or, simply, is *p-large*) otherwise.

Definition 2.2. For a given $q \in \mathbb{R}$, a function or a differential form or a bundle-valued differential form f has *p-balanced growth* (or, simply, is *p-balanced*) if f has one of the following: p-finite, p-mild, p-obtuse, p-moderate, or p-small growth, and has *p-imbalanced growth* (or, simply, is *p-imbalanced*) otherwise.

The above definitions of "p-balanced, p-finite, p-mild, p-obtuse, p-moderate, p-small" and their counter-parts "p-imbalanced, p-infinite, p-severe, p-acute, p-immoderate, p-large" growth depend on q, and q will be specified in the context in which the definition is used.

Theorem 2.1 ([21], Theorem 5.4.)**.** *For a given $q \in \mathbb{R}$, a function, or differential form or bundle-valued differential form f is*

$$p\text{-}moderate \ (2.4) \iff p\text{-}small \ (2.6) \implies p\text{-}mild \ (2.2) \implies p\text{-}obtuse \ (2.3)$$

or equivalently

$$p\text{-}acute \implies p\text{-}severe \implies p\text{-}large \iff p\text{-}immoderate \,.$$

Hence, for a given $q \in \mathbb{R}$, f is

$$p\text{-}balanced \implies \text{either } p\text{-}finite \ (2.1) \text{ or } p\text{-}obtuse \ (2.3)$$

$$p\text{-}imbalanced \implies \text{both } p\text{-}infinite \text{ and } p\text{-}immoderate \,.$$

If in addition $\int_{B(x_0;r)} |f|^q \, dv$ *is convex in* r*, then the following four types of growth are all equivalent:* f *is p-mild, p-obtuse, p-moderate, and p-small (resp. p-severe, p-acute, p-immoderate, and p-large), i.e.* f *is* (2.2) $\Leftrightarrow$ (2.3) $\Leftrightarrow$ (2.4) $\Leftrightarrow$ (2.6) *for the same value of* $q \in \mathbb{R}$.

In particular, we have

Corollary 2.1 ([21], Corollary 5.1)**.** *Every* L^q *function or differential form or bundle-valued differential form* f *on* M *has p-balanced growth,* $p \geq 0$*, and in fact, has p-finite, p-mild, p-obtuse, p-moderate, and p-small growth,* $p \geq 0$*, for the same value of* q.

In [25], among many different types of inequalities on a complete noncompact Reimannian manifold M, we have the following uniqueness property.

Theorem 2.2 (Liouville property for solutions of $f \operatorname{div}(|\nabla f|^{p-2}\nabla f) \geq 0$)**.** *Every* C^2 *solution* $f : M \to (-\infty, \infty)$ *of* $f \operatorname{div}(|\nabla f|^{p-2}\nabla f) \geq 0$ *is constant provided* f *is p-balanced, i.e.* f *is one of the following: p-finite, p-mild, p-obtuse, p-moderate, or p-small, for some* $q > p - 1$*. In particular, every* C^2*,* L^q *solution* f *of* $f \operatorname{div}(|\nabla f|^{p-2}\nabla f) \geq 0$ *is constant for any* $q > p - 1$.

3. Convexity and p-subharmonicity

For completeness, we prove theorems that link convexity and p-subharmonicity as follows.

Proof of Theorem 1.1. Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame field on M. Then the differential df satisfies

$$\sum_{i=1}^n df(e_i)e_i = \sum_{i=1}^n \langle \nabla f, e_i \rangle_M e_i = \nabla f \tag{3.1}$$

At point $x \in M$ where $|\nabla f| \neq 0$, we may assume, without loss of generality $e_1 = \dfrac{\nabla f}{|\nabla f|}$.

$$\begin{aligned}(\nabla_{\nabla f} df)(\nabla f) &= (\nabla df)(\nabla f, \nabla f) \\ &= (\nabla df)(|\nabla f|e_1, |\nabla f|e_1) \\ &= |df|^2 (\nabla df)(e_1, e_1)\,.\end{aligned} \tag{3.2}$$

At point $x \in M$ where $|\nabla f| = 0$,

$$(\nabla_{\nabla f} df)(\nabla f) = 0\,. \tag{3.3}$$

Let $j > 0$ be an integer. It follows from (3.1) and (3.2) that

$$\begin{aligned}&\operatorname{div}\Big(\big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} df\Big) \\ &\equiv \sum_{i=1}^n \nabla\Big(\big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} df\Big)(e_i, e_i) \\ &= \sum_{i=1}^n \Big(\nabla_{e_i}\Big(\big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} df\Big)\Big)(e_i)\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \Big(\Big(e_i \big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} \Big) df \Big)(e_i) \\
&\quad + \sum_{i=1}^{n} \Big(\big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} \nabla_{e_i} df \Big)(e_i) \\
&= \sum_{i=1}^{n} \Big((p-2)\big(|df|^2 + \frac{1}{j}\big)^{(p-4)/2} \langle \nabla_{e_i} df, df \rangle df \Big)(e_i) \\
&\quad + \sum_{i=1}^{n} \Big(\big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} \nabla df \Big)(e_i, e_i) \\
&= \sum_{i=1}^{n} (p-2)\big(|df|^2 + \frac{1}{j}\big)^{(p-4)/2} \langle \nabla_{e_i} df, df \rangle df(e_i) \\
&\quad + \sum_{i=1}^{n} \Big(\big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} \nabla df \Big)(e_i, e_i) \\
&= (p-2)\big(|df|^2 + \frac{1}{j}\big)^{(p-4)/2} \langle \nabla_{\nabla f} df, df \rangle \\
&\quad + \sum_{i=1}^{n} \Big(\big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} \nabla df \Big)(e_i, e_i) \\
&= (p-2)\big(|df|^2 + \frac{1}{j}\big)^{(p-4)/2} (\nabla_{\nabla f} df)(\nabla f) \\
&\quad + \sum_{i=1}^{n} \Big(\big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} \nabla df \Big)(e_i, e_i)
\end{aligned}
$$

At point $x \in M$ where $|\nabla f| \neq 0$, for every integer $j > 0$, the last expression

$$
\begin{aligned}
&(p-2)\big(|df|^2 + \frac{1}{j}\big)^{(p-4)/2} \big(\nabla_{\nabla f} df\big)(\nabla f) \\
&\quad + \sum_{i=1}^{n} \Big(\big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} \nabla df \Big)(e_i, e_i) \\
&= (p-2)\big(|df|^2 + \frac{1}{j}\big)^{(p-4)/2} |df|^2 (\nabla df)(e_1, e_1) \\
&\quad + \sum_{i=1}^{n} \big(|df|^2 + \frac{1}{j}\big)^{(p-4)/2} \big(|df|^2 + \frac{1}{j}\big) (\nabla df)(e_i, e_i) \\
&\geq (p-1)\big(|df|^2 + \frac{1}{j}\big)^{(p-4)/2} |df|^2 (\nabla df)(e_1, e_1) \\
&\quad + \sum_{i=2}^{n} \big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} (\nabla df)(e_i, e_i) \\
&\geq 0
\end{aligned} \tag{3.5}
$$

by (3.2), the convexity of f and $p \geq 1$.

At point $x \in M$ where $|\nabla f| = 0$, for every $j > 0$, the last expression in (3.4)

$$
\begin{aligned}
&(p-2)\big(|df|^2 + \frac{1}{j}\big)^{(p-4)/2} \big(\nabla_{\nabla f} df\big)(\nabla f) \\
&\quad + \sum_{i=1}^{n} \Big(\big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} \nabla df \Big)(e_i, e_i) \\
&\geq 0
\end{aligned} \tag{3.6}
$$

by (3.3), the convexity of f and $p \geq 1$. Combining (3.4), (3.5), and (3.6), we have for every integer $j > 0$,

$$\text{div}\Big(\big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} df\Big) \geq 0 \quad \text{everywhere in} \quad M\,. \tag{3.7}$$

It follows from the Stokes Theorem that for every integer $j > 0$,

$$\int_M \langle \big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} df, d\phi \rangle_M \, dx \leq 0 \tag{3.8}$$

whenever $\phi \in C_0^\infty(M)$ is nonnegative.

As Cauchy-Schwarz inequality yields

$$\begin{aligned}\langle \big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} df, d\phi \rangle_M &\leq \big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} |df||d\phi| \\ &\leq \big(|df|^2 + 1\big)^{(p-1)/2} |d\phi| \in L^1(M)\,,\end{aligned}$$

it follows from the fact that

$$\lim_{j\to\infty} \big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} df = |df|^{p-2} df \tag{3.9}$$

everywhere for $p > 1$, the dominated convergence theorem and (3.8), we obtain the desired

$$\int_M \langle |df|^{p-2} df, d\phi\rangle_M \, dx = \lim_{j\to\infty} \int_M \langle \big(|df|^2 + \frac{1}{j}\big)^{(p-2)/2} df, d\phi \rangle_M \, dx \leq 0\,. \tag{3.10}$$

Proof of Theorem 1.3. If f is a submersion, then for $p \geq 1$, (3.9) holds and hence (3.10) completes the proof.

Proof of Corollary 1.1. Since f is concave, $-f$ is convex and hence by Theorem 1.1, for every $p \geq 1$, $-f$ is a p-subharmonic function, or f is a p-superharmonic function.

Proof of Theorem 1.4. If f is a convex function on $\mathbb{R}^n$, then by Aleksandrov's Theorem (cf. [16, 4]), f has a second derivative $\mathcal{L}^n$ almost everywhere. Replacing M in the proofs of Theorem 1.1 and 1.3 with $\mathbb{R}^n$, and "everywhere" with "$\mathcal{L}^n$ a.e." complete the proof.

4. A Counter-Example

In this section, we show the optimality of $p \geq 1$ in Theorem 1.3 (in which $M = \mathbb{R}^n \backslash \{0\}$) and Theorem 1.4 (in which $|df| \neq 0$ $\mathcal{L}^n$ a.e.) by giving

Counter-example 4.1. The function $f(x) = e^{|x|^2}$ in Euclidean space $\mathbb{R}^n$ is a convex function and is not a p-subharmonic function for any $p < 1$.

Proof. By a straightforward computation, we have:

$$\begin{aligned}\operatorname{div}\big(|df|^{p-2}df\big) &= \operatorname{div}\big((2re^{r^2})^{p-2}2x_1e^{r^2},\cdots,(2re^{r^2})^{p-2}2x_ne^{r^2}\big)\\ &= 2^{p-1}\sum_{i=1}^{n}\big\{e^{(p-1)r^2}r^{p-2}+e^{(p-1)r^2}(p-2)r^{p-3}\,\frac{x_i^2}{r}\\ &\quad +e^{(p-1)r^2}(p-1)2r^{p-2}x_i^2\big\}\\ &= 2^{p-1}\big\{nr^{p-2}e^{(p-1)r^2}+(p-2)r^{p-2}e^{(p-1)r^2}\\ &\quad +2(p-1)r^pe^{(p-1)r^2}\big\}\\ &= (n+p-2+2(p-1)r^2)2^{p-1}e^{(p-1)r^2}r^{p-2}\\ &< 0\end{aligned}$$

for sufficiently large $r > 0$, if $p < 1$.

□

The above computation also shows that $f(x) = e^{|x|^2}$ in $\mathbb{R}^n$ is a p-subharmonic function for every $p \geq 1$.

5. Further Applications

In this section, we utilize the link between convex functions and p-subharmonic functions, and apply the estimates on the growth of p-subharmonic functions in [25] to prove Liouville type theorem for convex functions.

Theorem 5.1 (Louiville Type Theorem for Convex Functions)**.** *Every p-balanced nonnegative C^2 convex function on a complete noncompact Riemannian manifold M is constant for $p > 1$.*

Proof of Theorem 5.1. Since f is a C^2 convex function on M, Theorem 1.1 implies that f is a p-subharmonic function for $p > 1$. This is equivalent to f is a subsolution of the p-Laplace equation, i.e., $\operatorname{div}(|df|^{p-2})df \geq 0$. In view of $f \geq 0$, we have $f\operatorname{div}(|df|^{p-2})df \geq 0$. It follows from Theorem 2.2 that f is constant.

Corollary 5.1. *Every C^2, L^q convex function on a complete noncompact Riemannian manifold M is constant for any $q > p - 1 > 0$.*

Proof. This follows from Corollary 2.1 that if f is in L^q, then f is p-balanced, $p > 0$ for the same q (cf. [21]). So we can apply Theorem 5.1, and the result follows.
□

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